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Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology

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Abstract

We analyze the space of geometrically continuous piecewise polynomial functions, or splines, for rectangular and triangular patches with arbitrary topology and general rational transition maps. To define these spaces of G^1 spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. We determine the dimension of the space of G^1 spline functions which are of degree $\leqslant k$ on triangular pieces and of bi-degree $\leqslant (k,k)$ on rectangular pieces, for k big enough. A separability property on the edges is involved to obtain the dimension formula. An explicit construction of basis functions attached respectively to vertices, edges and faces is proposed; examples of bases of G^1 splines of small degree for topological surfaces with boundary and without boundary are detailed.

Keywords: geometrically continuous splines, dimension and bases of spline spaces, gluing data, polygonal patches, surfaces of arbitrary topology

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1. Introduction

The accurate and efficient representation of shapes is a major challenge in geometric modeling. To achieve high order accuracy in the representation of curves, surfaces or functions, piecewise polynomials models are usually employed. Parametric models with prescribed regularity properties are nowadays commonly used in Computer Aided Geometric Design (CAGD) to address these problems. They involve so-called spline functions, which are piecewise polynomial functions on intervals of $\mathbb{R}$ with continuity and differentiability constraints at some nodes. Extensions of these functions to higher dimension is usually done by taking tensor product spline basis functions. Curves, surfaces or volumes are represented as the image of parametric functions expressed in terms of spline basis functions. For instance, surface patches are described as the image of a piecewise polynomial (or rational) map from a rectangular domain of $\mathbb{R}^2$ to $\mathbb{R}^n$. But to represent objects with complex topology, such maps on rectangular parameter domains are not sufficient. One solution which is commonly used in Computer-Aided Design (CAD) is to trim the B-spline rectangular patches and to "stitch" together the trimmed pieces to create the complete shape representation. This results in complex models, which are not simple to use and to modify, since structural rigidity conditions cannot easily be imposed along the trimming curve between two trimmed patches.

To allow flexibility in the representation of shapes with complex topology, another technique called geometric continuity has been studied. Rectangular parametric surface patches are glued along their common boundary, with continuity constraints on the tangent planes (or on higher osculating spaces). In this way, smooth surfaces can be generated from quadrilateral meshes by gluing several simple parametric surfaces, forming surfaces with the expected smoothness property along the edges.

This approach built on the theory on differential manifolds, in works such as [7], [12], [9]. The idea of using transition maps or reparameterizations in connection with building smooth surfaces had been used for instance by DeRose [7] in CAGD, who gave one of the first general definitions of splines based on fixing a parametrization.

Since these initial developments, several works focused on the construction of such G^1 surfaces [18], [16], [24], [23], [5], [32], [11], [13], [10], [27], [26], [3], . . . with polynomial, piecewise polynomial, rational or special functions and on their use in geometric modeling applications such as surface fitting or surface reconstruction [8], [25], [15], ...

The problem of investigating the minimal degree of polynomial pieces has also been considered [19]. Other research investigate the construction of adapted rational transition maps for a given topological structure [1]. We refer to [20] for a review of these constructions. Constraints that the transition maps must satisfy in order to define regular spline spaces have also been identified [21]. But it has not yet been proved that these constraints are sufficient for the constructions.

The use of G^k spline functions to approximate functions over computational domains with arbitrary topology received recently a new attention for applications in isogeometric analysis. In this context, describing the space of functions, its dimension and adapted bases is of particular importance. A family of bi-cubic spline functions was recently introduced by Wu et al [31] for isogeometric applications, where constant transition maps are used, which induce singular spline basis functions at extraordinary vertices. Multi-patch representations of computational domains are also used in [4], with constant transition maps at the shared edges of rectangular faces, using an identification of Locally Refined spline basis functions. In [14], G^k continuous splines are described and the G^1 condition is transformed into a linear system of relations between the control coefficients. The case of two rectangular patches, which share an edge is analysed experimentally. In [2], the space of G^1 splines of bi-degree $\geqslant 4$ for rectangular decompositions of planar domains is analyzed. Minimal Determining Sets of points are studied, providing dimension formulae and dual basis for G^1 spline functions over planar rectangular meshes with linear gluing transition maps.

Our objective is to analyze the space of G^1 spline functions for rectangular and triangular patches with arbitrary topology and general rational transition maps. We are interested in determining the dimension of the space of G^1 spline functions which are of degree $\leqslant k$ on triangular pieces and of bi-degree $\leqslant (k, k)$ on rectangular pieces. To define the space of G^1 spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by the faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. A separability property is involved to obtain a dimension formula of the G^1 spline spaces of degree $\leqslant k$ on such topological surfaces, for k big enough. This leads to an explicit construction of basis functions attached respectively to vertices, edges and faces.

For the presentation of these results, we structure the paper as follows. The next section introduces the notion of topological surface $\mathcal{M}$, differentiable functions on $\mathcal{M}$ and constraints on the transition maps to have an ample space of differentiable functions. Section 3 deals with the space of spline functions which are piecewise polynomial and differentiable on $\mathcal{M}$. Section 4 analyzes the gluing conditions along an edge. Section 5 analyzes the gluing condition around a vertex. In Section 6, we give the dimension formula for the space of spline functions of degree $\leqslant k$ over a topological surface $\mathcal{M}$ and describe explicit basis constructions. Finally, in Section 7, we detail an example with boundary edges and another one with no boundary edges. We also provide an appendix with an algorithmic description of the basis construction.

2. Differentiable functions on a topological surface

Typically in CAGD, parametric patches are glued into surfaces by splines (i.e., polynomial maps) from polygons in $\mathbb{R}^2$. The simplest C^r construction is with the polygons in $\mathbb{R}^2$ situated next to each other, so that C^r continuity across patch edges comes from C^r continuity of the coordinate functions across the polygon edges. This is called *parametric continuity*. A more general construction to generate a C^r surface from polygonal patches is called *geometric continuity* [7], [20]. Inspired by differential geometry, attempts have been made [12], [28], [29] to define *geometrically continuous G^r surfaces* from a collection of polygons in $\mathbb{R}^2$ with additional data to glue their edges and differentiations. They are defined by parametrization maps from the polygons to $\mathbb{R}^3$ satisfying geometric regularity conditions along edges.

It is easy to define a C^0 surface from a collection of polygons and homeomorphisms between their edges.

Definition 2.1. Given a collection $\mathcal{M}_2$ of (possibly coinciding) polygons σ_i in $\mathbb{R}^2$, a *topological surface* $\mathcal{M}$ is defined by giving a set of homeomorphisms $\mu : \tau_i \to \tau_j$ between pairs of polygonal edges $\tau_i \subset \sigma_i$, $\tau_j \subset \sigma_j$ $(\sigma_i, \sigma_j \in \mathcal{M}_2)$. Each polygonal edge can be paired with at most one other edge, and it cannot be glued with itself.

A G^0-*continuous function* on the topological surface $\mathcal{M}$ is defined by assigning a continuous function f_i to each polygon σ_i, such that the restrictions to the polygonal edges are compatible with the homeomorphisms μ.

The topological surface $\mathcal{M}$ is the disjoint union of the polygons, with some points identified to equivalence classes by the homeomorphisms μ. The polygons are also called the faces of $\mathcal{M}$ and their set is denoted $\mathcal{M}_2$. Each homeomorphism μ identifies the edges τ_i, τ_j of the polygons σ_i, σ_j to an *interior edge* of $\mathcal{M}$. We say that the edge is shared by the faces σ_i and σ_j. An edge not involved in any homeomorphism μ is a *boundary edge* of $\mathcal{M}$. The edges of $\mathcal{M}$ are the equivalent classes of edges of the polygons of $\mathbb{R}^2$ identified by the homeomorphisms μ. Their set is denoted $\mathcal{M}_1$. Similarly, let $\mathcal{M}_0$ denote the set of $\mathcal{M}$-vertices, that is, equivalences classes of polygonal vertices. An *interior vertex* is an equivalence class of polygonal vertices $\gamma_0, \gamma_1, \ldots, \gamma_n = \gamma_0$ such that the adjacent vertices γ_i, γ_{i+1} are identified by an edge homeomorphism. The set of these equivalent classes of identified vertices of the polygons, or interior vertices of $\mathcal{M}$, is denoted $\mathcal{M}_0^\circ$.

2.1. Gluing data

The definition of a differential surface S typically requires an atlas of S, that is a collection $\{V_p, \psi_p\}_{p\in J}$ such that $\{V_p\}_{p\in J}$ is an open covering of S [30]. Each ψ_p is a homeomorphism $\psi_p : U_p \to V_p$, where U_p is an open set in $\mathbb{R}^2$. For distinct $p, q \in J$ such that $V_p \cap V_q \neq \emptyset$, let $U_{p,q} := \psi_p^{-1}(V_p \cap V_q)$ and $U_{q,p} := \psi_q^{-1}(V_p \cap V_q)$. Then the map $\psi_q^{-1} \circ \psi_p : U_{p,q} \to U_{q,p}$ is required to be a C^1-diffeomorphism. The maps $\phi_{pq} : \psi_q^{-1} \circ \psi_p$ are called transition maps. A differentiable function f on S is a function such that for any open set V_p, the composition $f_p = f \circ \psi_p^{-1} : U_p \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Our objective is to study the space of differentiable functions that can be constructed on a surface S associated to the topological surface $\mathcal{M}$. Instead of an atlas of a differential surface S, we consider a topological surface $\mathcal{M}$ together with *gluing data* given by maps (that we call *transition maps*) between the pairs of faces of $\mathcal{M}$ that share an edge in $\mathcal{M}$. We make this precise in the following definition.

Definition 2.2. For a topological surface $\mathcal{M}$, a gluing structure associated to $\mathcal{M}$ consists of the following:

- for each face $\sigma \in \mathcal{M}_2$ an open set U_σ of $\mathbb{R}^2$ containing σ;
- for each edge $\tau \in \mathcal{M}_1$ of a cell σ, an open set $U_{\tau,\sigma}$ of $\mathbb{R}^2$ containing τ;
- for each edge $\tau \in \mathcal{M}_1$ shared by two faces $\sigma_i, \sigma_j \in \mathcal{M}_2$, a C^1-diffeomorphism called the *transition map* $\phi_{\sigma_j,\sigma_i} \colon U_{\tau,\sigma_i} \to U_{\tau,\sigma_j}$ between the open sets U_{τ,σ_i} and U_{τ,σ_j}, and also its correspondent inverse map ϕ_{σ_i,σ_j};

- for each edge $\tau \in \mathcal{M}_1$ of a cell σ, the identity C^1-diffeomorphism that defines the identity transition map $\phi_{\sigma,\tau} = \mathrm{Id}$ between U_σ and $U_{\tau,\sigma}$.

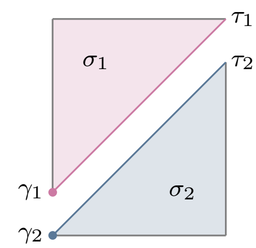


Figure 1: Topological surface constructed from two triangles.

A transition map as in Definition 2.2 differs from the usual notion of transition map in the context of differential manifolds (see [12]), since we do not require compatibility conditions at the vertices and across edges. The precise compatibility conditions that we need on these maps ϕ_{σ_j,σ_i} are given in Sections 2.3 ans 2.4.

Let $\tau = (\tau_1, \tau_2)$ be an edge shared by two faces $\sigma_1, \sigma_2 \in \mathcal{M}_2$ and let $\gamma = (\gamma_1, \gamma_2)$ be a vertex of τ corresponding to γ_1 in σ_1 and to γ_2 in σ_2, as in Figure 1. We denote by τ_1' (resp. τ_2') the second edge of σ_1 (resp. σ_2) through γ_1 (resp. γ_2) We associate to σ_1 and σ_2 two coordinate systems (u_1, v_1) and (u_2, v_2) such that $\gamma_1 = (0,0)$, $\tau_1 = \{(u_1, 0), u_1 \in [0,1]\}$, $\tau_1' = \{(0, v_1), v_1 \in [0,1]\}$ and $\gamma_2 = (0,0)$, $\tau_2 = \{(0, v_2), v_2 \in [0,1]\}$, $\tau_2' = \{(u_2, 0), u_2 \in [0,1]\}$. Using the Taylor expansion at $(0,0)$, a transition map from U_{τ,σ_1} to U_{τ,σ_2} is then of the form

$$\phi_{\sigma_2,\sigma_1} : (u_1, v_1) \longrightarrow (u_2, v_2) = \begin{pmatrix} v_1\, \mathfrak{b}_{\tau,\gamma}(u_1) + v_1^2 \rho_1(u_1, v_1) \\ u_1 + v_1\, \mathfrak{a}_{\tau,\gamma}(u_1) + v_1^2 \rho_2(u_1, v_1) \end{pmatrix} \qquad (1)$$

where $\mathfrak{a}_{\tau,\gamma}(u_1), \mathfrak{b}_{\tau,\gamma}(u_1), \rho_1(u_1, v_1), \rho_2(u_1, v_1)$ are C^1 functions. We will refer to it as the canonical form of the transition map ϕ_{σ_2,σ_1} at γ along τ. The functions $[\mathfrak{a}_{\tau,\gamma}, \mathfrak{b}_{\tau,\gamma}]$ are called the *gluing data* at γ along τ on σ_1.

Definition 2.3. An edge $\tau \in \mathcal{M}$ which contains the vertex $\gamma \in \mathcal{M}$ is called a *crossing edge at* γ if $\mathfrak{a}_{\tau,\gamma}(0) = 0$ where $[\mathfrak{a}_\tau, \mathfrak{b}_\tau]$ is the gluing data at γ along τ. We define $\mathfrak{c}_\tau(\gamma) = 1$ if τ is a crossing edge at γ and $\mathfrak{c}_\tau(\gamma) = 0$ otherwise. By convention, $\mathfrak{c}_\tau(\gamma) = 0$ for a boundary edge. If $\gamma \in \mathcal{M}_0$ is an interior vertex where all adjacent edges are crossing edges at γ, then it is called a *crossing vertex*. Similarly, we define $\mathfrak{c}_+(\gamma) = 1$ if γ is a crossing vertex and $\mathfrak{c}_+(\gamma) = 0$ otherwise.

2.2. Differentiable functions on a topological surface

We can now define the notion of differentiable function on $\mathcal{M}$:

Definition 2.4. A differentiable function f on the topological surface $\mathcal{M}$ is a collection $f = (f_\sigma)_{\sigma \in \mathcal{M}}$ of differentiable functions $f_\sigma : U_\sigma \to \mathbb{R}$ such that $\forall \gamma \in \tau = \sigma_1 \cap \sigma_2$, $\forall \mathbf{u} \in U_{\tau,\sigma_1}$,

$$J_\gamma(f_{\sigma_1})(\mathbf{u}) = J_\gamma(f_{\sigma_2} \circ \phi_{\sigma_2,\sigma_1})(\mathbf{u}) \tag{2}$$

where J_γ is the jet or Taylor expansion of order 1 at γ.

If f_1, f_2 are the functions associated to the faces $\sigma_1, \sigma_2 \in \mathcal{M}_2$ which are glued along the edge τ with a transition map of the form (1), the regularity condition (2) leads to the following relations:

- $f_1(u_1, 0) = f_2 \circ \phi_{\sigma_2,\sigma_1}(u_1, 0)$ for $u_1 \in [0, 1]$; that is
$$f_1(u_1, 0) = f_2(0, u_1) \tag{3}$$
- $\dfrac{\partial f_1}{\partial v_1}(u_1, 0) = \dfrac{\partial (f_2 \circ \phi)}{\partial v_1}(u_1, 0)$ for $\phi = \phi_{\sigma_2,\sigma_1}$ and $u_1 \in [0, 1]$, which translates to
$$\frac{\partial f_1}{\partial v_1}(u_1, 0) = \mathfrak{b}_{\tau,\gamma}(u_1)\frac{\partial f_2}{\partial u_2}(0, u_1) + \mathfrak{a}_{\tau,\gamma}(u_1)\frac{\partial f_2}{\partial v_2}(0, u_1) \tag{4}$$
for $u_1 \in [0, 1]$, with $\mathfrak{a}(u_1) = \frac{\partial \phi_1}{\partial v_1}(u_1, 0)$, $\mathfrak{b}(u_1) = \frac{\partial \phi_2}{\partial v_1}(u_1, 0)$, where ϕ_1 and ϕ_2 are the components of ϕ at the first and the second variable respectively.

A convenient way to describe this regularity condition is to express the relation (4) as a relation between differentials acting on the space of differential functions on the edge τ:

$$\mathfrak{a}_{\tau,\gamma}(u_1)\partial_{v_2} + \mathfrak{b}_{\tau,\gamma}(u_1)\partial_{u_2} - \partial_{v_1} = 0 \tag{5}$$

With this notation, at a crossing vertex γ with 4 edges we have $\mathfrak{b}_{\tau,\gamma}(0)\partial_{u_2} - \partial_{v_1} = 0$. The differentials along two opposite edges are "aligned", which explains the terminology of crossing vertex.

Definition 2.5. A subspace $\mathcal{D}$ of the vector space of differentiable functions on $\mathcal{M}$ is said to be *ample* if at every point γ of a face σ of $\mathcal{M}$, the space of values and differentials at γ, namely $\left[\, f(\gamma),\, \partial_{u_\sigma}(f)(\gamma),\, \partial_{v_\sigma} f(\gamma) \right]$ for $f \in \mathcal{D}$, is of dimension 3.

This definition does not depend on the choice of the face σ to which γ belongs, since for γ on a shared edge, the value and differentials coincide after transformation by the invertible transition map.

2.3. Compatibility condition at a vertex

Giving gluing data on the edges is not sufficient to ensure the existence of an ample space of differentiable functions on $\mathcal{M}$. At vertices shared by several edges and faces, additional conditions on the transition maps need to be satisfied. We describe them in this section, and show that they are sufficient to construct an ample space of splines on $\mathcal{M}$ in the following sections.

For a vertex $\gamma \in \mathcal{M}_0^\circ$, (see Fig. 2) which is common to faces $\sigma_1, \ldots, \sigma_F$ glued cyclically around γ, along the edges $\tau_i = \sigma_{i+1} \cap \sigma_i$ for $i = 1, \ldots, F$ (with $\sigma_{F+1} = \sigma_1$), we impose the following condition:

$$J_\gamma(\phi_{1,F}) \circ \cdots \circ J_\gamma(\phi_{3,2}) \circ J_\gamma(\phi_{2,1})(u, v) = (u, v), \tag{6}$$

where J_γ is the jet or Taylor expansion of order 1 at γ. We can assume that

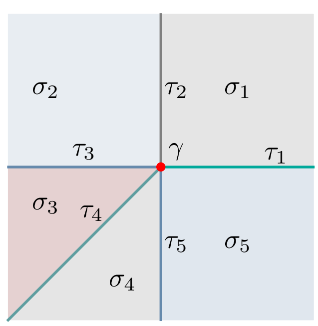


Figure 2: The faces σ_i for $i = 1, \ldots, 5$ are glued cyclically around a vertex γ.

for each $i = 1, \ldots, F$, the edge τ_i is defined (linearly) by $v_i = 0$ in σ_i. It is easy to check that the condition (6) on the Taylor expansion at γ leads to the following:

Condition 2.6. *If the vertex γ is on the faces $\sigma_1, \ldots, \sigma_F$ glued cyclically around γ, the gluing $[\mathfrak{a}_i,\ \mathfrak{b}_i]$ at γ on the edges τ_i between σ_{i-1} and σ_i satisfy*

$$\prod_{i=1}^{F} \begin{pmatrix} 0 & 1 \\ \mathfrak{b}_i(0) & \mathfrak{a}_i(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

This gives algebraic restrictions on the values $\mathfrak{a}_i(0)$, $\mathfrak{b}_i(0)$. At a crossing vertex γ (see Def. 2.3) of four incident edges, the equality (7) amounts to

$$\mathfrak{b}_1(0)\mathfrak{b}_3(0) = 1, \qquad \mathfrak{b}_2(0)\mathfrak{b}_4(0) = 1. \tag{8}$$

It turns out that Condition 2.6 is not sufficient around crossing vertices for ensuring an ample space of differentiable functions on $\mathcal{M}$. An obstruction was noticed in [21] in a setting of rectangular patches. We write this constrain in a general setting:

Condition 2.7. *If the vertex γ is a crossing vertex with 4 edges $\tau_1, \ldots, \tau_4$, the gluing data $[\mathfrak{a}_i, \mathfrak{b}_i]$ $i = 1 \ldots 4$ on these edges at γ satisfy*

$$\begin{aligned} \mathfrak{a}_1'(0) + \frac{\mathfrak{b}_4'(0)}{\mathfrak{b}_4(0)} &= -\mathfrak{b}_1(0)\left(\mathfrak{a}_3'(0) + \frac{\mathfrak{b}_2'(0)}{\mathfrak{b}_2(0)}\right), && (9)\\ \mathfrak{a}_2'(0) + \frac{\mathfrak{b}_1'(0)}{\mathfrak{b}_1(0)} &= -\mathfrak{b}_2(0)\left(\mathfrak{a}_4'(0) + \frac{\mathfrak{b}_3'(0)}{\mathfrak{b}_3(0)}\right). && (10)\end{aligned}$$

Lemma 2.8. *If the space of differentiable functions on $\mathcal{M}$ is ample and Condition 2.6 is satisfied, then the gluing data at every crossing vertex γ of 4 incident edges must also satisfy Condition 2.7.*

Proof. The value and first derivatives at every point $\gamma \in \mathcal{M}$ of all differentiable functions on $\mathcal{M}$ should span a space of dimension 3.

If γ is a crossing vertex, then we have 4 restrictions on the Taylor expansions of a spline components (f_1, f_2, f_3, f_4). Let us write the Taylor expansion of f_i at $\gamma = (0, 0)$ as $f_i = p_i + q_i u_i + r_i v_i + s_i u_i v_i + \ldots$. The gluing conditions imply the following. From (3),

$$p_1 = p_2 = p_3 = p_4, \quad q_1 = r_2, \quad q_2 = r_3, \quad q_3 = r_4 \quad \text{and} \quad q_4 = r_1 \tag{11}$$

this together with the Condition (4) on the first derivatives imply

$$\mathfrak{b}_2(0)q_1 - q_3 = 0, \qquad \mathfrak{b}_3(0)q_2 - q_4 = 0.$$

When we consider the derivative of f_i with respect to u_i, again applying (4), we get the conditions

$$\begin{aligned}
s_2 - \mathfrak{b}_2(0)s_1 &= \mathfrak{b}_2'(0)q_1 + \mathfrak{a}_2'(0)q_2,\\
s_3 - \mathfrak{b}_3(0)s_2 &= \mathfrak{b}_3'(0)q_2 + \mathfrak{a}_3'(0)q_3,\\
s_4 - \mathfrak{b}_4(0)s_3 &= \mathfrak{b}_4'(0)q_3 + \mathfrak{a}_4'(0)q_4,\\
s_1 - \mathfrak{b}_1(0)s_4 &= \mathfrak{b}_1'(0)q_4 + \mathfrak{a}_1'(0)q_1.
\end{aligned}$$

By combining the last four equations respectively with the weights $\mathfrak{b}_4(0)$, $\mathfrak{b}_1(0)\mathfrak{b}_4(0)$, $\mathfrak{b}_1(0)$, 1, together with Condition 2.6, we get

$$\begin{aligned}
q_1 \left(\mathfrak{a}_1'(0) + \mathfrak{b}_2(0)\mathfrak{b}_4'(0)\right) + q_3\, \mathfrak{b}_1(0)\big(\mathfrak{b}_2(0)\mathfrak{a}_3'(0) + \mathfrak{b}_2'(0)\big)+&\\
q_2 \left(\mathfrak{b}_1(0)\mathfrak{a}_2'(0) + \mathfrak{b}_1'(0)\right) + q_4\, \mathfrak{b}_2(0)\big(\mathfrak{a}_4'(0) + \mathfrak{b}_1(0)\mathfrak{b}_3'(0)\big) &= 0. \qquad (12)
\end{aligned}$$

This relation does not involve the cross derivatives s_1, s_2, s_3, s_4, but gives an unwanted relation between the first order derivatives. After replacing (11) and (8) in (12), we encounter conditions (9) and (10). Under these conditions, there is no relation between q_1, q_2, and there is one degree of freedom for (s_1, s_2, s_3, s_4). □

The restrictions (9) and (10) were noticed in [21] in the context of gluing tensor product rectangular patches with all $\mathfrak{b}_i(0) = -1$. The restrictions are then simply

$$\mathfrak{a}_1'(0) = \mathfrak{a}_3'(0), \qquad \mathfrak{a}_2'(0) = \mathfrak{a}_4'(0).$$

2.4. Topological restrictions

A guiding principle for the construction of geometric continuous functions is that G^1 properties are equivalent to C^1 properties in the plane after an adequate reparameterization of the problem. Gluing two faces along an edge is transformed locally via such reparameterization maps, into gluing two half-planes along a line. Each half-plane is in correspondence with the half-plane determined by one of the faces and the shared edge. A natural gluing is to have the half planes on each side of the line. In this case, the points of one face are mapped by the reparameterizations on one side of the line and the points of the other face on the other side of the line. This implies that the transition maps keep locally the points of a face on the same side of the edge and thus it should have a positive Jacobian at each point of the edge.

Therefore the first topological restriction that we ask for each edge τ, using the canonical form (1), is the following:

$$\forall u \in [0,1],\ \mathfrak{b}_\tau(u) < 0.$$

When the function $\mathfrak{b}_\tau$ is positive on the edge; the transition map identifies interiors of the polygons. It corresponds to two patches of surfaces virtually pasted at a sharp edge (i.e., at angle 0) rather than in a proper continuously smooth manner (i.e., at the angle π). In some CAGD applications, it may be useful to model surfaces with sharp wing-like edges by the G^1 continuity restrictions with $\mathfrak{b}_\tau > 0$. But typical G^1 continuity applications should require $\mathfrak{b}_\tau < 0$ on the whole edge to prevent this degeneration. The regularity property across edges, considered irrespective of orientation, is called weak geometric continuity [7, §6.7] and the restriction we consider corresponds to coherently oriented parametrizations in [7, §6.8].

Similarly, gluing the faces around a vertex γ should be equivalent to gluing sectors around a point in the plane, via the reparameterization maps. Such sectors should form a fan around the parameter point, which can be identified with the local neighborhood of the vertex γ on the surface. Thus these sectors should not overlap. If this fan is defined by vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{F'} \in \mathbb{R}^2$ ($\mathbf{u}_{i+1}$ is supposed to be outside the union of the sectors defined by two consecutive vectors $\mathbf{u}_{j-1}, \mathbf{u}_j$ for $2 < j < i$), we easily check that the coefficients $\mathfrak{a}_i(0)$, $\mathfrak{b}_i(0)$ of the transition map (1) across the edge τ_i at γ are such that:

$$\mathbf{u}_{i-1} = \mathfrak{a}_i(0)\mathbf{u}_i + \mathfrak{b}_i(0)\,\mathbf{u}_{i+1} \tag{13}$$

or equivalently

$$[\mathbf{u}_i, \mathbf{u}_{i-1}] = \begin{bmatrix} 0 & 1 \\ \mathfrak{b}_i(0) & \mathfrak{a}_i(0) \end{bmatrix} [\mathbf{u}_{i+1}, \mathbf{u}_i]$$

(see also the construction in the next section 2.5). If the sector angles are less than π (i.e. the sector $\mathbf{u}_i, \mathbf{u}_{i+1}$ coincides with the cone generated by $\mathbf{u}_i, \mathbf{u}_{i+1}$) the condition that the sectors form a fan and do not overlap translates as follows: the coefficients of the last row of

$$\prod_{i=j}^{k} \begin{pmatrix} 0 & 1 \\ \mathfrak{b}_i(0) & \mathfrak{a}_i(0) \end{pmatrix}$$

should not be both non-negative for $1 < j \leqslant k < F$.

A natural way to define transition maps at a vertex γ which satisfy this condition is to choose vectors in the plane that define a fan, as in Figure 2. Then the coefficients $\mathfrak{a}_i(0)$, $\mathfrak{b}_i(0)$ are uniquely determined from the relations (13).

The topological constraints could be dropped in some applications, for example, when modeling analytical surfaces with branching points, or surfaces with sharp wing-like interior edges, or with winding-up boundary. In these specific applications, the compatibility Condition 2.7 at *crossing vertex* might need to be extended, to allow winding up of 8, 12, etc., crossing edges, and take into account the sharp edges. Apart from this kind of consideration, the topological conditions do not essentially affect our algebraic dimension count.

The framework that we propose is more general than previous approaches used in Geometric Modeling to define G^1 splines (see e.g. [22, §3]) since it allows to define differentiable functions on topological surfaces such as a Möbius strip or a Klein bottle.

Moreover, it does not rely on the construction of manifold surfaces and atlas, but only on compatible transition maps.

2.5. Example

A simple way to define transition maps is to use a symmetric gluing as proposed in [12, §8.2] for rectangular patches. If $\tau = (\gamma_0, \gamma_1)$ is the shared edge between σ_1 and σ_2, the transition map can be of the form:

$$\phi(u,v) = \begin{pmatrix} -v \\ u + 2v\big(\mathfrak{d}_0(u)\cos\frac{2\pi}{n_0} - \mathfrak{d}_1(u)\cos\frac{2\pi}{n_1}\big) \end{pmatrix} \tag{14}$$

where n_0 (resp. n_1) is the number of edges at the vertex γ_0 (resp. γ_1). Additionally, if γ_0 corresponds to $u = 0$ and γ_1 to $u = 1$, the functions $\mathfrak{a}$ and $\mathfrak{b}$ interpolate 0 and 1: $\mathfrak{d}_0(0) = 1$, $\mathfrak{d}_0(1) = 0$, $\mathfrak{d}_1(0) = 0$, $\mathfrak{d}_1(1) = 1$ and their derivatives of order 1 should vanish at 0, 1. It corresponds to a symmetric gluing, where the angle of two consecutive edges at γ_i is $\frac{2\pi}{n_i}$. If $\mathfrak{d}_0(u)$ and $\mathfrak{d}_1(u)$ are polynomial functions, their degree must be at least 3. If $\mathfrak{d}_0(u)$ and $\mathfrak{d}_1(u)$ are rational functions with the same denominator, the maximal degree of the numerators and denominator must be at least 2. As we will see the dimension of the spline space decreases when the degree increases. Thus it is important to construct transition maps with low degree numerators and denominators. See e.g. [20, 21] for low degree constructions, which depend on the structure of $\mathcal{M}$.

A general construction of gluing data which satisfies the compatibility conditions is as follows.

(i) For all the vertices $\gamma \in \mathcal{M}_0$ and for all the edges $\tau_1, \ldots, \tau_F$ of $\mathcal{M}_1$ that contain γ, choose vectors $\mathbf{u}_1, \ldots, \mathbf{u}_F \in \mathbb{R}^2$ such that the cones generated by $\mathbf{u}_i, \mathbf{u}_{i+1}$ form a fan in $\mathbb{R}^2$ and such that the union of these cones is $\mathbb{R}^2$ when γ is an interior vertex.
Compute the transition map $\phi_{\sigma_i,\sigma_{i-1}}$ at $\gamma = (0,0)$ on the edge τ_i

$$\phi_{\sigma_i,\sigma_{i-1}}(0,0) = S \circ [\mathbf{u}_i, \mathbf{u}_{i+1}]^{-1} \circ [\mathbf{u}_{i-1}, \mathbf{u}_i] \circ S = \begin{bmatrix} 0 & \mathfrak{b}_{\tau_i}(0) \\ 1 & \mathfrak{a}_{\tau_i}(0) \end{bmatrix}$$

where $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $[\mathbf{u}_i, \mathbf{u}_{i+1}]$ is the matrix which columns are the vectors $\mathbf{u}_i, \mathbf{u}_{i+1}$, $|\mathbf{u}_i, \mathbf{u}_j|$ is the determinant of the vectors $\mathbf{u}_i, \mathbf{u}_j$ and

$$\mathfrak{a}_{\tau_i}(0) = \frac{|\mathbf{u}_{i-1}, \mathbf{u}_{i+1}|}{|\mathbf{u}_i, \mathbf{u}_{i+1}|}, \mathfrak{b}_{\tau_i}(0) = -\frac{|\mathbf{u}_{i-1}, \mathbf{u}_i|}{|\mathbf{u}_i, \mathbf{u}_{i+1}|}.$$

(ii) For all the edges $\tau \in \mathcal{M}_1$, define the rational functions $\mathfrak{a}_\tau = \frac{a_\tau}{c_\tau}, \mathfrak{b}_\tau = \frac{b_\tau}{c_\tau}$ on the edges τ by interpolation as follows: if there is no crossing edge in $\mathcal{M}_1$, then a linear interpolation of the value at the vertices is sufficient. If $\gamma_1, \ldots, \gamma_n$ is a sequence of crossing vertices, and $\gamma_0\gamma_1, \gamma_1\gamma_2, \ldots, \gamma_n\gamma_{n+1}$ is a sequence of edges passing "straight" through them, we can choose linear gluing data on one edge, and quadratic data on the remaining edges of the sequence so that the constraints (9) and (10) are satisfied.

Therefore, for general meshes, gluing data which satisfy the compatibility condition and the topological condition can be constructed in degree $\leqslant 2$.

3. Spline space on a topological surface

The main object of our study is the space of functions on the topological surface $\mathcal{M}$, which are differentiable and piecewise polynomial. Such functions are called *spline functions* on $\mathcal{M}$. Let $\mathcal{R}(\sigma) = \mathbb{R}[u_\sigma, v_\sigma]$ be the ring of polynomials in the variables (u_σ, v_σ) attached to the face σ. A spline function f is defined by assigning to each face $\sigma \in \mathcal{M}_2$ a polynomial $f_\sigma \in \mathcal{R}(\sigma)$, and by imposing the regularity conditions across the shared edges.

We also consider rational gluing data on the interior edges $\tau \in \mathcal{M}_1$:

$$\mathfrak{a}_\tau(u_1) = \frac{a_\tau(u_1)}{c_\tau(u_1)} \quad \text{and} \quad \mathfrak{b}_\tau(u_1) = \frac{b_\tau(u_1)}{c_\tau(u_1)} \tag{15}$$

with $a_\tau(u_1), b_\tau(u_1)$ and $c_\tau(u_1)$ polynomials in the variable u_1, where $b_\tau(u_1)$ and $c_\tau(u_1)$ do not vanish on τ (i.e. for $u_1 \in [0,1]$). As $b(u_1), c(u_1)$ do not vanish on τ, the transition map ϕ_{σ_2,σ_1} is a C^1-diffeomorphism in a neighborhood of the edge $\tau = (\tau_1, \tau_2)$ between σ_1 and σ_1. The polynomial vector $[a_\tau, b_\tau, c_\tau]$ is also called hereafter the gluing data of the edge τ. We assume hereafter that *the transition maps satisfy Conditions 2.6, 2.7 and all crossing vertices of $\mathcal{M}$ have 4 edges*.

We can now define the space of splines on $\mathcal{M}$:

Definition 3.1. We denote by $\mathcal{S}^1(\mathcal{M})$ the ($\mathbb{R}$-linear) space of differentiable functions on the topological surface $\mathcal{M}$ which are defined by assigning polynomials to the faces $\sigma \in \mathcal{M}_2$ satisfying the G^1 constrains (2). More precisely,

$$\mathcal{S}^1(\mathcal{M}) := \{f \in \oplus_{\sigma \in \mathcal{M}_2} \mathcal{R}(\sigma) \mid J_\gamma(f_{\sigma_1}) = J_\gamma(f_{\sigma_2} \circ \phi_{\sigma_2,\sigma_1}) \ \forall \gamma \in \tau = \sigma_1 \cap \sigma_2\},$$

where J_γ is the jet or Taylor expansion of order 1.

A spline $f \in \mathcal{S}^1(\mathcal{M})$ gives a piecewise-polynomial map, defined on every face of $\mathcal{M}$, and the jets of order 1 coincide on the shared edges. This definition implies that a spline function $f \in \mathcal{S}^1(\mathcal{M})$ is C^1 on a neighborhood of a shared edge $\tau = \sigma_1 \cap \sigma_2$ if we use the re-parametrization $f_{\sigma_2} \circ \phi_{\sigma_2,\sigma_1}$.

Definition 3.1 can be directly extended to splines of any order r but in this paper we only consider $r = 1$.

3.1. Polynomials on faces

On each face $\sigma \in \mathcal{M}_2$, we consider polynomials of degree bounded by $k \in \mathbb{N}$.

If $\sigma \in \mathcal{M}_2$ is a triangle (P, Q, R), we denote by $\mathcal{R}_k(\sigma)$ the finite dimensional vector space of polynomials in $\mathbb{R}[u_\sigma, v_\sigma]$ with total degree bounded by k.

After a change of coordinates, we may assume that the coordinate function $u = u_\sigma$ satisfies $u_\sigma(PR) = 0$ and $u_\sigma(Q) = 1$, while $v = v_\sigma$ satisfies $v_\sigma(PQ) = 0$ and $v_\sigma(R) = 1$. Introducing $w = 1 - u - v$, we can express any polynomial in $\mathcal{R}_k(\sigma)$ as a homogeneous polynomial of degree $\leqslant k$ in the *barycentric* coordinates u, v, w using the *Bernstein-Bézier basis*:

$$b^{\triangle}_{i,j}(u, v, w) = \frac{k!}{i!j!(k-i-j)!} \, u^i \, v^j \, w^{k-i-j},$$

for $0 \leqslant i+j \leqslant k$. We verify directly that for a function

$$f(u,v) = \sum_{0\leqslant i+j\leqslant k} c_{i,j} b^{\triangle}_{i,j}(u,v,1-u-v)$$

expressed in this basis, we have

$$\begin{aligned} f(0,0) &= c_{0,0}, \\ \partial_u f(0,0) &= k(c_{1,0}-c_{0,0}), \quad \partial_v f(0,0) = k(c_{0,1}-c_{0,0}), \\ \partial_u\partial_v f(0,0) &= k\,(k-1)\,(c_{1,1}-c_{1,0}-c_{0,1}+c_{0,0}). \end{aligned} \tag{16}$$

If $\sigma \in \mathcal{M}_2$ is a rectangle (P,Q,R,S), we will denote by $\mathcal{R}_k(\sigma)$ the finite dimensional vector space of polynomials in $\mathbb{R}[u_\sigma, v_\sigma]$ with partial degree in u_σ and v_σ bounded by k, where $u = u_\sigma$ is chosen such that $u_\sigma(PS) = 0$, $u_\sigma(QR) = 1$, and $v = v_\sigma$ is chosen such that $v_\sigma(PQ) = 0$, $v_\sigma(RS) = 1$. Introducing $\tilde{u} = 1-u$, $\tilde{v} = 1-v$, we can express any polynomial function of $\mathcal{R}_k(\sigma)$ as a bi-homogeneous polynomial of degree k in $u, \tilde{u}$ and degree k in $v, \tilde{v}$, using the *tensor product Bernstein-Bézier basis*

$$b^{\square}_{i,j}(u,\tilde{u},v,\tilde{v}) = \frac{k!k!}{i!j!(k-i)!(k-j)!}\, u^i\, \tilde{u}^{k-i}\, v^j\, \tilde{v}^{k-j}.$$

for $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant k$. We verify directly that for a function $f = \sum_{0\leqslant i,j\leqslant k} c_{i,j} b^{\square}_{i,j}$ expressed in this basis, we have

$$\begin{aligned} f(0,0) &= c_{0,0}, \\ \partial_u f(0,0) &= k(c_{1,0}-c_{0,0}), \quad \partial_v f(0,0) = k(c_{0,1}-c_{0,0}), \\ \partial_u\partial_v f(0,0) &= k^2\,(c_{1,1}-c_{1,0}-c_{0,1}+c_{0,0}). \end{aligned} \tag{17}$$

The finite dimensional vector space of spline functions $f = (f_\sigma)_{\sigma\in\mathcal{M}_2} \in \mathcal{S}(\mathcal{M})$ of degree bounded by $k \in \mathbb{N}$ on each face ($f_\sigma \in \mathcal{R}_k(\sigma)$) and of regularity r is denoted $\mathcal{S}^r_k(\mathcal{M})$ or simply $\mathcal{S}_k(\mathcal{M})$ when $r = 1$.

3.2. Taylor maps

An important tool that we are going to use intensively is the Taylor map associated to a vertex or to an edge of $\mathcal{M}$.

Let $\gamma \in \mathcal{M}_0$ be a vertex on a face $\sigma \in \mathcal{M}_2$ belonging to two edges $\tau, \tau' \in \mathcal{M}_1$ of σ. We define the *ring of* γ *on* σ by $\mathcal{R}^\sigma(\gamma) = \mathcal{R}(\sigma)/(\ell^2_\tau, \ell^2_{\tau'})$ where $(\ell^2_\tau, \ell^2_{\tau'})$ is the ideal generated by the squares of ℓ_τ and $\ell_{\tau'}$, the equations

$\ell_\tau(u,v)=0$ and $\ell_{\tau'}(u,v)=0$ are respectively the equations of τ and τ' in $\mathcal{R}(\sigma)=\mathbb{R}[u,v]$.

The *Taylor expansion at* γ *on* σ is the map

$$T_\gamma^\sigma : f \in \mathcal{R}(\sigma) \mapsto f \mod (\ell_\tau^2, \ell_{\tau'}^2) \text{ in } \mathcal{R}^\sigma(\gamma).$$

Choosing an adapted basis of $\mathcal{R}^\sigma(\gamma)$, one can defined T_γ^σ by

$$T_\gamma^\sigma(f) = \left[\, f(\gamma),\, \partial_u f(\gamma),\, \partial_v f(\gamma),\, \partial_u\partial_v f(\gamma)\,\right].$$

The map T_γ^σ can also be defined in another basis of $\mathcal{R}^\sigma(\gamma)$ in terms of the Bernstein coefficients by

$$T_\gamma^\sigma(f) = \left[\, c_{0,0}(f),\, c_{1,0}(f),\, c_{0,1}(f),\, c_{1,1}(f)\,\right]$$

where $c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1}$ are the first Bernstein coefficients associated to $\gamma = (0,0)$.

We define the Taylor map T_γ on all the faces σ that contain γ,

$$T_\gamma : f = (f_\sigma) \in \oplus_\sigma \mathcal{R}(\sigma) \to (T_\gamma^\sigma(f_\sigma)) \in \oplus_{\sigma\supset\gamma} \mathcal{R}^\sigma(\gamma).$$

Similarly, we define T_0 as the Taylor map at all the vertices on all the faces of $\mathcal{M}$.

For an edge $\tau \in \mathcal{M}_1$ on a face $\sigma \in \mathcal{M}_2$, we define the *ring of* τ *on* σ by $\mathcal{R}^\sigma(\tau) = \mathcal{R}(\sigma)/(\ell_\tau^2)$ where $\ell_\tau(u,v)=0$ is the equation of τ in $\mathcal{R}(\sigma)=\mathbb{R}[u,v]$. The *Taylor expansion along* τ *on* σ is defined by

$$T_\tau^\sigma : f \in \mathcal{R}(\sigma) \mapsto f \mod (\ell_\tau^2) \text{ in } \mathcal{R}^\sigma(\tau),$$

and the Taylor map on all the faces σ that contain τ is given by

$$T_\tau : f = (f_\sigma) \in \oplus_\sigma \mathcal{R}(\sigma) \to (T_\tau^\sigma(f_\sigma)) \in \oplus_{\sigma\supset\tau} \mathcal{R}^\sigma(\gamma).$$

Similarly, we define T_1 as the Taylor map along all the edges on all the faces of $\mathcal{M}$.

4. G^1 splines along an edge

To analyze the constraints imposed by gluing data along an edge, we consider first a simple topological surface $\mathcal{N}$ composed of two faces σ_1, σ_2 glued along an edge τ.

A spline function $f \in S_k^1(\mathcal{N})$ on $\mathcal{N}$ is represented by a pair of polynomials $f = (f_1, f_2)$ with $f_i \in \mathcal{R}(\sigma_i) = \mathbb{R}[u_i, v_i]$ for $i = 1, 2$.

By a change of coordinates, we assume that the edge τ is defined by $v_1 = 0$ and $u_1 \in [0,1]$ in σ_1 and by $u_2 = 0$ and $v_2 \in [0,1]$ in σ_2.

4.1. Splines and syzygies

With the transition map ϕ_{σ_2,σ_1} defined by the rational functions $\mathfrak{a} = \frac{a_\tau}{c_\tau}$ and $\mathfrak{b} = \frac{b_\tau}{c_\tau}$ as in (15), the differentiability Condition (4) along the interior edge τ becomes

$$a(u_1)A(u_1) + b(u_1)B(u_1) + c(u_1)C(u_1) = 0,$$

where

$$A(u_1) = \frac{\partial f_2}{\partial v_2}(0, u_1), \quad B(u_1) = \frac{\partial f_2}{\partial u_2}(0, u_1), \quad C(u_1) = -\frac{\partial f_1}{\partial v_1}(u_1, 0).$$

Thus, the G^1-smoothness condition along an interior edge is equivalent to the condition on (A, B, C) of being a *syzygy* of the polynomials $a(u_1), b(u_1), c(u_1)$.

More precisely, a G^1 spline (f_1, f_2) on $\mathcal{N}$ is constructed from a syzygy (A, B, C) of a, b, c by defining:

$$f_1(u_1, v_1) = c_0 + \int_0^{u_1} A(t)dt - v_1 C(u_1) + v_1^2 E_1(u_1, v_1), \tag{18}$$

$$f_2(u_2, v_2) = c_0 + \int_0^{v_2} A(t)dt + u_2 B(v_2) + u_2^2 E_2(u_2, v_2), \tag{19}$$

where $c_0 \in \mathbb{R}$ is any constant, and E_1, E_2 are (any) polynomials in $\mathbb{R}[u_i, v_i]$ for $i = 1, 2$, respectively.

We will use this representation for the splines on $\mathcal{N}$ to compute the dimension of the space of G^1 splines $\mathcal{S}_k^1(\mathcal{N})$, see Proposition 4.6 below. Before, we introduce some notation, both for the proof and the dimension formula.

The module of syzygies of $a(u_1), b(u_1), c(u_1)$ over the ring $\mathbb{R}[u_1]$ is denoted by $Z = \operatorname{Syz}(a, b, c)$. For $(A, B, C) \in Z$, the maximum of the degrees, $\max(\deg A, \deg B, \deg C)$ is called the *coefficient degree* of the syzygy.

Each of the faces σ_1 and σ_2 in $\mathcal{N}$ can be a triangle or a rectangle. Let us denote by $F_\square$ the number of rectangles and by F_Δ the number of triangles in $\mathcal{N}$.

Definition 4.1. As before, let σ_1, σ_2 be the faces of $\mathcal{N}$. We define

$$m = \min\big(F_\Delta(\sigma_1), F_\Delta(\sigma_2)\big),$$

where $F_\Delta(\sigma_i) = 1$ if σ_i is a triangle and 0 otherwise. For the polynomials $a, b, c \in \mathbb{R}[u_1]$ defining the gluing data along the edge τ, let $n = \max\big(\deg(a), \deg(b), \deg(c)\big)$,

$$d_a = n + 1, \quad d_b = n + F_\Delta(\sigma_2), \quad \text{and} \quad d_c = n + F_\Delta(\sigma_1),$$

and

$$e = \begin{cases} 0\,, & \text{if } \min\big(d_a - \deg(a), d_b - \deg(b), d_c - \deg(c)\big) = 0 \text{ and} \\ 1\,, & \text{otherwise.} \end{cases}$$

By the formulas (18) and (19) representing a spline $(f_1, f_2) \in \mathcal{S}^1_k(\mathcal{N})$, let us notice that we need to consider syzygies (A, B, C) of a, b, c such that $\deg(A) \leqslant k-1$, $\deg(B) \leqslant k - F_\Delta(\sigma_2)$, and $\deg(C) \leqslant k - F_\Delta(\sigma_1)$. The reason is that f_i is of bidegree at most (k, k) if σ_i is a rectangle and of total degree at most k if σ_i is triangle.

Definition 4.2. For $k \geqslant 0$, we will denote by Z_k the vector subspace of Z of syzygies of (a, b, c) defined as the set

$$\begin{aligned} Z_k = \{(A, B, C) \in Z \colon\ \deg(A) \leqslant k-1,\ \deg(B) \leqslant k - F_\Delta(\sigma_2),& \\ \text{and}\ \ \deg(C) \leqslant k - F_\Delta(\sigma_1)\}.& \end{aligned}$$

Let us consider the map

$$\begin{aligned} \Theta_\tau : Z &\ \to\ \mathcal{S}^1(\mathcal{N}) \\ (A, B, C) &\ \mapsto\ \left(\int_0^{u_1} A(t)dt - v_1 C(u_1),\ \int_0^{v_2} A(t)dt + u_2 B(v_2)\right). \end{aligned} \tag{20}$$

By construction, we have $\Theta_\tau(Z_k) \subset \mathcal{S}^1_k(\mathcal{N})$.

The dimension of Z_k, as a vector space over $\mathbb{R}$, will be deduced from classical results on graded modules over $S = \mathbb{R}[u_0, u_1]$. We will study the module $\mathrm{Syz}(\bar{a}, \bar{b}, \bar{c})$, where $\bar{a}, \bar{b}, \bar{c} \in S$ are the homogenization of a, b and c in degree d_a, d_b and d_c respectively. The elements in $\mathrm{Syz}(\bar{a}, \bar{b}, \bar{c})$ in degree $n + k$ will precisely lead to the syzygies in Z_k.

Lemma 4.3. *For polynomials $a, b, c \in \mathbb{R}[u_1]$, with $b, c \neq 0$, $\gcd(a, b, c) = 1$ and $Z = \mathrm{Syz}(a, b, c)$ as defined above,*

(*i*) *Z is a free $\mathbb{R}[u_1]$-module of rank 2.*

(*ii*) *The module Z is generated by vectors (A_1, B_1, C_1), (A_2, B_2, C_2) of coefficient degree μ and $\nu = n - \mu + 1 + F_\Delta - e - 2m$ where μ is the smallest possible coefficient degree.*

(*iii*) *For $k \in \mathbb{N}$, the dimension of Z_k as vector space over $\mathbb{R}$ is given by*

$$\dim Z_k = (k - \mu - m + 1)_+ + (k - n + \mu + m - F_\Delta + e)_+$$

where $t_+ = \max(0, t)$ for $t \in \mathbb{Z}$.

(*iv*) *The generators (A_1, B_1, C_1), (A_2, B_2, C_2) of Z can be chosen so that*

$$(a, b, c) = (B_1C_2 - B_2C_1, C_1A_2 - C_2A_1, A_1B_2 - A_2B_1).$$

Proof. We study the syzygy module $Z = \mathrm{Syz}(a, b, c)$ using results on graded resolutions. For this purpose, we homogenize a, b and c in degree $d_a = n+1$, $d_b = n + F_\Delta(\sigma_2)$, and $d_c = n + F_\Delta(\sigma_1)$, respectively, where $F_\Delta(\sigma_i)$ is as in Definition 4.1. Let u_0, u_1 be the homogeneous coordinates, and $\bar{a}, \bar{b}, \bar{c}$ the corresponding homogenizations of a, b, and c. We consider the module of homogeneous syzygies $\mathrm{Syz}(\bar{a}, \bar{b}, \bar{c})$ over the polynomial ring $S = \mathbb{R}[u_0, u_1]$.

Claim 4.4. *For any $k \geqslant 0$, the elements in Z_k are exactly the syzygies of degree $n + k$ in $\mathrm{Syz}(\bar{a}, \bar{b}, \bar{c})$ after dehomogenization by setting $u_0 = 1$.*

Proof. It is clear that if $\bar{A}\bar{a} + \bar{B}\bar{b} + \bar{C}\bar{c} = 0$, then by dehomogenization taking $u_0 = 1$, we get a syzygy (A, B, C) of (a, b, c). Moreover, if $\deg(\bar{A}\bar{a}) = \deg(\bar{B}\bar{b}) = \deg(\bar{C}\bar{c}) = n + k$, then $\deg(\bar{A}) = k - 1$, $\deg(\bar{B}) = k - F_\Delta(\sigma_2)$ and $\deg(\bar{C}) = k - F_\Delta(\sigma_1)$. It follows that $(A, B, C) \in Z_k$.

On the other hand, any syzygy $(A, B, C) \in Z_k$ is given by polynomials that satisfy the conditions in Definition 4.2. Thus $\max\{\deg A, \deg B, \deg C\} \leqslant k$, and since $n = \max\{\deg a, \deg b, \deg c\}$ then we may consider the homogenization of the polynomial $Aa + Bb + Cc$ in degree $n + k$. These polynomials satisfy

$$\begin{aligned} 0 &= u_0^{k+n}(Aa + Bb + Cc)(u_1/u_0) \\ &= u_0^{k-1} \cdot u_0^{n+1} Aa\big(u_1/u_0\big) + u_0^{k-F_\Delta(\sigma_2)} \cdot u_0^{n+F_\Delta(\sigma_2)} Bb\big(u_1/u_0\big) \\ &\qquad + u_0^{k-F_\Delta(\sigma_1)} \cdot u_0^{n+F_\Delta(\sigma_1)} Cc\big(u_1/u_0\big). \end{aligned}$$

It is easy to check that

$$\bar{A} = u_0^{k-1} A(u_1/u_0), \quad \bar{B} = u_0^{k-F_\Delta(\sigma_2)} B(u_1/u_0), \quad \bar{C} = u_0^{k-F_\Delta(\sigma_1)} C(u_1/u_0)$$

are all polynomials in $\mathbb{R}[u_1, u_0]$, and define a syzygy of $\bar{a}, \bar{b}, \bar{c}$ of degree $n+k$. Let us also notice that the polynomials

$$\bar{a} = u_0^{n+1} a(u_1/u_0),\ \bar{b} = u_0^{n+F_\Delta(\sigma_2)} b(u_1/u_0),\ \text{ and } \bar{c} = u_0^{n+F_\Delta(\sigma_1)} c(u_1/u_0)$$

are precisely the homogenization of a, b, c in degree d_a, d_b, d_c, respectively. $\square$

As $gcd(a, b, c) = 1$, we have $gcd(\bar{a}, \bar{b}, \bar{c}) = u_0$ if $e = 1$, and $gcd(\bar{a}, \bar{b}, \bar{c}) = 1$ otherwise.

Let $I = (\bar{a}, \bar{b}, \bar{c})$ be the ideal generated by $\bar{a}, \bar{b}, \bar{c}$ in S. If $gcd(\bar{a}, \bar{b}, \bar{c}) = 1$ then there exists $t_0 \in \mathbb{N}$ such that $\forall t \geqslant t_0$, $I_t = (u_0, u_1)^t$ and in that case, $\dim_{\mathbb{R}}(S/I)_t = 0$ for t sufficiently large. It follows that the Hilbert polynomial $HP_{S/I}$ of S/I is the zero polynomial.

For the second case, namely if $gcd(\bar{a}, \bar{b}, \bar{c}) = u_0$, since $\gcd(a, b, c) = 1$ then the polynomials $\bar{a}/u_0$, $\bar{b}/u_0$ and $\bar{c}/u_0$ have gcd equal to 1. Hence there exists $t_0 \in \mathbb{N}$ such that $\forall t \geqslant t_0$, $I_t = u_0\,(u_0, u_1)^{t-1}$. In this case $\dim_{\mathbb{R}}(S/I)_t = 1$ for t sufficiently large, and it follows that the Hilbert polynomial $HP_{S/I}$ is the constant polynomial equal to 1.

Then the exact sequence

$$0 \to I \to S \to S/I \to 0$$

implies that

$$HP_I(t) = HP_S(t) - HP_{S/I}(t) = \binom{t+1}{1} - e, \tag{21}$$

where HP_M is the Hilbert polynomial of the module M.

By the Graded Hilbert Syzygy Theorem, we get a resolution of the form

$$0 \longrightarrow S(-d_1) \oplus \cdots \oplus S(-d_L) \xrightarrow{\lambda} S(-d_a) \oplus S(-d_b) \oplus S(-d_c) \longrightarrow I \longrightarrow 0.$$

Notice that this resolution is not necessarily minimal. Since this is an exact sequence, then the Hilbert polynomial of the middle term is the sum of the other two Hilbert polynomials, and applying (21) we get

$$3t - (d_a + d_b + d_c) + 3 = (t - d_1 + 1) + \cdots + (t - d_L + 1) + (t+1) - e.$$

It follows that $L = 2$ which proves *(i)*. Furthermore, we have that the degrees d_1 and d_2 of the syzygies satisfy $d_1 + d_2 = d_a + d_b + d_c - e$.

The matrix Λ representing λ is a 3×2 matrix

$$\begin{pmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{B}_1 & \bar{B}_2 \\ \bar{C}_1 & \bar{C}_2 \end{pmatrix}$$

the first column corresponding to the generator of degree d_1 and the second of degree d_2. These two syzygies correspond to vectors of polynomial coefficients of degree $\mu = d_1 - \min(d_a, d_b, d_c)$ and $\nu = d_2 - \min(d_a, d_b, d_c)$. By Definition 4.1, $\min(d_a, d_b, d_c) = n + \min\big(1, F_\Delta(\sigma_1), F_\Delta(\sigma_2)\big) = n + m$, and also $d_a + d_b + d_c = 3n + F_\Delta + 1$. Let us assume that $d_1 \leqslant d_2$, then μ is the smallest degree of the coefficient vector of a syzygy of $(\bar{a}, \bar{b}, \bar{c})$, and $\nu = n - \mu + 1 + F_\Delta - e - 2\,m$.

By exactness, the two columns of Λ generate $\operatorname{Syz}(\bar{a}, \bar{b}, \bar{c})$. The dehomogenization (by setting $u_0 = 1$) of the syzygies in $\operatorname{Syz}(\bar{a}, \bar{b}, \bar{c})$ leads to syzygies of (a, b, c) over $\mathbb{R}[u_1]$. In particular, it is straightforward to show that the dehomogenization (A_i, B_i, C_i) of $(\bar{A}_i, \bar{B}_i, \bar{C}_i)$ for $i = 1, 2$ generate $Z = \operatorname{Syz}(a, b, c)$ as a module over $\mathbb{R}[u_1]$. This proves *(ii)*.

By Claim 4.4, the space Z_k is in correspondence with the space of homogeneous syzygies of degree $n+k$, which is spanned by the multiples of degree $n+k$ of $(\bar{A}_1, \bar{B}_1, \bar{C}_1)$ and $(\bar{A}_2, \bar{B}_2, \bar{C}_2)$. Therefore,

$$\begin{aligned} \dim Z_k &= (n + k - d_1 + 1)_+ + (n + k - d_2 + 1)_+ \\ &= (k - \mu - m + 1)_+ + (k - \nu - m + 1)_+ \end{aligned}$$

with $\nu = n - \mu + 1 + F_\Delta - e - 2\,m$. This proves *(iii)*.

The point *(iv)* is a consequence of Hilbert-Burch theorem. More details on this proof can be found in [6, Chapter 6, § 4.17]. □

Definition 4.5. For an interior edge τ in the topological surface $\mathcal{M}$ shared by the faces σ_1, σ_2 with gluing data $[a_\tau, b_\tau, c_\tau]$, we denote by $\mathcal{N}_\tau$ the topological surface formed by the cells σ_1, σ_2 glued along the edge τ with the same gluing data. Let μ_τ be the smallest coefficient degree among the two generators of the module $Z = \operatorname{Syz}(a_\tau, b_\tau, c_\tau)$. Let $\nu_\tau = n_\tau - \mu_\tau + 1 + F_\Delta(\tau) - e_\tau - 2\,m_\tau$ denote the complementary degree, where $n_\tau = \max(\deg(a_\tau), \deg(c_\tau), \deg(c_\tau))$, $m_\tau = \min(F_\Delta(\sigma_1), F_\Delta(\sigma_2))$, $e_\tau = \min(n_\tau + 1 - \deg(a_\tau), n_\tau + F_\Delta(\sigma_2) - \deg(b_\tau), n_\tau + F_\Delta(\sigma_1) - \deg(c_\tau))$ $F_\Delta(\tau) = F_\Delta(\sigma_1) + F_\Delta(\sigma_2)$. The corresponding basis of the syzygy module Z of $[a_\tau, b_\tau, c_\tau]$ is called the μ_τ-basis.

This construction allows us to determine the dimension of $S^1_k(\mathcal{N}_\tau)$.

Proposition 4.6. *For $F_\square$ (resp. F_Δ) the number of rectangles (resp. triangles) of $\mathcal{N}_\tau$,*

$$\begin{array}{rcl}\dim \mathcal{S}_k(\mathcal{N}_\tau) & = & 1 + (k^2-1)F_\square + \frac{1}{2}(k^2-k)F_\Delta + d_\tau(k) \\ \textit{where} \quad d_\tau(k) & = & (k-\mu_\tau - m_\tau + 1)_+ + (k - n_\tau + \mu_\tau + m_\tau - F_\Delta + e_\tau)_+\end{array}$$

Proof. Since the only constraints satisfied by the spline functions in $\mathcal{S}(\mathcal{N}_\tau)$ are the gluing conditions along the edge τ, the number of linearly independent splines on $\mathcal{N}_\tau$ can be easily counted by using (18) and (19), and the linearly independent terms in the Bernstein-Bézier representation of the polynomials f_1 and f_2 that conform a spline.

The gluing data and the smoothness along the edge τ impose conditions on the terms in f_1 and f_2 which are linear in v_1 and v_2, respectively. Thus, the dimension of the space of splines on $\mathcal{N}_\tau$ of degree exactly 1 in v_1 and v_2, is given by $\dim Z_k = d_\tau(k)$. The formula for d_τ follows from Lemma 4.3, by considering $Z = \mathrm{Syz}(a_\tau, b_\tau, c_\tau)$, where a_τ, b_τ, c_τ define the gluing data along τ. □

4.2. Separation of vertices

We analyze now the separability of the spline functions on an edge, that is when the Taylor map at the vertices separate the spline functions.

Let $f = (f_1, f_2) \in \mathcal{R}(\sigma_1) \oplus \mathcal{R}(\sigma_2)$ of the form $f_i(u_i, v_i) = p_i + q_i\, u_i + \tilde{q}_i\, v_i + s_i\, u_i v_i + r_i\, u_i^2 + \tilde{r}_i\, v_i^2 + \cdots$. Then

$$T_\gamma(f) = [p_1, q_1, \tilde{q}_1, s_1, p_2, q_2, \tilde{q}_2, s_2].$$

If $f = (f_1, f_2) \in \mathcal{S}_k^1(\mathcal{N}_\tau)$, then taking the Taylor expansion of the gluing condition (4) centered at $u_1 = 0$ yields

$$\begin{array}{rcl}\tilde{q}_1 + s_1\, u_1 & = & (\mathfrak{a}(0) + \mathfrak{a}'(0)u_1 + \cdots)\,(\tilde{q}_2 + 2\,\tilde{r}_2\, u_1 + \cdots) \\ & & +(\mathfrak{b}(0) + \mathfrak{b}'(0)u_1 + \cdots)\,(q_2 + s_2\, u_1 + \cdots)\end{array} \tag{22}$$

Combining (22) with (3) yields

$$\begin{array}{rcl}p_1 & = & p_2 \\ q_1 & = & \tilde{q}_2 \\ r_1 & = & \tilde{r}_2 \\ \tilde{q}_1 & = & \mathfrak{a}(0)\,\tilde{q}_2 + \mathfrak{b}(0)\,q_2 \\ s_1 & = & 2\,\mathfrak{a}(0)\,r_2 + \mathfrak{b}(0)\,s_2 + \mathfrak{a}'(0)\,\tilde{q}_2 + \mathfrak{b}'(0)\,q_2.\end{array}$$

Let $\mathcal{H}(\gamma)$ be the linear space spanned by the vectors $[p_1, q_1, \tilde{q}_1, s_1, p_2, q_2, \tilde{q}_2, s_2]$, which are solution of these equations.

If $\mathfrak{a}(0) \neq 0$, it is a space of dimension 5 otherwise its dimension is 4. Thus $\dim \mathcal{H}(\gamma) = 5 - \mathfrak{c}_\tau(\gamma)$.

Proposition 4.7. *For $k \geqslant \nu_\tau + m_\tau + 1$, $T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau)) = \mathcal{H}(\gamma)$. Its dimension is* $\dim T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau)) = 5 - \mathfrak{c}_\tau(\gamma)$.

Proof. Let $G(\gamma) = T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau))$. By construction $G(\gamma) \subset \mathcal{H}(\gamma)$. We are going to prove that for $k \geqslant \nu_\tau + m_\tau + 1$, $G(\gamma)$ and $\mathcal{H}(\gamma)$ have the same dimension and thus are equal.

By the decompositions (18) and (19), the elements of $T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau))$ are of the form

$$[c_0, A(0), -C(0), -C'(0), c_0, B(0), A(0), B'(0)]$$

where $c_0 \in \mathbb{R}$ and $[A, B, C] \in Z_k$. By Lemma 4.3, an element of Z_k is of the form $[A, B, C] = P\,[A_1, B_1, C_1] + Q\,[A_2, B_2, C_2]$ with $P, Q \in \mathbb{R}[u]$, $\deg(P) \leqslant k - \mu_\tau - m_\tau$ and $\deg(Q) \leqslant k - \nu_\tau - m_\tau$. By removing the repeated columns, reordering and changing some signs, we see that $G(\gamma) = T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau))$ is isomorphic to the space spanned by the elements

$$\begin{bmatrix} f_1(\gamma) \\ \partial_{u_1} f_1(\gamma) \\ \partial_{u_2} f_2(\gamma) \\ -\partial_{v_1} f_1(\gamma) \\ \partial_{u_2}\partial_{v_2} f_2(\gamma) \\ -\partial_{u_1}\partial_{v_1} f_1(\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A_1(0) & A_2(0) & 0 & 0 \\ 0 & B_1(0) & B_2(0) & 0 & 0 \\ 0 & C_1(0) & C_2(0) & 0 & 0 \\ 0 & B_1'(0) & B_2'(0) & B_1(0) & B_2(0) \\ 0 & C_1'(0) & C_2'(0) & C_1(0) & C_2(0) \end{bmatrix} \begin{bmatrix} c_0 \\ P(0) \\ Q(0) \\ P'(0) \\ Q'(0) \end{bmatrix} \qquad (23)$$

for $P, Q \in \mathbb{R}[u]$ with $\deg(P) \leqslant k - \mu_\tau - m_\tau$ and $\deg(Q) \leqslant k - \nu_\tau - m_\tau$. Let us assume that $k \geqslant \nu_\tau + m_\tau + 1$ so that $k - \mu_\tau - m_\tau \geqslant k - \nu_\tau - m_\tau \geqslant 1$.

As $A_1 B_2 - A_2 B_1 = c$ and $A_1(0)B_2(0) - A_2(0)B_1(0) = c(0) \neq 0$, we deduce that $[B_1(0), B_2(0)] \neq [0, 0]$ and that $\dim G(\gamma) \geqslant 4$.

If $\mathfrak{c}_\tau(\gamma) = 0$, then $a(0) = B_1(0)C_2(0) - B_2(0)C_1(0) \neq 0$ and $\dim G(\gamma) = 5 = 5 - \mathfrak{c}_\tau(\gamma) = \dim \mathcal{H}(\gamma)$.

If $\mathfrak{c}_\tau(\gamma) = 1$, then $a(0) = B_1(0)C_2(0) - B_2(0)C_1(0) = 0$ and $\dim G(\gamma) = 4 = 5 - \mathfrak{c}_\tau(\gamma) = \dim \mathcal{H}(\gamma)$.

In both cases, we have $\dim G(\gamma) = \dim \mathcal{H}(\gamma)$, which implies that $G(\gamma) = \mathcal{H}(\gamma)$. This completes the proof of the proposition. □

If γ' is the other end point of τ, we have a Taylor map for each γ and γ', that we join together. Let

$$\begin{array}{rcl} T_{\gamma,\gamma'} : \mathcal{R}(\sigma_1) \oplus \mathcal{R}(\sigma_2) & \to & \mathcal{R}^{\sigma_1}(\gamma) \oplus \mathcal{R}^{\sigma_2}(\gamma) \oplus \mathcal{R}^{\sigma_1}(\gamma') \oplus \mathcal{R}^{\sigma_2}(\gamma') \\ f = (f_1, f_2) & \mapsto & (T_\gamma(f), T_{\gamma'}(f)) \end{array} \quad (24)$$

and let $G(\tau) = T_{\gamma,\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau))$.

Proposition 4.8. *For $k \geqslant \nu_\tau + m_\tau + 4$, we have $T_{\gamma,\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau)) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$ and*

$$\dim T_{\gamma,\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau)) = 10 - \mathfrak{c}_\tau(\gamma) - \mathfrak{c}_\tau(\gamma').$$

Proof. By a change of coordinates, we can assume that the coordinates of γ (resp. γ') are $(0,0)$ (resp. $(1,0)$) in σ_1 and $(0,0)$ (resp. $(0,1)$) in σ_2.

Similarly to the proof of the previous proposition, $T_{\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau))$ is spanned by the vectors

$$[c_0 + \int_0^1 A(u)du, A(1), -C(1), C'(1), c_0 + \int_0^1 A(u)du, B(1), A(1), B'(1)]$$

for $c_0 \in \mathbb{R}$ and $[A, B, C] = P\,[A_1, B_1, C_1] + Q\,[A_2, B_2, C_2] \in Z_k$ with $\deg(P) \leqslant k - \mu_\tau - m_\tau$ and $\deg(Q) \leqslant k - \nu_\tau - m_\tau$.

For $k \geqslant \nu_\tau + m_\tau + 4$, we can find polynomials $P = p_0(1-3u^2+2u^3) + p_1(u-2u^2-u^3) + p_2u^2(1-u)^2$, $Q = q_0(1-3u^2+2u^3) + q_1(u-2u^2-u^3) + q_2u^2(1-u)^2$, of degree $\leqslant 4$ such that $P(0) = p_0, P'(0) = p_1$, $Q(0) = q_0$, $Q'(0) = q_1$, $P(1) = 0, P'(1) = 0$, $Q(1) = 0$, $Q'(1) = 0$ and $\int_0^1 (PA_1 + QA_2)(u)du = -c_0$.

This implies that $(\mathcal{H}(\gamma), 0) \subset G(\tau)$. By symmetry, we also have $(0, \mathcal{H}(\gamma')) \subset G(\tau)$. By construction $G(\tau) \subset (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$, therefore we have $G(\tau) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$ and $\dim G(\tau) = \dim \mathcal{H}(\gamma) + \dim \mathcal{H}(\gamma')$. We deduce the dimension formula from Proposition 4.7. □

Definition 4.9. The separability $\mathfrak{s}(\tau)$ of the edge τ is the minimal k such that $T_{\gamma,\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau)) = (T_\gamma(\mathcal{S}_k^1(\mathcal{N}_\tau)), T_{\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau)))$.

Remark 4.10. The bound $\nu_\tau + m_\tau + 4 \geqslant \mathfrak{s}(\tau)$ is not necessarily the minimal degree of separability. Separability can be attained as soon as $d_\tau(k) \geqslant 9 - \mathfrak{c}_\tau(\gamma) - \mathfrak{c}_\tau(\gamma')$.

4.3. Decompositions and dimension

Let $\tau \in \mathcal{M}_1$ be an interior edge τ shared by the cells $\sigma_1, \sigma_2 \in \mathcal{M}_2$. Let $K_1 = (v_1^2) \cap \mathcal{R}_k(\sigma_1)$ and $K_2 = (u_2^2) \cap \mathcal{R}_k(\sigma_2)$ be the polynomials of $\mathcal{R}_k(\sigma_1)$ (resp. $\mathcal{R}_k(\sigma_2)$) divisible by v_1^2 (resp. u_2^2). Let L be the subspace of polynomials of $\mathcal{R}_k(\sigma_1) \oplus \mathcal{R}_k(\sigma_2)$ spanned by the Bernstein basis functions on σ_1 and σ_2, which are not divisible by v_1^2 or u_2^2 and let π_L be the projection of $\mathcal{R}_k(\sigma_1) \oplus \mathcal{R}_k(\sigma_2)$ on L along $(K_1, 0) \oplus (0, K_2)$. The functions in L are said to have *their support along* τ. By construction, we have $\mathcal{R}_k(\sigma_1) \oplus \mathcal{R}_k(\sigma_2) = (K_1, 0) \oplus (0, K_2) \oplus L$. The elements of (K_1, K_2) are obviously in $\mathcal{S}_k^1(\mathcal{N}_\tau)$ since they vanish at the order 1 along τ.

Let $W_k(\tau) = \pi_L(\Theta_\tau(Z_k))$ where Θ_τ is defined in (20). Notice that $W_k(\tau) \subset \mathcal{S}_k^1(\mathcal{N}_\tau)$ since $\ker \pi_L \subset \mathcal{S}_k^1(\mathcal{N}_\tau)$. Moreover, since $\ker \pi_L$ does not intersect $\Theta_\tau(Z_k)$ and Θ_τ is injective, the spaces $W_k(\tau)$, $\Theta_\tau(Z_k)$ and Z_k have the same dimension. Therefore, we have $\dim(W_k(\tau)) = d_\tau(k)$ and $W_k(\tau) \neq \{0\}$ when $k \geqslant \mu_\tau + m$ (Lemma 4.3 (iii)).

From the relations (18) and (19), we deduce the following decomposition:

$$\mathcal{S}_k^1(\mathcal{N}_\tau) = (K_1, 0) \oplus (0, K_2) \oplus \mathbb{R}\,\mathbf{u} \oplus W_k(\tau) \tag{25}$$

where $\mathbf{u} = \pi_L((1,1))$. The sum of these spaces is direct, since the supports of the functions of each space do not intersect.

The map $T_{\gamma,\gamma'}$ defined in (24) induces the exact sequence

$$0 \to \mathcal{K}_k(\tau) \to \mathcal{S}_k^1(\mathcal{N}_\tau) \xrightarrow{T_{\gamma,\gamma'}} G(\tau) \to 0$$

where $\mathcal{K}_k(\tau) = \ker T_{\gamma,\gamma'}$ and $G(\tau) = T_{\gamma,\gamma'}(\mathcal{S}_k^1(\mathcal{N}_\tau))$. It is clear that $(K_1, K_2) \subset \mathcal{K}_k(\tau)$.

Definition 4.11. For an interior edge $\tau \in \mathcal{M}_1^o$, let $\mathcal{E}_k(\tau) = \ker(T_{\gamma,\gamma'}) \cap W_k(\tau)$ be the set of splines in $\mathcal{S}_k^1(\mathcal{N}_\tau)$ with their support along τ and with vanishing Taylor expansions at γ and γ'. For a boundary edge $\tau' = (\gamma, \gamma')$, which belongs to a face σ, we also define $\mathcal{E}_k(\tau')$ as the set of elements of $\mathcal{R}_k(\sigma)$ with their support along τ' and with vanishing Taylor expansions at γ and γ'.

Notice that the elements of $\mathcal{E}_k(\tau)$ have their support along τ and their Taylor expansion at γ and γ' vanish. Therefore, their Taylor expansion along all (boundary) edges of $\mathcal{N}_\tau$ distinct from τ also vanish.

Lemma 4.12. *For an interior edge $\tau \in \mathcal{M}_1^o$, we have $\mathcal{K}_k(\tau) = (K_1, 0) \oplus (0, K_2) \oplus \mathcal{E}_k(\tau)$.*

Proof. As $(K_1, 0), (0, K_2) \subset \ker T_{\gamma,\gamma'} = \mathcal{K}_k(\tau)$ and $\mathcal{K}_k(\tau) \cap (W_k(\tau) \oplus \mathbb{R}\,\mathbf{u}) = \mathcal{K}_k(\tau) \cap W_k(\tau) = \mathcal{E}_k(\tau)$, we have

$$\begin{aligned}\mathcal{K}_k(\tau) &= (K_1, 0) \oplus (0, K_2) \oplus ((W_k(\tau) \oplus \mathbb{R}\,\mathbf{u}) \cap \mathcal{K}_k(\tau)) \\ &= (K_1, 0) \oplus (0, K_2) \oplus \mathcal{E}_k(\tau).\end{aligned}$$

□

Corollary 4.13. *For an interior edge $\tau \in \mathcal{M}_1^o$ and for $k \geqslant \mathfrak{s}(\tau)$, the dimension of $\mathcal{E}_k(\tau)$ is*

$$\dim \mathcal{E}_k(\tau) = d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma').$$

Proof. By Lemma 4.12, we have

$$\dim \mathcal{E}_k(\tau) = \dim \mathcal{K}_k(\tau) - \dim K_1 - \dim K_2.$$

As $\mathcal{K}_k(\tau)$ is the kernel of $T_{\gamma,\gamma'}$ and $G(\tau)$ is its image, we have

$$\dim \mathcal{K}_k(\tau) = \dim S_k^1(\mathcal{N}_\tau) - \dim G(\tau).$$

As $\dim(W_k(\tau)) = d_\tau(k)$, we deduce from the decomposition (25) that $\dim S_k^1(\mathcal{N}_\tau) = 1 + d_\tau(k) + \dim K_1 + \dim K_2$. Using Proposition 4.8, $G(\tau) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$ and we obtain

$$\begin{aligned}\dim \mathcal{E}_k(\tau) &= \dim S_k^1(\mathcal{N}_\tau) - \dim G(\tau) - \dim K_1 - \dim K_2 \\ &= d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma').\end{aligned}$$

□

Remark 4.14. When τ is a boundary edge, which belongs to the face $\sigma_1 \in \mathcal{M}_2$, we have $\mathcal{S}_k(\mathcal{N}_\tau) = \mathcal{R}_k(\sigma_1)$, $\mathcal{K}_k(\tau) = K_1 \oplus \mathcal{E}_k(\tau)$ and for $k \geqslant \mathfrak{s}(\tau) = 3 + F_\Delta(\sigma_1)$, $\dim G(\tau) = 8$ and $\dim \mathcal{E}_k(\tau) = k + 1 + (k + 1 - F_\Delta(\sigma_1)) - 8 = 2\,k - F_\Delta(\sigma_1) - 6$.

Notice that this is also what we obtain if we attach a virtually rectangular face along τ with constant gluing data: $n = \mu = 0$, $m = 0$, $e = 2$, $\mathfrak{c}_\tau(\gamma) = \mathfrak{c}_\tau(\gamma') = 0$ and

$$d_\tau(k) = 2\,k + 3 - F_\Delta(\sigma_1),$$

so that $\dim \mathcal{E}_k(\tau) = d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma')$.

5. G^1 splines around a vertex

We consider now a topological surface $\mathcal{O}$ composed of faces $\sigma_1, \ldots, \sigma_F \in \mathcal{O}_2$ sharing a single vertex γ, and such that σ_i and σ_{i+1} share the edge $\tau_{i+1} = (\gamma, \delta_{i+1})$. In particular τ_i, τ_{i+1} are the two edges of σ_i containing the vertex γ. The number of edges containing γ is denoted F'. All the vertices of $\mathcal{O}$ different from γ are boundary vertices. The vertex γ is an interior vertex, iff σ_F and σ_1 share the edge τ_1. In this case, we identify the indices modulo F and we have $F' = F$, otherwise we have $F' = F + 1$. The gluing data for the interior edge τ_i is $\mathfrak{a}_i = \frac{a_i}{c_i}, \mathfrak{b}_i = \frac{b_i}{c_i}$.

The coordinates in the ring $\mathcal{R}(\sigma_i)$ are chosen so that the coordinates of γ are $(0,0)$ and τ_i is defined by $v_i = 0$, $u_i \in [0,1]$ and by $u_{i-1} = 0$, $v_{i-1} \in [0,1]$ in $\mathcal{R}(\sigma_{i-1})$. The canonical form of the transition map at γ across the edge τ_i is then

$$\phi_{\tau_i} : (u,v) \longrightarrow \begin{pmatrix} v_i\, \mathfrak{b}_i(u_i) \\ u_i + v_i\, \mathfrak{a}_i(u_i) \end{pmatrix}$$

Let $f = (f_i)_{i=1,\ldots,F} \in \mathcal{S}^1(\mathcal{O})$. The gluing condition (4) implies that the Taylor expansion of f_i at γ is of the form

$$f_i(u_i, v_i) = p + q_i\, u_i + q_{i+1}\, v_i + s_i\, u_i v_i + r_i\, u_i^2 + r_{i+1}\, v_i^2 + \cdots$$

for $p, q_i, s_i, r_i \in \mathbb{R}$, $i = 1, \ldots, F$ (see Fig. 3).

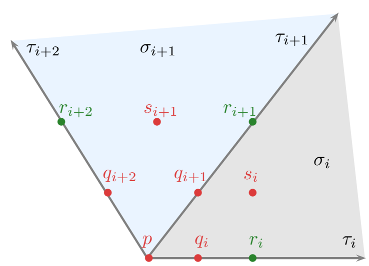


Figure 3: Taylor coefficients around a vertex.

By a computation similar to (22), Condition (4) implies that

$$\begin{aligned} q_{i+1} &= \mathfrak{a}_i(0)\, q_i + \mathfrak{b}_i(0)\, q_{i-1} && (i = 2, \ldots, F) && (26) \\ s_i &= 2\, \mathfrak{a}_i(0)\, r_i + \mathfrak{b}_i(0)\, s_{i-1} + \mathfrak{a}_i'(0)\, q_i + \mathfrak{b}_i'(0)\, q_{i-1} && (i = 2, \ldots, F) && (27) \end{aligned}$$

Let $\mathcal{H}(\gamma)$ be the vector space spanned by the vectors $\mathbf{h} = [p, q_1, \ldots, q_{F'}, s_1, \ldots, s_F]$ for $\mathbf{h}' = [p, q_1, \ldots, q_{F'}, s_1, \ldots, s_F, r_1, \ldots, r_{F'}]$ a solution of the linear system (26), (27).

Proposition 5.1.

$$\dim \mathcal{H}(\gamma) = 3 + F(\gamma) - \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_+(\gamma)$$

where $F = F(\gamma)$ is the number of faces around the vertex γ.

Proof. Notice that $\mathcal{H}(\gamma)$ is isomorphic to the projection of the solution set of system (26), (27) on the space of the variables $[p, \mathbf{q}, \mathbf{s}] = [p, q_1, \ldots, q_{F'}, s_1, \ldots, s_F]$.

The solutions in $\mathbf{q} = (q_1, \ldots, q_{F'})$ of the first set of equations satisfy the induction relations

$$\begin{pmatrix} q_i \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathfrak{b}_i(0) & \mathfrak{a}_i(0) \end{pmatrix} \begin{pmatrix} q_{i-1} \\ q_i \end{pmatrix} \text{ for } i = 2, \ldots, F.$$

As we have the compatibility condition 2.6 at an interior vertex, the solutions of (26) span a linear space of dimension 2, parametrized for instance by q_1, q_2.

The system (27) is formed by linearly independent equations which involve r_k and q_i, s_j when $\mathfrak{a}_k(0) \neq 0$ and by equations which only involve s_i, s_{i-1} and q_j when $\mathfrak{a}_i(0) = 0$.

Therefore the projection of the solution set of (27) on the space corresponding to the variables $[p, \mathbf{q}, \mathbf{s}]$ is defined by the equations which only involve s_i, s_{i-1} and q_i, q_{i-1} when $\mathfrak{a}_i(0) = 0$.

If one of the edges around γ is not a crossing edge, then the codimension of this space is $\sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma)$, with the convention that $\mathfrak{c}_\tau(\gamma) = 0$ if τ is a boundary edge.

If all the edges around γ are crossing edges, then the compatibility conditions 2.7 at γ imply that one of these equations is dependent from the other. Therefore, the codimension of this space is $\sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) - \mathfrak{c}_+(\gamma)$.

By intersecting it with the solution space of (26), we deduce that the dimension of $\mathcal{H}(\gamma)$ is precisely given by

$$1 + 2 + F - \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_+(\gamma).$$ □

Let $T_\gamma = \prod_{\sigma \ni \gamma} T_\gamma^\sigma$ be the Taylor map at γ on $\mathcal{O}$ and let $T_{\partial \mathcal{O}} = \prod_{\tau \not\ni \gamma} T_\tau^\sigma$ be the Taylor map along all the boundary edges which do not contain γ.

For $k \in \mathbb{N}$, we define $\mathcal{V}_k(\gamma) = \ker T_{\partial \mathcal{O}} \cap \mathcal{S}_k^1(\mathcal{O})$ the set of G^1 spline functions on $\mathcal{O}$ which vanish at the first order along the boundary edges (which do not contain γ).

Proposition 5.2. *For $k \geqslant \max_{i=1,\ldots,F}(\mathfrak{s}(\tau_i))$, $T_\gamma(\mathcal{V}_k(\gamma)) = \mathcal{H}(\gamma)$.*

Proof. By construction, the elements of $\mathcal{V}_k(\gamma)$ satisfy the equations (26), (27). This implies that $T_\gamma(\mathcal{V}_k(\gamma)) \subset \mathcal{H}(\gamma)$.

Consider an element $\mathbf{h} = (h_1, \ldots, h_F) \in \mathcal{H}(\gamma)$. By Proposition 4.8, for $k \geqslant \mathfrak{s}(\tau_i)$, there exists $(f_i, \tilde{f}_i) \in \mathcal{S}_k^1(\mathcal{N}_{\tau_i})$ such that $T_\gamma(f_i, \tilde{f}_i) = (h_i, h_{i-1})$ and $T_{\delta_i}(f_i, \tilde{f}_i) = 0$. Let $v_i = 0$ (resp. $u_{i-1} = 0$) be the equation of τ_i in σ_i (resp. σ_{i-1}). As for any polynomials $p \in (v_i^2) \cap \mathcal{R}_k(\sigma_i), q \in (u_{i-1})^2 \cap \mathcal{R}_k(\sigma_{i-1})$, $T_\gamma(p, q) = 0$, we can assume that $(f_i, \tilde{f}_i)$ has its support in $\mathcal{R}^{\sigma_i}(\tau_i) \oplus \mathcal{R}^{\sigma_{i-1}}(\tau_i)$.

By construction, we have $T_\gamma^{\sigma_i}(f_i) = T_\gamma^{\sigma_i}(\tilde{f}_{i+1}) = h_i$. Thus, there exist $g_i \in \mathcal{R}_k(\sigma_i)$ supported in $\mathcal{R}^{\sigma_i}(\tau_i) + \mathcal{R}^{\sigma_i}(\tau_{i+1})$ such that $T_{\tau_i}^{\sigma_i}(g_i) = f_i$, $T_{\tau_{i+1}}^{\sigma_i}(g_i) = \tilde{f}_{i+1}$. It is constructed by taking the coefficients of f_i on $\mathcal{R}^{\sigma_i}(\tau_i)$ and those of $\tilde{f}_{i+1}$ on $\mathcal{R}^{\sigma_i}(\tau_{i+1})$, the coefficients in $\mathcal{R}^{\sigma_i}(\tau_i) \cap \mathcal{R}^{\sigma_i}(\tau_{i+1})$ coinciding (see Fig. 4). As $T_{\delta_i}^{\sigma_i}(f_i) = T_{\delta_i}^{\sigma_i}(g_i) = 0$, $T_{\delta_{i+1}}^{\sigma_i}(\tilde{f}_i) = T_{\delta_{i+1}}^{\sigma_i}(g_i) = 0$ and g_i is supported in $\mathcal{R}^{\sigma_i}(\tau_i) + \mathcal{R}^{\sigma_i}(\tau_{i+1})$, we have $T_\tau^{\sigma_i}(g_i) = 0$ for any edge τ of the face σ_i, which does not contain γ.

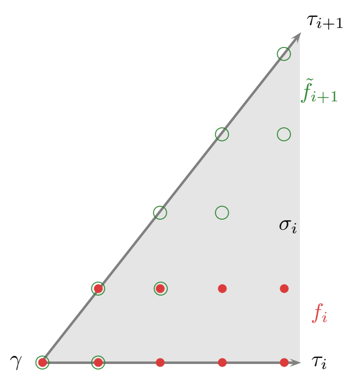


Figure 4: Lifting the edge functions.

Let $\mathbf{g} = [g_1, \ldots, g_F] \in \oplus_{\sigma_i \ni \gamma} \mathcal{R}_k(\sigma_i)$. By construction, $\mathbf{g}$ vanishes at the first order along all the boundary edges of $\mathcal{O}$, which do not contain γ. Moreover, $T_{\tau_i}(\mathbf{g}) = (f_i, \tilde{f}_i) \in \mathcal{S}_k^1(\mathcal{N}_{\tau_i})$, thus $\mathbf{g}$ satisfies the gluing conditions along the edge τ_i. We also have $T_\tau(\mathbf{g}) = 0$ for any edge τ, which does not contain γ. Thus $\mathbf{g}$ satisfies the gluing conditions along all the edges and its image by $T_{\partial\mathcal{O}}$ vanishes, i.e. $\mathbf{g} \in \mathcal{S}_k^1(\mathcal{O}) \cap \ker T_{\partial\mathcal{O}} = \mathcal{V}_k(\gamma)$. By construction, $T_\gamma(\mathbf{g}) = \mathbf{h}$. This shows that $\mathcal{H}(\gamma) \subset T_\gamma(\mathcal{V}_k(\gamma))$ and concludes the proof. □

6. G^1 splines on a general mesh

We consider now a general mesh $\mathcal{M}$ with an arbitrary number of faces, possibly with boundary edges.

We denote by $T_0 = \prod_{\gamma\in\mathcal{M}_0} T_\gamma$ the Taylor map at all the vertices of $\mathcal{M}$ and $\mathcal{H} = T_0(S_k^1(\mathcal{M}))$. We have the following exact sequence:

$$0 \to \mathcal{K}_k \to \mathcal{S}_k^1(\mathcal{M}) \xrightarrow{T_0} G \to 0$$

where $\mathcal{K}_k = \ker T_0 \cap \mathcal{S}_k^1(\mathcal{M})$ and $G = T_0(\mathcal{S}_k^1(\mathcal{M}))$. Let $\mathfrak{s}^* = \max\{\mathfrak{s}(\tau) \mid \tau \in \mathcal{M}_1\}$. We have $\mathfrak{s}^* \leqslant \max\{\nu_\tau + m_\tau + 4 \mid \tau \in \mathcal{M}_1\}$

6.1. Splines at a vertex

Let $\gamma \in \mathcal{M}_0$ be a vertex of $\mathcal{M}$ and let $\mathcal{O}_\gamma$ be the sub-mesh associated to the faces of $\mathcal{M}$ which contain γ. Let $\mathcal{V}_k(\gamma)$ be the set of spline functions in $\mathcal{S}_k^1(\mathcal{M})$ supported on the faces of $\mathcal{O}_\gamma$, which vanish at the first order along the edges that do not contain γ.

Proposition 6.1. *For $k \geqslant \mathfrak{s}^*$, $T_0(\mathcal{S}_k^1(\mathcal{M})) = \prod_\gamma \mathcal{H}(\gamma)$ and*

$$\dim T_0(\mathcal{S}_k^1(\mathcal{M})) = \sum_{\gamma\in\mathcal{M}_0} (F(\gamma)+3) - \sum_{\gamma\in\mathcal{M}_0}\sum_{\tau\ni\gamma} \mathfrak{c}_\tau(\gamma) + \sum_{\gamma\in\mathcal{M}_0} \mathfrak{c}_+(\gamma),$$

where $F(\gamma)$ is the number of faces of $\mathcal{M}$ that contain the vertex $\gamma \in \mathcal{M}_0$.

Proof. By proposition 5.2, for $k \geqslant \mathfrak{s}^*$ the image of $\mathcal{V}_k(\gamma)$ by T_0 is $G(\gamma)$, and $T_{\gamma'}(\mathcal{V}_k(\gamma)) = 0$ for any other vertex $\gamma' \neq \gamma$.

This shows that $T_0(\mathcal{S}_k^1(\mathcal{M})) = \prod_\gamma G(\gamma) = \prod_\gamma \mathcal{H}(\gamma)$. We deduce the dimension formula from Proposition 5.1. $\square$

6.2. Splines on edges

For an interior edge $\tau = (\gamma, \gamma') \in \mathcal{M}_1$, let $\mathcal{N}_\tau$ be the sub-mesh made of the faces σ_1, σ_2 of $\mathcal{M}$ containing τ. Let $\mathcal{E}_k(\tau) = \ker T_{\gamma,\gamma'} \cap \Theta_\tau(Z_k)$ (see Definition 4.11). The elements of $\mathcal{E}_k(\tau)$ correspond to splines of $\mathcal{S}_k^1(\mathcal{N}_\tau)$, which are in the kernel of $T_{\gamma,\gamma'}$ and with a support in $\mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau)$. Thus, $\mathcal{E}_k(\tau) \subset \ker T_{\tau'}$ for any edge $\tau' \in \mathcal{M}_1$, distinct from τ. We deduce that any element of $\mathcal{E}_k(\tau)$ satisfies the gluing condition along all edges of $\mathcal{M}_1$, and thus corresponds to a spline function in $\mathcal{S}_k^1(\mathcal{M})$. In other words, we have $\mathcal{E}_k(\tau) \subset \mathcal{S}_k^1(\mathcal{M}) \cap \ker T_0 = \mathcal{K}_k$. The elements of $\mathcal{E}_k(\tau)$ have a support in $\mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau)$ and their Taylor coefficients at the end points of τ vanish.

Thus the support of the elements of $\mathcal{E}_k(\tau)$ and $\mathcal{E}_k(\tau')$ for two distinct edges τ, τ' do not intersect, and their sum is direct. Let $\mathcal{E}_k = \oplus_{\tau\in\mathcal{M}_1}\mathcal{E}_k(\tau)$.

Let $\mathcal{F}_k = \ker T_1 \cap S_k^1(\mathcal{M})$ be the set of spline functions, which Taylor expansions along all edges vanish.

Proposition 6.2.

$$\mathcal{K}_k = \mathcal{F}_k \oplus \mathcal{E}_k$$

Proof. Let $f \in \mathcal{K}_k$ and take an interior edge $\tau \in \mathcal{M}_1$. Let σ_1, σ_2 be the two faces of $\mathcal{N}_\tau$.

Then $(f_{\sigma_1}, f_{\sigma_2}) \in \mathcal{S}_k^1(\mathcal{N}_\tau) \cap \ker T_0 = \mathcal{K}_k(\tau)$. By Lemma 4.12, $(f_{\sigma_1}, f_{\sigma_2}) = s_\tau + (k_1, k_2)$ with $s_\tau \in \mathcal{E}_k(\tau)$ and $(k_1, k_2) \in (K_1, K_2)$. As s_τ lifts to a spline $\in S_k^1(\mathcal{M})$, $f - s_\tau$ is an element of $S_k^1(\mathcal{M})$, which image by the Taylor expansion T_τ along the edge τ vanishes.

If τ is a boundary edge of $\mathcal{M}$, which belongs to the face σ_1, we have a similar decomposition $f_1 = s_\tau + k_1$ with $s_\tau \in \mathcal{E}_k(\tau)$ and $k_1 \in K_1$, using the convention of Remark 4.14. Similarly s_τ lifts to a spline $\in S_k^1(\mathcal{M})$, $f - s_\tau$ is an element of $S_k^1(\mathcal{M})$ in the kernel of T_τ.

Repeating this process for all edges $\tau \in \mathcal{M}_1$, we can construct an element $\tilde{f} = f - \sum_{\tau\in\mathcal{M}_1} s_\tau$ such that $\forall \tau \in \mathcal{M}_1, T_\tau(\tilde{f}) = 0$, i.e. $\tilde{f}$ belongs to $\ker T_1 = \mathcal{F}_k$. This shows that $\mathcal{K}_k \subset \mathcal{F}_k + \sum_{\tau\in\mathcal{M}_1} \mathcal{E}_k(\tau)$. By construction, we have $\mathcal{F}_k \subset \mathcal{K}_k$ and $\mathcal{E}_k = \oplus_{\tau\in\mathcal{M}_1}\mathcal{E}_k(\tau) \subset \mathcal{K}_k$. Considering the support of the functions in $\mathcal{F}_k$ and $\mathcal{E}_k$, we deduce that their sum is direct and equal to $\mathcal{K}_k$. $\square$

6.3. The dimension formula

We can now determine the dimension of $\mathcal{S}_k^1(\mathcal{M})$.

Theorem 6.3. *Let $\mathfrak{s}^* = \max\{\mathfrak{s}(\tau) \mid \tau \in \mathcal{M}_1\}$. Then, for $k \geqslant \mathfrak{s}^*$,*

$$\begin{aligned} \dim \mathcal{S}_k^1(\mathcal{M}) \;=\; & (k-3)^2 F_\square + \tfrac{1}{2}(k-5)(k-4)F_\Delta \\ & + \textstyle\sum_{\tau\in\mathcal{M}_1} d_\tau(k) + 4F_\square + 3F_\Delta - 9F_1 + 3F_0 + F_+ \end{aligned}$$

where

- *$d_\tau(k)$ is the dimension of the syzygies of the gluing data along τ in degree $\leqslant k$,*
- *$F_\square$ is the number of rectangular faces, F_Δ is the number of triangular faces,*

- *F_1 is the number of edges,*
- *F_0 (resp. F_+) is the number of (resp. crossing) vertices,*

Proof. By definition, we have

$$\dim \mathcal{S}_k^1(\mathcal{M}) = \dim \mathcal{H} + \dim \mathcal{K}_k.$$

By Proposition 6.2, we have

$$\begin{aligned}
\dim \mathcal{K}_k &= \dim\, \mathcal{F}_k + \dim\, \mathcal{E}_k = \dim \mathcal{F}_k + \sum_{\tau \in \mathcal{M}_1} \dim \mathcal{E}_k(\tau) \\
&= (k-3)^2 F_\square + \frac{1}{2}(k-5)(k-4)F_\Delta + \sum_{\tau \in \mathcal{M}_1} (d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_{\tau'}(\gamma'))
\end{aligned}$$

From Proposition 6.1, we deduce that

$$\begin{aligned}
\dim \mathcal{S}_k^1(\mathcal{M}) &= \dim\, \mathcal{K}_k + \dim\, \mathcal{H} \\
&= (k-3)^2 F_\square + \frac{1}{2}(k-5)(k-4)F_\Delta \\
&\quad + \sum_{\tau=(\gamma,\gamma')\in \mathcal{M}_1} (d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_{\tau'}(\gamma')) \\
&\quad + \sum_{\gamma \in \mathcal{M}_0} (F(\gamma) + 3) - \sum_{\gamma \in \mathcal{M}_0} \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) + \sum_{\gamma \in \mathcal{M}_0} \mathfrak{c}_+(\gamma) \\
&= F_\square (k-3)^2 + F_\Delta \frac{1}{2}(k-5)(k-4) + \sum_{\tau \in \mathcal{M}_1} d_\tau(k) - 9F_1 \\
&\quad + 4F_\square + 3F_\Delta + 3F_0 + F_+
\end{aligned}$$

since $\sum_{\tau=(\gamma,\gamma')\in \mathcal{M}_1} (\mathfrak{c}_\tau(\gamma) + \mathfrak{c}_{\tau'}(\gamma')) = \sum_{\gamma \in \mathcal{M}_0} \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma)$ and $\sum_{\gamma \in \mathcal{M}_0} F(\gamma) = 4F_\square + 3F_\Delta$. □

As a direct corollary, we obtain the following result:

Corollary 6.4. *If $\mathcal{M}$ is a topological surface with gluing data satisfying the compatibility Conditions 2.6-2.7 and if all crossing vertices of $\mathcal{M}$ have 4 edges, then $\mathcal{S}_k^1(\mathcal{M})$ is an ample space of differentiable functions on $\mathcal{M}$ for $k \geqslant \mathfrak{s}^*$.*

6.4. Basis

We are going now to describe an explicit construction of spline functions which form a basis of $\mathcal{S}_k^1(\mathcal{M})$. An algorithmic description of the computation of the Bernstein coefficients of these basis functions is provided in Appendix A.

We assume that k is bigger than the separability $\mathfrak{s}^*$ of all edges.

6.4.1. Basis functions associated to a vertex

Let $\gamma \in \mathcal{M}_0$ be a vertex and $\sigma_1, \ldots, \sigma_F$ be the faces of $\mathcal{M}_2$ adjacent to γ. We also assume that σ_i and σ_{i-1} share the edge $\tau_i \in \mathcal{M}_1$ and that τ_1 is not a crossing edge at γ if such an edge exists.

To compute the basis functions attached to γ, we compute first the taylor coefficients of $f_{\sigma_i} = p + q_i u_i + q_{i+1} v_i + s_i u_i v_i + \cdots$ solutions of the system (26)-(27) and then lift these Taylor coefficients to define a spline function with support in $\mathcal{O}_\gamma$. This leads to the following type of basis functions:

- 1 basis function attached to the value at γ: $p = 1$, $q_i = 0$, $s_i = 0$
- 2 basis functions attached to the derivatives at γ: $p = 0$, $[q_1, q_2] \in \{[1, 0], [0, 1]\}$ and $s_i = 0$ if τ_i is not a crossing edge at γ and determined by the relations (26)-(27) if τ_i is a crossing edge at γ.
- $F(\gamma) - \sum_{i=1}^{F'} \mathfrak{c}_{\tau_i}(\gamma) + \mathfrak{c}_+(\gamma)$ basis functions attached to the free cross derivatives, with $p = 0$, $q_i = 0$ and $s_i \in \{0, 1\}$ if τ_i is not a crossing edge and determined by the relations (26)-(27) if τ_i is a crossing edge at γ.

6.4.2. Basis functions associated to an edge

Let τ be an edge of $\mathcal{M}_1$ shared by two faces σ_1, σ_2 with vertices γ, γ'. Let us assume that the coordinates of these points in the face σ_1 are $\gamma = (0, 0)$ and $\gamma' = (1, 0)$.

The elements of $\mathcal{E}_k(\tau)$ are the image by Θ_τ of the elements of Z_k of the form $P\,[A_1, B_1, C_1] + Q\,[A_2, B_2, C_2]$ with degree $\deg(P) \leqslant k - \mu_\tau - m_\tau$, $\deg(Q) \leqslant k - \nu_\tau - m_\tau$ which are in the kernel of T_γ and $T_{\gamma'}$.

From the relation (23), we deduce that $P(0) = 0$, $Q(0) = 0$. That is, P and Q are divisible by u.

- If $\mathfrak{c}_\tau(\gamma) = 0$, i.e. γ is not a crossing vertex, we have $B_1(0)C_2(0) - B_2(0)C_1(0) = a(0) \neq 0$ and the relation (23) implies that $P'(0) = 0$, $Q'(0) = 0$. That is $P = u^2\tilde{P}$, $Q = u^2\tilde{Q}$.

- If $\mathfrak{c}_\tau(\gamma) = 1$, then the kernel of T_γ is generated by polynomials such that $P(0) = 0$, $Q(0) = 0$, $P'(0) = \lambda C_2(0)$, $Q'(0) = -\lambda C_1(0)$. That is $P = u\,(\lambda C_2(0) + u\tilde{P})$, $Q = u\,(-\lambda C_1(0) + u\,\tilde{Q})$.

That is

$$P = u\,\left(\lambda\,\mathfrak{c}_\tau(\gamma)\,C_2(0) + u\,\tilde{P}\right), Q = u\,\left(-\lambda\,\mathfrak{c}_\tau(\gamma)\,C_1(0) + u\,\tilde{Q}\right).$$

By symmetry at γ', we see that P and Q are of the form:

$$\begin{aligned} P &= u\,(1-u)\left(\lambda\,\mathfrak{c}_\tau(\gamma)C_2(0)\,(1-u) + \lambda'\,\mathfrak{c}_\tau(\gamma')\,C_2(1)\,u + u\,(1-u)\,\tilde{P}\right), \\ Q &= -u\,(1-u)\left(\lambda\,\mathfrak{c}_\tau(\gamma)\,C_1(0)\,(1-u) + \lambda'\,\mathfrak{c}_\tau(\gamma')\,C_1(1)\,u + u\,(1-u)\,\tilde{Q}\right), \end{aligned}$$

with $\lambda, \lambda' \in \mathbb{R}$, $\deg(\tilde{P}) \leqslant k - \mu - m - 4$, $\deg(\tilde{Q}) \leqslant k - \nu - m - 4$.

We construct a basis of $\mathcal{E}_k(\tau)$ by taking the image by Θ_τ of a maximal set of linearly independent elements of this form (see Section 4.3). This yields $d_\tau(k) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma')$ spline basis functions.

6.4.3. Basis functions associated to a face

Finally, we define the basis functions attached to a face $\sigma \in \mathcal{M}_2$ as the 2-interior Bernstein basis functions in degree $\leqslant k$. There are $(k-3)^2$ such basis spline functions for a rectangular face and $\binom{k-4}{2}$ for a triangular face.

7. Examples

7.1. Splines on flat triangular tilings

We consider a subdivision of a planar domain $\Omega \subset \mathbb{R}^2$ into a partition of triangles and the topological surface $\mathcal{M}$ induced by this subdivision.

For two faces $\sigma_1, \sigma_2 \in \mathcal{M}_2$, which share an edge $\tau \in \mathcal{M}_1$ at a vertex γ, there is a linear map ϕ_{σ_2,σ_1}, which transforms the variables (u_1, v_1) attached to σ_1 into the variables (u_2, v_2) attached to σ_2.

With $\gamma = (0, 0)$ and $v_1 = u_2$, the transition map ϕ_{σ_2,σ_1} is given by

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

where $a, b \in \mathbb{R}$ and $b \neq 0$. We choose these constant transition maps to define the space of splines $\mathcal{S}^1(\mathcal{M})$. The gluing conditions along the edges correspond then to C^1 conditions for the polynomials expressed in the same coordinate system. In this case, $\mathcal{S}^1(\mathcal{M})$ is the vector space of piecewise polynomial functions on $\mathcal{M}$, which are C^1 on Ω, that is, the classical C^1-spline functions on Ω.

If $a = 0$, the edge of σ_2 at γ distinct from τ is aligned with the edge of σ_1 at γ distinct from τ. The vertex γ is a crossing vertex ($\mathfrak{c}_+(\gamma) = 1$; all the coefficients a in the transition maps around γ vanish) if there are 4 edges at γ, which are pair-wise aligned.

As for any interior edge $\tau \in \mathcal{M}_1$ the transition map is constant, we have $n_\tau = 0$, $\mu_\tau = 0$, $\nu_\tau = 0$, $m_\tau = 1$, $\mathfrak{s}_\tau \leqslant 5$ and $d_\tau(k) = 2\,k$. For the boundary edges, we have $d_\tau(k) = 2\,k + 2$ (see Remark 4.14).

We deduce from Theorem 6.3 that for $k \geqslant 5$, we have

$$\begin{aligned}\dim \mathcal{S}_k^1 &= \frac{1}{2}(k-5)(k-4)F_\Delta + 2k\,F_1^o + (2k+2)\,F_1^b + 3F_\Delta - 9\,F_1 + 3\,F_0 + F_+\\ &= \frac{1}{2}(k+2)(k+1)\,F_\Delta - 6\,(k-2)\,F_\Delta + (2k-9)\,F_1^o + (2k-7)\,F_1^b + 3\,F_0 + F_+\end{aligned}$$

where F_1^o (resp. F_1^b) is the number of interior (resp. boundary) edges. Using the relations $3F_\Delta = 2F_1^o + F_1^b$ (counting the edges per triangle, we count twice the interior edges shared by two triangles and once the boundary edges), $F_1^b = F_0^b$ and $F_0 = F_0^o + F_0^b$ where F_0^o (resp. F_0^b) is the number of interior (resp. boundary) vertices, we obtain

$$\dim \mathcal{S}_k^1 = \frac{1}{2}(k+2)(k+1)\,F_\Delta - (2k+1)\,F_1^o + 3\,F_0^o + F_+.$$

This coincides with the dimension formula of C^1 piecewise polynomials of degree $k \geqslant 5$ on a triangular planar mesh, given in [17]. Here F_+ counts the number of crossing vertices, also called singular vertices in [17].

The basis functions constructed as in Section 6.4 are as follows:

- For each vertex γ, there are 3 basis functions associated to the evaluation and derivatives in x, y at γ. There are $F(\gamma) - \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_+(\gamma)$ basis functions associated to the free cross-derivatives on the triangles containing γ.

- For each interior edge $\tau = (\gamma, \gamma')$, there are $2k - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma')$ basis functions associated to $k + 1 - 6 = k - 5$ free interior Bernstein coefficients $b_{3,0}, \ldots, b_{k-3,0}$ on the edge, $k - 4$ free interior Bernstein coefficients $b_{2,1}, \ldots, b_{k-2,1}$ on one triangle σ which contains τ and $b_{2,0}$ (resp. $b_{k-2,0}$) if τ is a crossing vertex at γ (resp. γ').

- For each boundary edge τ', there are $2k-7$ basis functions associated to $k-3$ free interior Bernstein coefficients $b_{2,0},\ldots,b_{k-2,0}$ on the edge τ', and $k-4$ free interior Bernstein coefficients $b_{2,1},\ldots,b_{k-2,1}$ on the triangle σ which contains τ'.

- For each triangle σ, there are $\binom{k-4}{2}$ basis functions associated to the interior Bernstein coefficients $b_{i,j}$ with $2 \leqslant i,j \leqslant k-2$ and $0 \leqslant i+j \leqslant k$.

This basis description involves the Bernstein coefficients of polynomial on the triangles. The basis differs from the nodal basis proposed in [17]. From the listed Bernstein coefficients, we can however recover the nodal basis of [17], dual to the evaluation and derivatives at the vertices and at interior points of the edges and the triangles.

7.2. A round corner

We consider a mesh $\mathcal{M}$ composed of 3 rectangles $\sigma_1,\sigma_2,\sigma_3$ glued around an interior vertex γ, along the 3 interior edges τ_1,τ_2,τ_3. There are 6 boundary edges and 6 boundary vertices.

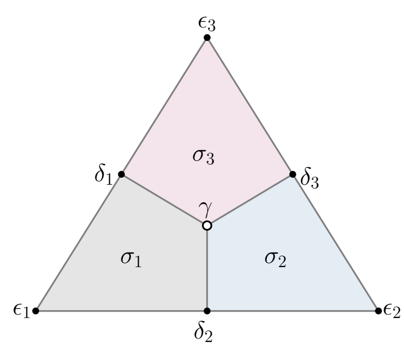


Figure 5: Smooth corner.

We take symmetric gluing data at γ and at the crossing boundary vertices δ_i. The transition map across the interior edge τ_i is given by the polynomials: $[a,b,c] = [(u-1),-1,1]$ where γ is the end point with $u=0$ and δ_i is the end point with $u=1$. The generating syzygies are

$$S_1 = [0,1,1], S_2 = [1,u,1].$$

For the interior edges τ_i, we have $n=1$, $m=0$, $\mu=0$, $\nu=1$ and $d_{\tau_i}(k) = k+1+k = 2k+1$. For the boundary edges τ', we have $n=0$, $m=0$, $\mu=0$, $\nu=0$ and $d_{\tau'}(k) = 2\,(k+1)$.

As $a(0) = -1$ (resp. $a(1) = 0$), γ is not a crossing vertex ($\mathfrak{c}_{\tau_i}(\gamma) = 0$) and δ_i is a crossing vertex of τ_i ($\mathfrak{c}_{\tau_i}(\delta_i) = 1$).

We check that the separability of all the interior edges is 4. For $k = 4$, the dimension of $S_k^1(\mathcal{M})$ is

$$3 \times (4-3)^2 + 3 \times (2 \times 4 + 1) + 6 \times (2 \times 4 + 2) + 4 \times 3 - 9 \times 9 + 3 \times 7 + 6 = 48.$$

The basis functions are constructed as in Section 6.4, using the algorithms of Appendix A.

- The number of basis functions attached to γ is $6 = 1 + 2 + 3$.
  - The basis function associated to the value at γ is

    $$[b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}, b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}, b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}]$$

  - The two basis functions associated to the derivatives at γ are

    $$\begin{array}{l} \big[\frac{1}{4}\, b_{1,0} + \frac{1}{4}\, b_{1,1} + \frac{7}{12}\, b_{2,0} + \frac{7}{12}\, b_{2,1} + \frac{1}{8}\, b_{1,2}, \\ \quad -\frac{1}{4}\, b_{0,1} - \frac{1}{4}\, b_{1,1} - \frac{1}{8}\, b_{2,1} - \frac{7}{12}\, b_{0,2} - b_{1,2}, \\ \quad \frac{1}{4}\, b_{0,1} - \frac{1}{4}\, b_{1,0} - \frac{7}{12}\, b_{2,0} - \frac{11}{24}\, b_{2,1} + \frac{7}{12}\, b_{0,2} + \frac{7}{8}\, b_{1,2}\big] \\ \big[\frac{1}{4}\, b_{0,1} + \frac{1}{4}\, b_{1,1} + \frac{1}{8}\, b_{2,1} + \frac{7}{12}\, b_{0,2} + b_{1,2}, \\ \quad -\frac{1}{4}\, b_{0,1} + \frac{1}{4}\, b_{1,0} + \frac{7}{12}\, b_{2,0} + \frac{11}{24}\, b_{2,1} - \frac{7}{12}\, b_{0,2} - \frac{7}{8}\, b_{1,2}, \\ \quad -\frac{1}{4}\, b_{1,0} - \frac{1}{4}\, b_{1,1} - \frac{7}{12}\, b_{2,0} - \frac{7}{12}\, b_{2,1} - \frac{1}{8}\, b_{1,2}\big] \end{array}$$

  - The three basis functions associated to the cross derivatives at γ are

    $$\begin{array}{l} \big[\frac{1}{16}\, b_{1,1} - \frac{1}{12}\, b_{2,0} - \frac{1}{24}\, b_{2,1} - \frac{1}{12}\, b_{0,2} - \frac{1}{8}\, b_{1,2}, \\ \quad -\frac{1}{12}\, b_{2,0} - \frac{1}{12}\, b_{2,1}, -\frac{1}{12}\, b_{0,2} - \frac{1}{6}\, b_{1,2}\big] \\ \big[-\frac{1}{12}\, b_{0,2} - \frac{1}{6}\, b_{1,2}, \\ \quad \frac{1}{16}\, b_{1,1} - \frac{1}{12}\, b_{2,0} - \frac{1}{24}\, b_{2,1} - \frac{1}{12}\, b_{0,2} - \frac{1}{8}\, b_{1,2}, -\frac{1}{12}\, b_{2,0} - \frac{1}{12}\, b_{2,1}\big] \\ \big[-\frac{1}{12}\, b_{2,0} - \frac{1}{12}\, b_{2,1}, \\ \quad -\frac{1}{12}\, b_{0,2} - \frac{1}{6}\, b_{1,2}, \frac{1}{16}\, b_{1,1} - \frac{1}{12}\, b_{2,0} - \frac{1}{24}\, b_{2,1} - \frac{1}{12}\, b_{0,2} - \frac{1}{8}\, b_{1,2}\big] \end{array}$$

- The number of basis functions attached to δ_i is $4 = 1 + 2 + 2 - 1$. Here are the 4 basis functions associated to δ_1:

  $$\begin{array}{l} [b_{3,0} + b_{3,1} + b_{4,0} + b_{4,1}, 0, b_{0,3} + b_{1,3} + b_{0,4} + b_{1,4}], \\ [b_{3,0} + b_{3,1}, 0, b_{0,3} + b_{1,3}], \\ [b_{3,1} + b_{4,1}, 0, -b_{1,3} - b_{1,4}], \\ [b_{3,1}, 0, -b_{1,3}]. \end{array}$$

The basis functions associated to the other boundary points δ_2, δ_3 are obtained by cyclic permutation.

- The number of basis functions attached to the remaining boundary points is $4 = 1 + 2 + 1$. For ϵ_1, the 4 basis functions are

$$[b_{3,3} + b_{3,4} + b_{4,3} + b_{4,4}, 0, 0],\ [b_{3,3} + b_{4,3}, 0, 0],\ [b_{3,3} + b_{3,4}, 0, 0],\ [b_{3,3}, 0, 0]$$

  The basis functions associated to the other boundary points are obtained by cyclic permutation.

- The number of basis functions attached to the edge τ_i is $2 \times 4 - 7 = 1$. For the edge τ_1, it is
$$[b_{2,1}, 0, -b_{1,2}].$$
  The basis functions associated to the other interior edges are obtained by cyclic permutation.

- The number of basis functions attached to the boundary edges is $2(4 - 3) = 2$. For the boundary edge (ϵ_1, δ_1) of σ_1, the two basis functions are
$$[b_{3,2}, 0, 0], [b_{4,2}, 0, 0].$$

- The number of basis functions attached to a face σ_i is $(4 - 3)^2 = 1$. The basis function associated to σ_1 is

$$[b_{2,2}, 0, 0]$$

  and the two other ones are obtained by cyclic permutation.

7.3. A pruned octahedron

We consider a mesh $\mathcal{M}$ with 6 triangular faces AEF, CEF, ABE, BCE, ADF, CDF and one rectangular face $ABCD$, depicted in Figure 6 as the *Schlegel diagram* of a convex polyhedron in $\mathbb{R}^3$. It is an octahedron where an edge BD is removed and two triangular faces are merged into a rectangular face (see [29] for the complete octahedron, which involves only triangular faces).

We are going to use the following notation for the variables on the faces of $\mathcal{M}$. For $X, Y \in \mathbb{R}^2$ two vertices defining an edge XY of a face σ, let $u_X^Y : \mathbb{R}^2 \to \mathbb{R}$ be the linear function with $u_X^Y(X) = 0$, $u_X^Y(Y) = 1$ and $u_X^Y(Z) = 0$

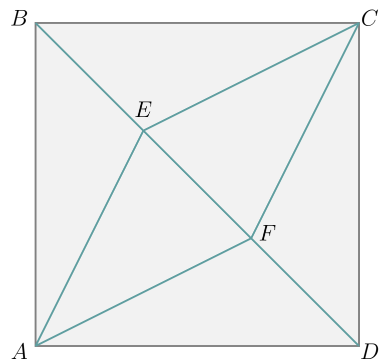


Figure 6: A pruned octahedron

for all the points Z on the edge of σ through X and distinct from XY. We will use these linear functions u_E^B, u_F^A, etc., as variables on the different faces. As the restriction on a share edge XY of the two functions defined on the faces adjacent to XY coincide, there is no ambiguity in evaluating these linear functions on $\mathcal{M}$. For a triangular face XYZ, we have $u_X^Y + u_Y^Z + u_Z^X = 1$. For a rectangular face $XYZW$, we have $u_X^Y = u_W^Z$ and $u_X^W = u_Y^Z$. We denote by ∂_{XY} the derivative with respect to the variable u_X^Y. It is such that $\partial_{XY}(u_X^Y) = 1$. On a triangular face XYZ, we have $\partial_{XY} + \partial_{YZ} + \partial_{ZX} = 0$.

We use a symmetric gluing at all the vertices and therefore the vertices A, C, E, F are crossing vertices. Let us describe how we construct the gluing data on the edges by interpolation at the vertices, in a smaller degree than the degree associated to the gluing (14) proposed in [12].

In terms of differentials (see relation (5)), the symmetric gluing at the vertices translates as

$$\begin{array}{lll} \partial_{EA} + \partial_{EC} = 0, & \partial_{EB} + \partial_{EF} = 0 & \text{at } E, \\ \partial_{FA} + \partial_{FC} = 0, & \partial_{FD} + \partial_{FE} = 0 & \text{at } F, \\ \partial_{AB} + \partial_{AF} = 0, & \partial_{AE} + \partial_{AD} = 0 & \text{at } A, \\ \partial_{CB} + \partial_{CF} = 0, & \partial_{CE} + \partial_{CD} = 0 & \text{at } C, \end{array} \tag{28}$$

and around the vertices of order 3:

$$\partial_{BA} + \partial_{BC} + \partial_{BE} = 0 \quad \text{at } B, \qquad \partial_{DA} + \partial_{DC} + \partial_{DF} = 0 \quad \text{at } D.$$

For gluing the triangles EFA and EFC along EF, we interpolate the following relations between the derivatives:

$$\partial_{EA} + \partial_{EC} = 0 \text{ at } E, \quad \partial_{EA} + \partial_{EC} = 2\,\partial_{EF} \text{ at } F,$$

where the second expression is $\partial_{FA} + \partial_{FC} = 0$ rewritten using $\partial_{FA} = \partial_{EA} - \partial_{EF}, \partial_{FC} = \partial_{EC} - \partial_{EF}$. We choose the linear interpolation

$$\partial_{EA} + \partial_{EC} = 2u_E^F \, \partial_{EF}.$$

Thereby we have $\mathfrak{a}_{EF} = 2u_E^F$, $\mathfrak{b}_{EF} = -1$ and the gluing data for the edge EF is $[2\, u_E^F, -1, 1]$.

For the edge EB between the triangles EBA and EBC, we interpolate the following relations:

$$\partial_{EA} + \partial_{EC} = 0 \text{ at } E, \qquad \partial_{EA} + \partial_{EC} = 3\, \partial_{EB} \text{ at } B,$$

where the latter relation is $\partial_{BA} + \partial_{BC} + \partial_{BE} = 0$ rewritten using $\partial_{BA} = \partial_{EA} - \partial_{EB}, \partial_{BC} = \partial_{EC} - \partial_{EB}, \partial_{BE} = -\partial_{EB}$. Additionally, we have to take into account the compatibility conditions (9)-(10) since E is a crossing vertex. It translates as $\partial_{EB}(\mathfrak{a}_{EB}) = \partial_{EF}(\mathfrak{a}_{EF})$ and $\partial_{EC}(\mathfrak{a}_{EC}) = \partial_{EA}(\mathfrak{a}_{EA})$ at the vertex E. This leads to the following gluing data on the edge EB:

$$EB : [2u_E^B + (u_E^B)^2, -1, 1].$$

Similarly, the gluing data of the edge FD is

$$FD : [2u_F^D + (u_F^D)^2, -1, 1].$$

The edges EA, EC, FA, FC connect cross vertices just as EF and yield linear gluing data

$$\begin{array}{rlrl} EF: & [2u_E^A, -1, 1], & EC: & [2u_E^C, -1, 1], \\ FA: & [2u_F^A, -1, 1], & FC: & [2u_F^C, -1, 1] \end{array}$$

We check that the compatibility conditions (9)-(10) are satisfied across EA, EC and FA, FC. The gluing data along AB, AD, CB, CD looks the same:

$$\begin{array}{rlrl} AB: & [2u_A^B, -1, 1], & AD: & [2u_A^D, -1, 1], \\ CB: & [2u_C^B, -1, 1], & CD: & [2u_C^D, -1, 1]. \end{array}$$

We have linear gluing data everywhere except on the edges EB and FD. Let us analyze the syzygies associated this data.

- For the edges EB and FD with one crossing vertex, the gluing data is of the form $[2u+u^2,-1,1]$. We have $n=2$ and $m=1$ since the edge is connecting two triangles, $\mu=0$ and $\nu=2$ and $d(k)=2k-2$. The μ-basis is $[0,1,1]$, $[-1,-2u-u^2,0]$. The separability is achieved in degree $k\geqslant 6$ and not 5 as it could be expected ($d(5)\geqslant 8$).

- For the edges EA, EA, FA, FC, EF between triangular faces, with two crossing vertices, the linear gluing data is of the form $[2u,1,-1]$. We have $n=1$ and $m=1$, $\mu=0$, $\nu=1$ and $d(k)=2k-1$. The μ-basis is $[0,1,1]$, $[-1,-2u,0]$. The separability is achieved in degree $k\geqslant 4$.

- For the edges AB, AD, CB, CD between a triangular face and a rectangular face, with one crossing vertices, the linear gluing data is of the form $[2u,1,-1]$. We have $n=1$ and $m=0$, $\mu=1$ since the degree of the homogeneization $[d_a,d_b,d_c]$ (see Definition 4.1) is $[3,2,1]$ or $[3,1,2]$, $\nu=1$ and $d(k)=2k$. The μ-basis is $[0,1,1]$, $[-1,-2u,0]$. The separability is also achieved in degree $k\geqslant 4$.

Now we count how many splines do we have in degree $k\geqslant 6$:

- For the four crossing vertices A,C,E,F we have $1+2+1=4$ dimensions and $1+2+3=6$ dimensions for B and for D. In total we have $4\cdot 4+2\cdot 6=28$ degrees of freedom around the vertices of $\mathcal{M}$.

- For the edges EB and FD, we have $2(k-2)-8=2k-12$ dimensions. For the edges EA, EA, FA, FC, EF, we have $2k-1-7=2k-8$ dimensions. For the edges AB, AD, CB, CD, we have $2k-8$.

- For the 6 triangular faces, we have $\binom{k-4}{2}$ dimensions and for the rectangular face $(k-3)^2$.

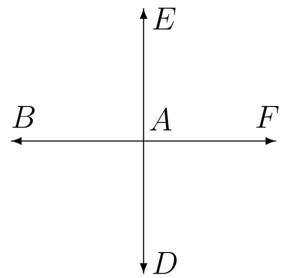
The dimension formula in degree $k\geqslant 6$ is then

$$28-4+11\cdot 2\,(k-4)+6\binom{k-4}{2}+(k-3)^2=(2k-3)^2+k-4.$$

For $k=6$, the dimension is 83. It turns out that this formula also holds for degree $k=4$, $k=5$.

The construction of basis functions can be done as described in Section 6.4. Let us give the basis functions associated to the value and first derivatives at the point A. Here are the Bernstein coefficients in degree 4 of the basis function for the value at A with the vertex A represented in the center and the edges represented by horizontal and vertical central lines (in bold):

$$
\begin{array}{ccccccc}
 & & & \mathbf{0} & & & \\
 & \iddots & 0 & \mathbf{\frac{1}{2}} & 0 & \ddots & \\
 & 0 & 1 & \mathbf{1} & 1 & 0 & \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{\frac{1}{2}} & \mathbf{0} \\
0 & -1 & 1 & \mathbf{1} & 1 & 0 & \\
\vdots & 0 & -1 & \mathbf{0} & 0 & \iddots & \\
 & \cdots & 0 & \mathbf{0} & & &
\end{array}
$$



This gives the following specializations to the polygons ABE, AEF, AFD, $ABCD$ (respectively), selectively de-homogenized:

$$
\begin{aligned}
&(u^A_E)^2\,(1+3u^B_A)(1+2u^E_A-u^B_A),\\
&(u^A_E)^2\,(1+2u^F_A+2u^E_A+6u^F_Au^E_A),\\
&(u^A_F)^2\,(1+3u^D_A)(1+2u^F_A-u^D_A),\\
&(u^A_Bu^A_D)^2\,\big(u^A_Bu^A_D(1+3u^D_A)(1+3u^B_A)-24u^D_Au^B_A(u^D_{AB}u^A_B+u^B_Au^A_D)\big),
\end{aligned}
$$

and 0 on the other faces. The basis function associated to the first derivative in one of the directions at the cross vertices is:

$$
\begin{array}{ccccccc}
 & & & \mathbf{0} & & & \\
 & \iddots & 0 & \mathbf{\frac{1}{6}} & 0 & \ddots & \\
 & 0 & \frac{1}{3} & \mathbf{\frac{1}{4}} & \frac{1}{3} & 0 & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & \frac{7}{24} & -\frac{5}{16} & -\mathbf{\frac{1}{4}} & -\frac{1}{3} & 0 & \\
\vdots & 0 & -\frac{1}{6} & \mathbf{0} & 0 & \iddots & \\
 & \cdots & 0 & \mathbf{0} & & &
\end{array}
\tag{29}
$$

The non-zero specializations to ABE, AEF, AFD, $ABCD$ are, respectively:

$$
\begin{aligned}
&(u^A_E)^2\,u^E_A\,(1+3u^B_A),\ (u^A_E)^2\,u^E_A\,(1+3u^F_A),\ -(u^A_F)^2\,u^D_A\,(u^A_F+4u^F_A),\\
&(u^A_Bu^A_D)^2\,u^D_A\,\big(-u^A_D(1+3u^B_A)+7u^B_Au^A_Bu^D_A\big)\,.
\end{aligned}
$$

The basis function for the derivative in the other direction is obtained by a mirror image of (29).

The basis function corresponding to the cross derivatives is realized by

$$
\begin{array}{ccccccc}
 & & & \mathbf{0} & & & \\
 & \iddots & 0 & \mathbf{0} & 0 & \ddots & \\
 & 0 & -\frac{1}{12} & \mathbf{0} & \frac{1}{12} & 0 & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & \frac{1}{24} & \frac{1}{16} & \mathbf{0} & -\frac{1}{12} & 0 & \\
\vdots & 0 & \frac{1}{24} & \mathbf{0} & 0 & \iddots & \\
 & \cdots & 0 & \mathbf{0} & & &
\end{array}
$$

In this spline, we could modify the 0 entry next to two $\frac{1}{24}$ entries to $\frac{1}{36}$, so to lower the degree of the specialization to the rectangle. After the modification, the 4 non-zero specializations would be

$$-(u_E^A)^2\, u_A^E\, u_A^B,\ (u_E^A)^2\, u_A^E\, u_A^F,\ -(u_F^A)^2\, u_A^D\, u_A^F,\ (u_B^A u_D^A)^2\, u_A^D\, u_A^B.$$

The local splines around C look the same. The local splines around other vertices involve the edges EB and FD, and we would need degree 6 splines.

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Appendix A. Algorithms for the basis construction

Our input data is the topological surface $\mathcal{M}$ and the gluing data. For each edge τ of $\mathcal{M}$, we are given the μ_τ-basis of Z_k:

$$S_1^\tau = [A_1^\tau, B_1^\tau, C_1^\tau], \quad S_2^\tau = [A_2^\tau, B_2^\tau, C_2^\tau].$$

The rational map is then described by

$$\mathfrak{a}_\tau = \frac{a_\tau}{c_\tau}, \mathfrak{b}_\tau = \frac{b_\tau}{c_\tau}$$

with $a_\tau = B_1^\tau C_2^\tau - B_2^\tau C_1^\tau$, $b_\tau = A_2^\tau C_1^\tau - A_1^\tau C_2^\tau$, and $c_\tau = A_1^\tau B_2^\tau - A_2^\tau B_1^\tau$.

The spline basis functions $f = (f_\sigma)$ are represented on each face σ by their coefficients in the Bernstein basis of the face in degree k:

$$f_i = \sum_{l,m} c_{l,m}^i b_{l,m}^\sigma(u_i, v_i)$$

Let $e_\sigma(k) = k^2$ if σ is a rectangular face and $e_\sigma(k) = k\,(k-1)$ if σ is a triangular face.

Appendix A.1. Vertex basis functions

Let γ be a vertex of $\mathcal{M}$ shared by the faces $\sigma_1, \ldots, \sigma_F$ and such that σ_i and σ_{i+1} share the edge τ_{i+1}. We compute the Bernstein coefficients $\mathbf{c}^i = [c_{0,0}^i, c_{1,0}^i, c_{0,1}^i, c_{1,1}^i, \ldots]$ of the basis functions attached to a vertex σ_i, using the equations (26), (27) and the relation between the Bernstein coefficients and the Taylor coefficients of the function at $(0,0)$, see (16), (17).

If $c_{0,0}^i$ corresponds to the point γ, with coordinates $(0,0)$ in the face σ_i and $f_{\sigma_i} = p + q_i u_i + q_{i+1} v_i + s_i u_i v_i + \cdots$ are $\mathbf{c}^i$, we use the relations $p = c_{0,0}^i, q_i = k\, c_{1,0}^i,\ q_{i+1} = k\, c_{0,1}^i,\ s_i = e_k\,(c_{1,1}^i - c_{1,0}^i - c_{0,1}^i)$.

Basis function for the value at vertex γ

for *i in [1,F]* **do**
  let $c^i_{0,0} := 1, c^i_{1,0} := 1, c^i_{0,1} := 1, c^i_{0,1} := 1$ and $c^i_{l,m} := 0$ for $(l,m) \notin \{(0,0),(1,0),(0,1),(1,1)\}$;
end

Basis functions for the derivatives at vertex γ

for $[c^1_{0,0}, c^1_{0,1}]$ *in* $\{[1,0],[0,1]\}$ **do**
  for *i in [2,F]* **do**

$$\begin{bmatrix} c^i_{1,0} \\ c^i_{0,1} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ \mathfrak{b}_{\tau_i}(0) & \mathfrak{a}_{\tau_i}(0) \end{pmatrix} \begin{bmatrix} c^{i-1}_{1,0} \\ c^{i-1}_{0,1} \end{bmatrix}$$

  end
  if *all edges* τ_i *are crossing edges at* γ **then**
    let $c^1_{1,1} := 0$;
  end
  for *i in [2,F]* **do**
    if τ_i *is a crossing edge at* γ **then**

$$\begin{aligned} c^i_{1,1} \quad = \quad & c^i_{1,0} + c^i_{0,1} \\ & + \tfrac{1}{e_{\sigma_i}(k)} \big(e_{\sigma_{i-1}}(k) \mathfrak{b}_{\tau_i}(0) \, (c^{i-1}_{1,1} - c^{i-1}_{1,0} - c^{i-1}_{0,1}) \\ & + k \, \mathfrak{a}'_{\tau_i}(0) \, c^{i-1}_{0,1} + k \, \mathfrak{b}'_{\tau_i}(0) \, c^{i-1}_{1,0} \big) \end{aligned}$$

    else
      $c^i_{1,1} := 0$;
    end
  end
  for *i in [1,F']* **do**
    lift$(\mathbf{c}_{i-1}, \mathbf{c}_i, \tau_i)$
  end
end

Basis functions for the cross derivatives around γ

for *i in [1,F]* **do**
  let $c^i_{0,0} := 0, c^i_{1,0} := 0, c^i_{0,1} := 0$;
end
if *all edges* τ_i *are crossing edges at* γ **then**
  let $c^1_{1,1} := 1$;
  for *i in [2,F]* **do**
    $c^i_{1,1} = \frac{e_{\sigma_{i-1}}}{e_{\sigma_i}(k)} \mathfrak{b}_{\tau_i}(0)\, c^{i-1}_{1,1}$;
  end
  for *i in [1,L]* **do**
    lift($\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i$);
  end
else
  for *j in [1,F] such that* τ_j *is not a crossing edge at* γ **do**
    let $c^j_{1,1} = 1$ and $c^l_{1,1} = 0$ for $l \neq j$;
    for *i in [1,F']* **do**
      if τ_i *is a crossing edge* **then**
        $c^i_{1,1} = \frac{e_{\sigma_{i-1}}}{e_{\sigma_i}(k)} \mathfrak{b}_{\tau_i}(0)\, c^{i-1}_{1,1}$
      end
    end
    for *i in [1,L]* **do**
      lift($\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i$);
    end
  end
end

The function lift($\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i$) used in the algorithm consists in computing the coefficient of a spline function with support along the edge τ_i, from its first Taylor coefficients on the faces σ_{i-1}, σ_i.

LIFT$(\mathbf{c}_i, \mathbf{c}_{i-1},\ \tau_i)$

for *i in [1,F]* **do**

solve the systems:

$$\begin{bmatrix} k\, c^i_{1,0} \\ k\, c^{i-1}_{1,0} \end{bmatrix} = \begin{bmatrix} A_1(0) & A_2(0) \\ B_1(0) & B_2(0) \end{bmatrix} \begin{bmatrix} p^i_0 \\ q^i_0 \end{bmatrix}$$

and

$$\begin{bmatrix} e_{\sigma_{i-1}}(k)\,(c^{i-1}_{1,1} - c^{i-1}_{1,0} - c^{i-1}_{0,1}) \\ -e_{\sigma_i}(k)\,(c^i_{1,1} - c^i_{1,0} - c^i_{0,1}) \end{bmatrix} - \begin{bmatrix} B'_1(0) & B'_2(0) \\ C'_1(0) & C'_2(0) \end{bmatrix} \begin{bmatrix} p^i_0 \\ q^i_0 \end{bmatrix}$$
$$= \begin{bmatrix} B_1(0) & B_2(0) \\ C_1(0) & C_2(0) \end{bmatrix} \begin{bmatrix} p^i_1 \\ q^i_1 \end{bmatrix}.$$

compute $P^i := p^i_0(1 - 3u_i^2 + 2u_i^3) + p^i_1(u - 2u_i^2 + u_i^2)$,
$Q^i := q^i_0(1 - 3u_i^2 + 2u_i^3) + q^i_1(u - 2u_i^2 + u_i^2)$;

compute the image $(g_i, \tilde{g}_i)$ of $P^i S^i_1 + Q^i S^i_2$ by Θ_{τ_i} and update the coefficients of $\mathbf{c}_{i-1}, \mathbf{c}_i$;

end

As $A_1(0)B_2(0) - A_2(0)B_1(0) = c(0) \neq 0$, the first system has a unique solution. When $B_1(0)C_2(0) - B_2(0)C_1(0) = a(0) \neq 0$ (i.e. when τ_i is not a crossing edge at γ), the second system has a unique solution. When $a(0) = 0$ (i.e. when τ_i is a crossing edge at γ), the second system is degenerate, but it still has a (least square) solution.

The polynomials P^i (resp. Q^i) are constructed so that $P^i(0) = p^i_0, P^{i'}(0) = p^i_1,\ P^i(1) = 0, P^{i'}(1) = 0$ (resp. $Q^i(0) = q^i_0, Q^{i'}(0) = q^i_1,\ Q^i(1) = 0, Q^{i'}(1) = 0$).

By construction, the Taylor expansions of their image by Θ_{τ_i} vanish at γ' and coincide with $[c^{i-1}_{0,0}, c^{i-1}_{1,0}, c^{i-1}_{0,1}, e_{\sigma_{i-1}}(k)(c^{i-1}_{1,1} - c^{i-1}_{1,0} - c^{i-1}_{0,1})]$, $[c^i_{0,0}, c^i_{1,0}, c^i_{0,1}, e_{\sigma_i}(k)(c^i_{1,1} - c^i_{1,0} - c^i_{0,1})]$ at γ respectively on σ_{i-1} and σ_i.

Appendix A.2. Edge basis functions

Basis functions for the edge τ

Input: $[A_1^\tau, B_1^\tau, C_1^\tau]$, $[A_2^\tau, B_2^\tau, C_2^\tau]$ the μ-basis of the syzygy module $Z(\tau)$;

if $\mathfrak{c}_\tau(\gamma) = 1$ **then**
| compute the image by Θ_τ of
| $u(1-u)^2 \left(C_2^\tau(0)\,[A_1^\tau, B_1^\tau, C_1^\tau] - C_1^\tau(0)\,[A_2^\tau, B_2^\tau, C_2^\tau]\right)$
end
if $\mathfrak{c}_\tau(\gamma') = 1$ **then**
| compute the image by Θ_τ of
| $u^2(1-u) \left(C_2^\tau(1)\,[A_1^\tau, B_1^\tau, C_1^\tau] - C_1^\tau(1)\,[A_2^\tau, B_2^\tau, C_2^\tau]\right)$
end
Let $\Delta = u^2(1-u)^2$;
for *i in [0,*$k - \mu - m - 4$*]* **do**
| compute the image by Θ_τ of $u^i\Delta\,[A_1^\tau, B_1^\tau, C_1^\tau]$.
end
for *i in [0,*$k - \nu - m - 4$*]* **do**
| compute the image by Θ_τ of $u^i\Delta\,[A_2^\tau, B_2^\tau, C_2^\tau]$.
end

Appendix A.3. Face basis functions

Basis functions for the face σ

for $2 \leqslant i \leqslant k-2$, $2 \leqslant j \leqslant k-2$ *(and* $i + j \leqslant k - 2$ *if* σ *is a triangle)* **do**
| let $c_{i,j} := 1$ and $c_{i',j'} = 0$ for $i' \neq i$ or $j' \neq j$.
end