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# MODERATE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL UNDER MINIMAL RESTRICTIONS ON THE POTENTIAL

MOSHE MARCUS AND VITALY MOROZ

ABSTRACT. We study semilinear elliptic equations with Hardy potential  
(E) 
$$-\mathcal{L}_\mu u + u^q = 0$$

in a bounded smooth domain  $\Omega \subset \mathbb{R}^N$ . Here  $q > 1$ ,  $\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$  and  $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$ . Assuming that  $0 \leq \mu < C_H(\Omega)$ , boundary value problems with measure data and discrete boundary singularities for positive solutions of (E) have been studied in [10]. In the case of *convex* domains  $C_H(\Omega) = 1/4$ . In this case similar problems have been studied in [8]. In the present paper we study these problems, in *arbitrary domains*, assuming only  $-\infty < \mu < 1/4$ , even if  $C_H(\Omega) < 1/4$ . We recall that  $C_H(\Omega) \leq 1/4$  and, in general, strict inequality holds. The key to our study is the fact that, if  $\mu < 1/4$  then in smooth domains there exist local  $\mathcal{L}_\mu$ -superharmonic functions in a neighborhood of  $\partial\Omega$  (even if  $C_H(\Omega) < 1/4$ ). Using this fact we extend the notion of *normalized boundary trace* introduced in [10], to arbitrary domains, provided that  $\mu < 1/4$ . Further we study the b.v.p. with normalized boundary trace  $\nu$  in the space of positive finite measures on  $\partial\Omega$ . We show that existence depends on two critical values of the exponent  $q$  and discuss the question of uniqueness. Part of the paper is devoted to the study of the linear operator: properties of local  $\mathcal{L}_\mu$ -subharmonic and superharmonic functions and the related notion of moderate solutions. Here we extend and/or improve results of [5] and [10] which are later used in the study of the nonlinear problem.

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Introduction.** On bounded smooth domains  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) we study semilinear elliptic equations with Hardy potential of the form,

$$(P_\mu) \quad -\Delta u - \frac{\mu}{\delta_\Omega^2} u + |u|^{q-1} u = 0 \quad \text{in } \Omega,$$

where  $q > 1$ ,  $-\infty < \mu < 1/4$  and

$$\delta_\Omega(x) := \text{dist}(x, \partial\Omega).$$

Equations  $(P_0)$  had been extensively studied in the past two decades and by now the structure of the set of positive solution of such equations is well understood, see [11] and further references therein. Equation  $(P_\mu)$  with Hardy potential, i.e. with  $\mu \neq 0$ , had been first considered in [5], where a

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classification of positive solutions had been introduced and conditions for the existence and nonexistence of *large* solutions for  $(P_\mu)$  had been derived.

The study and classification of positive solutions of equation  $(P_\mu)$  relies on the properties of the associated linear equation

$$(1.1) \quad -\mathcal{L}_\mu h = 0 \quad \text{in } \Omega,$$

where

$$\mathcal{L}_\mu := \Delta + \frac{\mu}{\delta_\Omega^2}.$$

Denote

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}$$

and note that  $\alpha_+ + \alpha_- = 1$ . For  $\rho > 0$  and  $\varepsilon \in (0, \rho)$  we use the notation

$$\begin{aligned} \Omega_\rho &:= \{x \in \Omega : \delta(x) < \rho\}, & \Omega_{\varepsilon, \rho} &:= \{x \in \Omega : \varepsilon < \delta(x) < \rho\} \\ D_\rho &:= \{x \in \Omega : \delta(x) > \rho\}, & \Sigma_\rho &:= \{x \in \Omega : \delta(x) = \rho\}. \end{aligned}$$

A function  $w \in L^1_{\text{loc}}(G)$  is a  $\mathcal{L}_\mu$ -subharmonic in  $\Omega$  if  $\mathcal{L}_\mu w \leq 0$  in the distribution sense, i.e.,

$$\int_G w(-\Delta\varphi) dx - \int_G \frac{\mu}{\delta_\Omega^2} w\varphi dx \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega).$$

We say that  $w$  is a *local*  $\mathcal{L}_\mu$ -subharmonic function if there exists  $\rho > 0$  such that  $w \in L^1_{\text{loc}}(\Omega_\rho)$  is subharmonic in  $\Omega_\rho$ . Similarly, (local)  $\mathcal{L}_\mu$ -superharmonic functions are defined with “ $\geq$ ” in the above inequality.

**1.2. The role of the Hardy constant.** The existence and properties of positive  $\mathcal{L}_\mu$ -harmonic and superharmonic functions in  $\Omega$  are controlled by the Hardy constant of the domain, defined as

$$(1.2) \quad C_H(\Omega) := \inf_{C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{\delta_\Omega^2} dx}.$$

For a bounded Lipschitz domain it is known that  $C_H(\Omega) \in (0, 1/4]$ . If  $\Omega$  is convex then  $C_H(\Omega) = 1/4$ . In general,  $C_H(\Omega)$  varies with the domain and could be arbitrary small (see, e.g. [9, Theorem I and Section 4]) for a discussion and examples).

Denote the *local* Hardy constant in  $\Omega_\rho$  relative to  $\partial\Omega$  by

$$(1.3) \quad C_H^{\partial\Omega}(\Omega_\rho) := \inf_{C_c^\infty(\Omega_\rho) \setminus \{0\}} \frac{\int_{\Omega_\rho} |\nabla u|^2 dx}{\int_{\Omega_\rho} \frac{u^2}{\delta_\Omega^2} dx}.$$

Note the difference between  $C_H^{\partial\Omega}(\Omega_\rho)$  and  $C_H(\Omega_\rho)$ : the distance involved in the first one is  $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$  while in the second it is  $\delta_{\Omega_\rho}(x) = \text{dist}(x, \partial\Omega_\rho)$ . Obviously  $C_H^{\partial\Omega}(\Omega_\rho) \geq C_H(\Omega_\rho)$ .

The following lemma shows that in contrast to the “global” Hardy constant  $C_H(\Omega)$  the value of the “local” Hardy constant  $C_H^{\partial\Omega}(\Omega_\rho)$  does not depend on the shape of  $\Omega$ , provided that  $\rho$  is sufficiently small.

**Lemma 1.1.** (LOCAL HARDY INEQUALITY) *There exists  $\bar{\rho} = \bar{\rho}(\Omega) > 0$  such that for every  $\rho \in (0, \bar{\rho}]$  one has  $C_H^{\partial\Omega}(\Omega_\rho) = C_H(\Omega_\rho) = 1/4$ .*

The fact that  $C_H^{\partial\Omega}(\Omega_\rho) = 1/4$  is due to [9, p.3246], while  $C_H(\Omega_\rho) = 1/4$  follows from [6, Lemma 1.2].

The relation between Hardy constant and the existence of positive  $\mathcal{L}_\mu$ -superharmonics is explained by the following classical result, cf. [9, p.3246].

**Lemma 1.2.** *Equation (1.1) admits a positive  $\mathcal{L}_\mu$ -superharmonic function in  $\Omega$  if and only if  $\mu \leq C_H(\Omega)$ .*

*Equation (1.1) admits a positive  $\mathcal{L}_\mu$ -superharmonic in  $\Omega_\rho$  with  $\rho \in (0, \bar{\rho})$  if and only if  $\mu \leq 1/4$ .*

Thus, according to Lemma 1.1, if  $C_H(\Omega) < 1/4$  then, for  $\mu \in [C_H(\Omega), 1/4)$ , there exist local positive  $\mathcal{L}_\mu$ -superharmonic functions but no “global” positive  $\mathcal{L}_\mu$ -superharmonic functions in  $\Omega$ .

**1.3. Moderate solutions and normalised boundary trace.** In this work we study *moderate* positive solutions of nonlinear equation  $(P_\mu)$  in the range  $\mu < 1/4$ , including negative values of  $\mu$ . Recall that in the classical theory of equations  $(P_\mu)$  with  $\mu = 0$ , moderate solution is a solution which is dominated by a positive harmonic function, cf. [11, pp.66-69]. This concept had been extended to equations  $(P_\mu)$  with  $0 \leq \mu < C_H(\Omega)$  in [10], where  $\mathcal{L}_\mu$ -moderate solution is defined as a solution dominated by a positive  $\mathcal{L}_\mu$ -harmonic function. This definition is not applicable in the range  $\mu \in [C_H(\Omega), 1/4)$ , when the set of positive  $\mathcal{L}_\mu$ -harmonic function is empty. Therefore we modify it as follows:

**Definition 1.3.** A solution  $u \in L_{loc}^1(\Omega)$  of equation  $(P_\mu)$  is  $\mathcal{L}_\mu$ -moderate if there exists a local positive  $\mathcal{L}_\mu$ -harmonic function  $h$  such that  $|u| \leq h$  in  $\Omega_\rho$  for some  $\rho \in (0, \bar{\rho}]$ .

We are going to show that equation  $(P_\mu)$  admits  $\mathcal{L}_\mu$ -moderate solutions, with prescribed (normalized) boundary data, in the *entire* domain  $\Omega$  for every  $\mu < 1/4$ , even when  $C_H(\Omega) < 1/4$ . The *existence* of a certain class of positive solutions was observed in [5, Lemma 4.15].

More specifically, we study the generalised boundary trace problem

$$(P_\mu^\nu) \quad \begin{cases} -\mathcal{L}_\mu u + |u|^{q-1}u = 0 & \text{in } \Omega, \\ \text{tr}_{\partial\Omega}^*(u) = \nu, \end{cases}$$

where  $\mu < 1/4$ ,  $q > 1$ ,  $\nu \in \mathcal{M}^+(\partial\Omega)$  and  $\text{tr}_{\partial\Omega}^*(u)$  denotes the *normalised boundary trace* of a positive Borel function  $u$  on  $\partial\Omega$ . A function  $u \in L_{loc}^q(\Omega)$  is a solution of  $(P_\mu^\nu)$  if it satisfies the equation in the distribution sense and attains the indicated boundary data.

The concept of normalised boundary trace was introduced in [10] in order to classify positive moderate solutions of  $(P_\mu^\nu)$  in terms of their behaviour at the boundary, when  $0 < \mu < C_H(\Omega)$ .<sup>1</sup> It is defined as follows.

A nonnegative Borel function  $u : \Omega \rightarrow \mathbb{R}$  possesses a *normalised boundary trace*  $\nu \in \mathfrak{M}^+(\partial\Omega)$  if,

<sup>1</sup>Actually, the assumption  $\mu > 0$  was introduced in [10] only for simplicity: the normalised boundary trace is well-defined and the related results remain valid for any  $\mu < C_H(\Omega)$ .

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^\Omega[\nu]| dS = 0$$

where  $K_\mu^\Omega$  is the Martin kernel of  $\mathcal{L}_\mu$  in  $\Omega$ . If, for a given  $u$  there exists a measure  $\nu$  as above then it is unique.

By Ancona [2], if  $\mu < C_H(\Omega)$  there is a (1-1) correspondence between the set of positive  $\mathcal{L}_\mu$ -harmonic functions in  $\Omega$  and  $\mathfrak{M}^+(\partial\Omega)$ ; the  $\mathcal{L}_\mu$ -harmonic function  $v$  corresponding to a measure  $\nu$  has the representation  $v = K_\mu^\Omega[\nu]$ . (For details and notation see Subsection 2.1 below.)

We point out that, except in the case  $\mu = 0$ ,  $\text{tr}_{\partial\Omega}^*(u)$  is *not* the standard measure boundary trace of  $u$ . In fact, when  $\mu > 0$ , the measure boundary trace of any  $\mathcal{L}_\mu$ -harmonic function is zero.

In order to extend the definition of normalised boundary trace to arbitrary  $\mu < 1/4$  we pick  $\rho \in (0, \bar{\rho}]$  (with  $\bar{\rho}$  as in Lemma 1.1) and employ (1.4) with  $K_\mu^{\Omega_\rho}$  instead of  $K_\mu^\Omega$ . Since  $C_H(\Omega_\rho) = 1/4$ ,  $K_\mu^{\Omega_\rho}$  is well defined for every  $\mu < 1/4$ .

We show that if, for some  $\rho$  as above, there exists  $\nu \in \mathfrak{M}_+(\partial\Omega)$  such that

$$(1.5) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^{\Omega_\rho}[\nu]| dS = 0$$

then (1.5) holds for every  $\rho \in (0, \bar{\rho}]$  and the measure  $\nu$  is independent of  $\rho$ .

In addition we show that a positive solution of equation  $(P_\mu)$  possesses a normalised boundary trace if and only if it is a moderate solution.

**1.4. Main results.** We start with a few results about the linear operator.

**Theorem 1.4.** *Let  $\mu < 1/4$ . Suppose that  $u$  is positive and  $\mathcal{L}_\mu$ -subharmonic in  $\Omega_{\bar{\rho}}$ . Then  $u$  has a normalized boundary trace on  $\partial\Omega$  if and only if  $u$  is dominated in  $\Omega_\rho$  (for some  $\rho \in (0, \bar{\rho})$ ) by an  $\mathcal{L}_\mu$ -harmonic function.*

**Theorem 1.5.** *Let  $\mu < 1/4$ . Suppose that  $u$  is a non-negative,  $\mathcal{L}_\mu$ -subharmonic function in  $\Omega_{\bar{\rho}}$ . In addition assume that, for some  $\rho \in (0, \bar{\rho})$   $u$  is dominated in  $\Omega_\rho$  by an  $\mathcal{L}_\mu$ -harmonic function. Then, either*

(i)  $\text{tr}_{\partial\Omega}^* u = 0$ , in which case, for every  $\beta \in (0, \rho)$  there exists a constant  $c_\beta > 0$  such that

$$(1.6) \quad u(x) \leq c_\beta \delta(x)^{\alpha+} \quad \text{in } \Omega_\beta$$

or

(ii)  $\text{tr}_{\partial\Omega}^* u > 0$ , in which case, for every  $\beta$  as above,

$$(1.7) \quad \frac{1}{c_\beta} \beta(x)^{\alpha-} \leq \int_{\Sigma_\beta} u dS \leq c_\beta \beta^{\alpha-} \quad \text{in } \Omega_\beta.$$

**Theorem 1.6.** *Let  $\mu < 1/4$ . Suppose that  $u$  is positive and  $\mathcal{L}_\mu$ -superharmonic in  $\Omega_{\bar{\rho}}$ . If  $\text{tr}_{\partial\Omega}^* u \neq 0$  then (1.7) holds.*

**Corollary 1.7.** *Suppose that  $u$  is non-negative and  $\mathcal{L}_\mu$ -subharmonic in  $\Omega_{\bar{\rho}}$ . Then either (1.6) holds or*

$$(1.8) \quad 0 < \limsup_{\beta \rightarrow 0} \frac{1}{\beta^{\alpha-}} \int_{\Sigma_\beta} u dS.$$

*Remark 1.8.* The corollary is an improved version of [5, Thm. 2.9]. Since we do not assume that  $u$  is dominated by an  $\mathcal{L}_\mu$ -harmonic function the alternative to (1.6) is not necessarily (1.7) but only (1.8) which is nothing more than the negation of the statement  $\text{tr}_{\partial\Omega}^* u = 0$ .

Clearly every positive subsolution of the nonlinear equation  $(P_\mu)$  is  $\mathcal{L}_\mu$ -subharmonic so that the above results apply to it.

We turn to the nonlinear problem.

**Theorem 1.9.** *Let  $\mu < 1/4$  and  $\nu \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$ . Assume that  $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$  for some  $\rho \in (0, \bar{\rho}]$ . Then the boundary value problem  $(P_\mu^\nu)$  admits a positive solution  $u$ .*

We emphasise that if  $C_H(\Omega) < 1/4$  then for  $\mu \in [C_H(\Omega), 1/4)$  an  $\mathcal{L}_\mu$ -harmonic extension of  $\nu$  exists only locally in a strip  $\Omega_\rho$ . Nevertheless, problem  $(P_\mu^\nu)$  has a positive solution in  $\Omega$ , for any  $\mu < 1/4$ .

When  $\mu < C_H(\Omega)$  problem  $(P_\mu^\nu)$  admits at most one solution for every  $\nu \in \mathfrak{M}_+(\partial\Omega)$  [10]. However, if  $C_H(\Omega) < \mu < 1/4$  then uniqueness fails. Indeed, it was proved in [5, Theorem 5.3] that in the latter case there exists a positive solution of  $(P_\mu^\nu)$  with  $\nu = 0$ . An alternative, more direct proof, of this result is presented in Appendix A.

**Theorem 1.10.** *Let  $u$  be a positive solution of  $(P_\mu)$ . Then,*

(i)  *$u$  has a normalized boundary trace if and only if  $u \in L^q(\Omega; \delta^{\alpha_+})$ .*

(ii) *If  $u$  has normalized boundary trace  $\nu$  then*

$$(1.9) \quad \lim_{x \rightarrow y} \frac{u(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \text{non-tangentially, for } \nu\text{-a.e. } y \in \partial\Omega.$$

**Theorem 1.11.** *Let  $\nu \in \mathfrak{M}_+(\partial\Omega)$ . If  $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega; \delta^{\alpha_+})$  then  $(P_\mu^\nu)$  has a solution.*

In general, the existence of a solution of  $(P_\mu^\nu)$  does not imply that  $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega; \delta^{\alpha_+})$ . In fact, for any  $\mu > 0$  and  $q > 1$ , one can construct functions  $f \in L^1(\partial\Omega)$  such that  $\mathbb{K}_\mu^{\Omega_\rho}[f] \notin L^q(\Omega; \delta^{\alpha_+})$  while  $(P_\mu^\nu)$  has a solution whenever  $\nu = f \in L^1(\partial\Omega)$ .

Let

$$(1.10) \quad q_{\mu,c} := \frac{N + \alpha_+}{N - 1 - \alpha_-} \quad \forall \mu < 1/4.$$

The next result has been obtained in [10, Theorems E and F] for  $\mu \in (0, C_H(\Omega))$ . A similar result is presented in [8, Theorems D and E], under the assumption that  $\Omega$  is a convex domain, in which case it is known that  $C_H(\Omega) = 1/4$ .

**Proposition 1.12.** *Let  $\mu < 1/4$ . If  $1 < q < q_{\mu,c}$  then the boundary value problem  $(P_\mu^\nu)$  has a solution for every Borel measure  $\nu \in \mathfrak{M}^+(\partial\Omega)$ . Moreover, if  $q \geq q_{\mu,c}$  then problem  $(P_\mu^\nu)$  has no solution when  $\nu$  is the Dirac measure.*

In the next proposition, the existence statement is a consequence of Theorem 1.11. The non-existence part is more subtle.

**Proposition 1.13.** (i) For every  $\mu < 1/4$  put

$$q_\mu^* = \begin{cases} \infty & \text{if } \mu \geq 0 \\ 1 - \frac{2}{\alpha_-} & \text{if } \mu < 0. \end{cases}$$

If  $1 < q < q_\mu^*$  then problem  $(P_\mu^\nu)$  has a solution for every measure  $\nu = fdS$ ,  $f \in L^1(\partial\Omega)$ .

(ii) If  $q \geq q_\mu^*$  then problem  $(P_\mu^\nu)$  has no solution for any  $\nu \in \mathfrak{M}_+(\partial\Omega) \setminus \{0\}$ .

*Remark 1.14.* If  $\mu < 0$  then  $\alpha_- < 0$  so that  $q_\mu^* > 1$  and  $q_{\mu,c} < q_\mu^*$ .

The paper is organised as follows. In Section 2 we study the linear problem. We derive estimates of the Green and Martin kernels of  $\mathcal{L}_\mu$  in  $\Omega_\rho$  and discuss the boundary behavior of local positive  $\mathcal{L}_\mu$ -sub and superharmonic functions in terms of the normalized trace.

In Section 3 these results are applied to the study of the nonlinear boundary value problem  $(P_\mu^\nu)$ .

## 2. LINEAR EQUATION AND NORMALISED BOUNDARY TRACE

**2.1. The local behavior of Green and Martin kernels.** We recall some results concerning Schrödinger equations, that are needed in what follows. The results are due to Ancona [2]. Let  $D$  be a bounded Lipschitz domain and consider the Schrödinger operator  $\mathcal{L}^V = \Delta + V$  where  $V \in C(D)$  is a potential such that, for some constant  $a > 0$ ,  $|V(x)| \leq a \operatorname{dist}(x, \partial D)^{-2}$  and  $\mathcal{L}^V$  possesses a positive supersolution. (If  $V \leq 0$  there is always a supersolution namely,  $u = 1$ .) Then  $\mathcal{L}^V$  has a Green function  $G^V$  and Martin kernel  $K^V$  in  $D$ . The Martin boundary coincides with  $\partial D$  and the following holds,

**Theorem 2.1** (Representation Theorem). For every  $\nu \in \mathfrak{M}^+(\partial D)$  the function

$$\mathbb{K}^V[\nu](x) := \int_{\partial D} K^V(x, y) d\nu(y), \quad x \in D,$$

is  $\mathcal{L}^V$ -harmonic in  $D$ . Conversely, if  $u$  is a positive  $\mathcal{L}^V$ -harmonic function in  $D$  then there exists a unique measure  $\nu \in \mathfrak{M}^+(\partial D)$  such that  $u = \mathbb{K}^V[\nu]$ .

In order to state the boundary Harnack principle we need additional notation. Let  $y \in \partial D$  and let  $\xi = \xi^y$  be a local set of coordinates centered at  $y$  such that the  $\xi_1$ -axis is in the direction of an interior pseudo normal  $\mathbf{n}_y$ . (If  $D$  is a  $C^1$  domain we may take  $\mathbf{n}_y$  to be the interior unit normal.) Denote

$$T_y(r, \rho) = \{\xi = (\xi_1, \xi') : |\xi_1| < \rho, \quad |\xi'| < r\}.$$

Assume that  $r$  and  $\rho$  are so chosen that

$$\omega_y := T_y(r, \rho) \cap D = \{\xi : F_y(\xi') < \xi_1 < \rho, \quad |\xi'| < r\}$$

where  $F_y$  is a Lipschitz function in  $\mathbb{R}^{N-1}$ , with Lipschitz constant  $\Lambda$ ,  $F_y(0) = 0$  and  $12\Lambda < \rho/r$ . Since  $D$  is a bounded Lipschitz domain  $\Lambda, r, \rho$  can be chosen independently of  $y \in \partial D$ .

Let  $A \in T(r, \rho)$  be the point such that  $\xi(A) = (\rho/2, 0)$ . Then the boundary Harnack principle reads as follows: If  $u, v$  are positive  $\mathcal{L}_\mu$ -harmonic functions in  $\omega_y$  vanishing continuously on  $\partial\Omega \cap T_y(r, \rho)$  then

$$(2.1) \quad C^{-1} \frac{u(A)}{v(A)} \leq \frac{u(\xi)}{v(\xi)} \leq C \frac{u(A)}{v(A)} \quad \forall \xi \in T_y(r/2, \rho/2) \cap D,$$

where the constant  $C$  depends only on  $N, M, \rho/r$  and the Lipschitz constant of  $F_y$ , say  $\Lambda$ . ( $\Lambda$  may be taken to be independent of  $y \in \partial D$ .)

We also need the following consequence of the boundary Harnack principle (c.f. Ancona [1, Lemma 3.5]): there exist positive numbers  $c, t_0$  such that

$$(2.2) \quad c^{-1}|x - y|^{2-N} \leq K^V(x, y)G^V(x, x_0) \leq c|x - y|^{2-N}$$

for every  $y \in \partial\Omega'$  and  $x$  on the interior pseudo normal at  $y$  such that  $|x - y| \leq t_0$ .

Recall that if  $V(x) = \mu \text{dist}(x, \partial D)^{-2}$  and  $\mu < C_H(D)$  then  $\mathcal{L}^V$  has a positive supersolution. In particular, if  $D = \Omega_{\bar{\rho}}$  then  $C_H(D) = 1/4$ . Therefore, in this case, the above results apply to the operator  $\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$  for every  $\mu < 1/4$ .

*Notation.* Let  $D$  be a subdomain of  $\Omega$  and denote

$$\mathcal{L}_{\mu, D} = \Delta + \frac{\mu}{\delta_D^2} \quad \text{where} \quad \delta_D(x) = \text{dist}(x, \partial D).$$

Assume that  $\mu < C_H(D)$  and let  $D'$  be a subdomain of  $D$ . Obviously  $C_H(D') \geq C_H(D)$ . Denote the Green kernel (resp. the Martin kernel) of  $\mathcal{L}_\mu$  in  $D$  by  $G_\mu^D$  (resp.  $K_\mu^D$ ). Denote the Green kernel (resp. the Martin kernel) of  $\mathcal{L}_{\mu, D}$  in  $D'$  by  $G_{\mu, D}^{D'}$  (resp.  $K_{\mu, D}^{D'}$ ).

**Lemma 2.2.** *Assume that  $\mu < 1/4$ . Let  $\bar{\rho}$  be as in Lemma 1.1 and  $t \in (0, \bar{\rho})$ . Put  $U = \Omega_{\bar{\rho}} = [\delta(x) < \bar{\rho}]$ ,  $\Omega_t = [\delta(x) < t]$ ,  $U_t = [\bar{\rho} > \delta(x) > t]$ . Then,*

$$(2.3) \quad \begin{aligned} G_\mu^{\Omega_{t/2}}(x, y) &\leq \\ C(t) \inf(|x - y|^{2-N}, \delta(x)^{\alpha+} \delta(y)^{\alpha+} |x - y|^{2\alpha-N}) &\quad \forall x, y \in \Omega_{t/2} \end{aligned}$$

*Proof.* Note that  $\mathcal{L}_\mu = \mathcal{L}_{\mu, U}$  in  $\Omega_{t/2}$ . Hence

$$G_\mu^{\Omega_{t/2}} = G_{\mu, U}^{\Omega_{t/2}}.$$

It is well-known that the Green function is monotone with respect to the domain. Therefore  $G_{\mu, U}^{\Omega_{t/2}} < G_{\mu, U}^{\Omega_t}$  which implies

$$(2.4) \quad G_\mu^{\Omega_{t/2}}(x, y) \leq c G_{\mu, U}^{\Omega_t}(x, y) \quad \forall x, y \in \Omega_{t/2}.$$

By (2.4) and the estimate of the Green function of  $\mathcal{L}_{\mu, U}$  (see [7] and [10, (2.6)]),

$$(2.5) \quad \begin{aligned} G_\mu^{\Omega_{t/2}}(x, y) &\leq c G_{\mu, U}^{\Omega_t}(x, y) \leq c G_{\mu, U}^U(x, y) \\ &\sim \inf(|x - y|^{2-N}, \delta(x)^{\alpha+} \delta(y)^{\alpha+} |x - y|^{2\alpha-N}) \end{aligned}$$

for every  $x, y \in \Omega_{t/2}$ . This implies (2.3).  $\square$



**Theorem 2.3.** *Assume that  $\mu < 1/4$ , let  $\bar{\rho}$  be as in Lemma 1.1 and let  $t \in (0, \bar{\rho}/2)$ . Using the notations of the previous lemma, pick  $x_t \in U_t$  and  $x'_t \in \Omega_t$  such that  $\delta(x_t) = (t + \bar{\rho})/2$  and  $\delta(x'_t) = t/2$ . As usual  $G_0^U$  denotes the Green function for  $-\Delta$  in  $U$ . A similar notation is employed for the corresponding Martin kernels. Then,*

$$(2.6) \quad \begin{aligned} c_1(t)^{-1} G_{\mu,U}^U(x, x_t) &\leq G_\mu^U(x, x_t) \leq c_1(t) G_{\mu,U}^U(x, x_t) \quad \forall x \in \Omega_t \\ c_2(t)^{-1} G_0^U(x, x'_t) &\leq G_\mu^U(x, x'_t) \leq c_2(t) G_0^U(x, x'_t) \quad \forall x \in U_t, \end{aligned}$$

and

$$(2.7) \quad \begin{aligned} c_3(t)^{-1} K_{\mu,U}^U(x, y) &\leq K_\mu^U(x, y) \leq c_3(t) K_{\mu,U}^U(x, y) \quad \forall (x, y) \in \Omega_t \times \partial\Omega, \\ c_4(t)^{-1} K_0^U(x, y) &\leq K_\mu^U(x, y) \leq c_4(t) K_0^U(x, y) \quad \forall (x, y) \in U_t \times \Sigma_{\bar{\rho}}. \end{aligned}$$

*Proof.* Note that  $\mathcal{L}_\mu = \mathcal{L}_{\mu,U}$  in  $\Omega_{\bar{\rho}/2}$ . Hence both  $G_\mu^U(\cdot, x_t)$  and  $G_{\mu,U}^U(\cdot, x_t)$  are  $\mathcal{L}_\mu$ -harmonic in  $\Omega_t$  and vanish on  $\partial\Omega$ . Therefore, by the boundary Harnack principle they are equivalent in a strip  $S$  along  $\partial\Omega$ . In addition they are continuous and bounded away from zero in  $\Omega_t \setminus S$ . This implies the first inequality in (2.6). For the second inequality:  $G_\mu^U(\cdot, x'_t)$  is  $\mathcal{L}_\mu$ -harmonic in  $U_t$ ,  $G_0^U(\cdot, x'_t)$  is  $\Delta$  harmonic in  $U_t$  and  $\mathcal{L}_\mu - \Delta = \mu/\delta(x)^2$  is bounded in  $U_t$ . Therefore, since they both vanish on  $\Sigma_{\bar{\rho}}$ , we can still apply the boundary Harnack principle (c.f. Ancona [4]) to deduce that they are equivalent in the strip  $U_t$ . This implies the second inequality in (2.6).

Recall that,  $G_{\mu,U}^U(x, x_t) \sim \delta_U(x)^{\alpha+}$  in  $\Omega_t$  for  $t \in (0, \rho)$ . (Of course the constants involved in this relation depend on  $t$ .) Since  $\delta_\Omega \sim \delta_U$  in  $\Omega_t$ , this fact and (2.6) imply,

$$(2.8) \quad G_\mu^U(x, x_t) \sim \delta_\Omega(x)^{\alpha+} \quad \forall x \in \Omega_t.$$

In what follows we use the notation introduced for the statement of the boundary Harnack principle. Let  $y \in \partial\Omega$  and let  $\xi = \xi_y$  be a local set of coordinates at  $y$  relative to  $U$ . Thus

$$\omega_y = T_y(r, \rho) \cap U = \{\xi : F_y(\xi') < \xi_1 < \rho, |\xi'| < r\}.$$

We assume that  $\gamma = \rho/r > 12\Lambda$ .

Since  $K_\mu^U(\cdot, y)$  and  $G_\mu^U(\cdot, x_t)$  satisfy the (classical) Harnack inequality (2.2) remains valid in  $\mathcal{C}_y(b) \cap T_y(r, \rho)$ . Therefore, assuming that  $\rho < t < \bar{\rho}$ ,

$$(2.9) \quad K_\mu^U(\xi, y) G_\mu^U(\xi, x_t) \sim K_\mu^U((\xi_1, 0), y) G_\mu^U((\xi_1, 0), x_t) \sim |\xi|^{2-N}$$

for every  $\xi \in \mathcal{C}_y(b) \cap T_y(r, \rho)$ . By (2.8) and (2.9),

$$(2.10) \quad K_\mu^U(\xi, y) \sim |\xi|^{2-N} \delta(\xi)^{-\alpha+} \quad \forall \xi \in \mathcal{C}_y(b) \cap T_y(r, \rho).$$

Let  $\eta$  be a point in  $\mathbb{R}^{N-1}$  such that  $0 < |\eta| < r/2$  and denote by  $P$  the point  $(F_y(\eta), \eta)$  in the local coordinates  $\xi_y$ . Then  $P \in \partial\Omega$  and  $\xi_P := \xi_y - P$  is a standard set of local coordinates at  $P$ . Choose  $r_P, \rho_P$  such that  $r_P = |\eta|/2$  and  $\rho_P/r_P = \gamma$ . Then,

$$|x - y| = |\xi_y| \sim |\xi'_y| \sim r_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P).$$

Let  $A_P = (\rho_P/2, 0)$  in  $\xi_P$  coordinates, i.e.,  $A_P = (F_y(\eta) + \gamma r_P/2, \eta)$  in  $\xi_y$  coordinates. Pick  $b$  such that  $\Lambda < b < 2\Lambda$ . Then

$$F_y(\eta) + \rho_P/2 \geq -\Lambda|\eta| - \gamma r_P/2 = |\eta|(-\Lambda + \gamma/4) > 2\Lambda|\eta|.$$

Consequently,  $F_y(\eta) < b|\eta| < F_y(\eta) + \rho_P/2$ , which implies

$$A_P \in \mathcal{C}_y(b) := \{\xi_y = (\xi_1, \xi') : \xi_1 > b|\xi'|\}.$$

Observe that

$$\delta_\Omega(A_P) \sim \rho_P/2, \quad |\xi_y(A_P)| = |A_P - y| \sim (\rho_P^2 + r_P^2)^{1/2} \sim r_P.$$

Therefore, by (2.10),

$$K_\mu^U(A_P, y) \sim r_P^{2-N-\alpha+}.$$

In fact,

$$|x - y| = |\xi_y| \sim r_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P).$$

Therefore applying (2.1) in  $\Omega \cap T_P(r_P, \rho_P)$  with  $u(x) = K_\mu^U(x, y)$  we obtain,

$$(2.11) \quad \begin{aligned} K_\mu^U(x, y) &\sim K_\mu^U(A_P, y) \frac{G_\mu^U(x, x_t)}{G_\mu^U(A_P, x_t)} \sim r_P^{2-N-\alpha+} (\delta(x)/r_P)^{\alpha+} \\ &\sim |x - y|^{2-N-2\alpha+} \delta(x)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha- - N} \end{aligned}$$

for every  $x \in \Omega \cap T_P(r_P/2, \rho_P/2)$ . Combining (2.10) and (2.11), we obtain,

$$(2.12) \quad K_\mu^U(x, y) \sim |x - y|^{2-N-\alpha+} (\delta(x)/|x - y|)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha- - N}$$

for every  $x \in T_y(r/2, \rho/2)$ . As (2.12) holds uniformly with respect to  $y \in \partial\Omega$  we conclude that there exists  $r' > 0$  such that this relation holds for every  $(x, y) \in \Omega_{r'} \times \partial\Omega$ . Consequently, for every  $t \in (0, \bar{\rho})$ ,

$$(2.13) \quad K_\mu^U(x, y) \sim |x - y|^{2-N-\alpha+} (\delta(x)/|x - y|)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha- - N}$$

for every  $(x, y) \in \Omega_t \times \partial\Omega$  with similarity constants depending on  $t$ . Since  $K_{\mu, U}^U$  behaves precisely in the same way (see [10, Sec. 2.2]) we obtain the first inequality in (2.7). The second inequality is proved in a similar way.  $\square$

We state below two key results concerning the operator  $\mathcal{L}_\mu$  in  $U = \Omega_{\bar{\rho}}$ . These have been recently proved in [10], with respect to the operator  $\mathcal{L}_\mu$  in  $\Omega$  under the assumption that  $0 < \mu < C_H(\Omega)$ . (In fact, the condition  $\mu > 0$  is redundant and does not affect the proofs.) Since  $C_H(\Omega_{\bar{\rho}}) = 1/4$ , the results apply to the operator  $\mathcal{L}_{\mu, \Omega_{\bar{\rho}}}$  for every  $\mu < 1/4$ . In view of the relation between the Martin kernels and Green functions of  $\mathcal{L}_{\mu, \Omega_{\bar{\rho}}}$  and  $\mathcal{L}_\mu$  in  $\Omega_{\bar{\rho}}$ , these results also apply to the operator  $\mathcal{L}_\mu$  in  $\Omega_{\bar{\rho}}$ .

**Theorem 2.4.** (i) If  $\nu_0 \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$  then there exist positive numbers  $c$  and  $\rho_0 < \bar{\rho}$  such that,

$$(2.14) \quad c^{-1} \|\nu_0\| \leq \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_0] dS \leq c \|\nu_0\|, \quad \varepsilon \in (0, \rho_0).$$

(ii) Let  $\rho \in (0, \bar{\rho})$  and let  $\tau$  be a Radon measure in  $\Omega_{\bar{\rho}}$ . Denote

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) := \int_{\Omega_\rho} G_\mu^{\Omega_\rho}(x, y) d\tau(y), \quad x \in \Omega_\rho.$$

If  $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho)$  then for every  $0 < \varepsilon < \rho' < \rho$ ,

$$(2.15) \quad \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau] dS_x \leq c \int_{\Omega_\rho} \delta^{\alpha_+} d\tau,$$

where  $c$  is a constant depending on  $\mu, \rho'$ , but not on  $\varepsilon$ . Moreover,

$$(2.16) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau] dS = 0.$$

*Remark 2.5.* If  $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x') < \infty$  for some point  $x' \in \Omega_\rho$  then  $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho)$  and  $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) < \infty$  for every  $x \in \Omega_\rho$ . This follows from the fact that there exists  $c > 0$  such that for every fixed  $x \in \Omega_\rho$ ,

$$\frac{1}{c} \delta(y)^{\alpha_+} \leq G_\mu^{\Omega_\rho}(x, y) \leq c \delta(y)^{\alpha_+} \quad \forall y \in \Omega_{\delta(x)/2}.$$

*Proof.* In view of (2.13), inequality (2.14) follows from [10, Corollary 2.11].

The proof of (2.15) and (2.16) is similar to that of [10, Proposition 2.12]. However several modifications are needed; therefore we provide the proof of these statements in detail.

We may assume that  $\tau > 0$ . Denote  $v := \mathbb{G}_\mu^{\Omega_\rho}[\tau]$ . We start with the proof of (2.15).

By Fubini's theorem and (2.6),

$$\begin{aligned} \int_{\Sigma_\beta} v dS_x &\leq c \left( \int_{\Omega} \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x - y|^{2-N} dS_x d\tau(y) \right. \\ &\quad \left. + \beta^{\alpha_+} \int_{\Omega} \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x - y|^{2\alpha_- - N} dS_x \delta^{\alpha_+}(y) d\tau(y) \right) = I_1(\beta) + I_2(\beta). \end{aligned}$$

Note that, if  $x \in \Sigma_\beta$  and  $|x - y| \leq \beta/2$  then  $\beta/2 \leq \delta(y) \leq 3\beta/2$ . Therefore

$$\begin{aligned} I_1(\beta) &\leq c_1 \beta^{-\alpha_+} \int_{\Sigma_\beta \cap B_{\frac{\beta}{4}}(y)} |x - y|^{2-N} dS_x \int_{\Omega_\rho} \delta(y)^{\alpha_+} d\tau(y) \\ &\leq c'_1 \beta^{1-\alpha_+} \int_{\Omega_\rho} \delta(y)^{\alpha_+} d\tau(y) = c'_1 \beta^{\alpha_-} \int_{\Omega_\rho} \delta(y)^{\alpha_+} d\tau(y) \end{aligned}$$

and

$$I_2(\beta) \leq c_2 \beta^{\alpha_+} \int_{\beta/4}^{\infty} r^{2\alpha_- - N} r^{N-2} dr \int_{\Omega_\rho} \delta(y)^{\alpha_+} d\tau \leq c'_2 \beta^{\alpha_-} \int_{\Omega_\rho} \delta(y)^{\alpha_+} d\tau.$$

This implies (2.15).

Given  $\ell \in (0, \|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)})$  and  $\beta_1 \in (0, \beta_0)$  put  $\tau_1 = \tau \chi_{\bar{D}_{\beta_1}}$  and  $\tau_2 = \tau - \tau_1$ . Pick  $\beta_1 = \beta_1(\ell)$  such that

$$(2.17) \quad \int_{\Omega_{\beta_1}} \delta(y)^{\alpha_+} d\tau \leq \ell.$$

Thus the choice of  $\beta_1$  depends on the rate at which  $\int_{\Omega_\beta} \delta_+^\alpha d\tau$  tends to zero as  $\beta \rightarrow 0$ .

Put  $v_i = \mathbb{G}_\mu^\Omega[\tau_i]$ . Then, for  $0 < \beta < \beta_1/2$ ,

$$\int_{\Sigma_\beta} v_1 dS_x \leq c_3 \beta^{\alpha_+} \beta_1^{2\alpha_- - N} \int_{\Omega_\rho} \delta^{\alpha_+}(y) d\tau_1(y).$$

Thus,

$$(2.18) \quad \lim_{\beta \rightarrow 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_1 dS_x = 0.$$

On the other hand, by (2.15) (replacing  $\Omega_\rho$  by  $\Omega_{\beta_1}$ ) and (2.17),

$$(2.19) \quad \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_2 dS_x \leq cl \quad \forall \beta < \beta_1.$$

This proves (2.16).  $\square$

**Corollary 2.6.** *Let  $\rho \in (0, \bar{\rho}]$  and assume that  $h$  is a nonnegative  $\mathcal{L}_\mu$ -harmonic function in  $\Omega_\rho$  such that*

$$(2.20) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} h dS = 0.$$

*Then: (i)  $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho]$  for some measure  $\nu_\rho \in \mathfrak{M}^+(\Sigma_\rho)$  and (ii) For  $t \in (0, \bar{\rho})$ ,*

$$(2.21) \quad h \sim \delta_\Omega^{\alpha_+} \quad \text{in } \Omega_t,$$

*with the similarity constant depending on  $t$ .*

*Proof.* (i) By the Representation Theorem,  $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$  for some  $\nu \in \mathfrak{M}(\partial\Omega_\rho)$ . By (2.14) and (2.20),  $\nu_0 := \nu \mathbf{1}_{\partial\Omega} = 0$ . Thus  $\nu = \nu_\rho := \nu \mathbf{1}_{\Sigma_\rho}$ .

(ii) This is a consequence of (i) and (2.13).  $\square$

**Corollary 2.7.** *If  $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho) \setminus \{0\}$  then there exists a positive constant  $c = c(\tau)$  such that*

$$(2.22) \quad \mathbb{G}_\mu^{\Omega_\rho}[\tau](x) \geq c\delta(x)^{\alpha_+} \quad \forall x \in \Omega_\rho,$$

and

$$(2.23) \quad \liminf_{x \rightarrow \partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\tau](x)}{\delta(x)^{\alpha_-}} < \infty.$$

*Proof.* Let  $t \in (0, \rho)$  be a number such that  $\tau(\Omega_\rho \setminus \Omega_t) > 0$ . Let  $\tau' \in \mathfrak{M}_+(\Omega_\rho)$  be defined by:  $\tau' = \tau$  in  $\Omega_\rho \setminus \Omega_t$  and  $\tau' = 0$  in  $\Omega_t$ . Then

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau] \geq G_\mu^{\Omega_\rho}[\tau'] := h.$$

Since  $h$  is  $\mathcal{L}_\mu$ -harmonic in  $\Omega_t$ , (2.22) is a consequence of (2.21)

Inequality (2.23) follows from (2.15).  $\square$

The next result was proved in [10] for  $\mathcal{L}_\mu$  in a domain  $\Omega$  such that  $\mu < C_H(\Omega)$ .

**Theorem 2.8.** *Let  $w$  be a nonnegative  $\mathcal{L}_\mu$ -subharmonic function in  $\Omega_\rho$ . If  $w$  is dominated by an  $\mathcal{L}_\mu$ -superharmonic function in  $\Omega_\rho$  then  $\mathcal{L}_\mu w = \lambda \in \mathfrak{M}_{\delta^+}^+(\Omega_\rho)$  and there exists  $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$  such that*

$$(2.24) \quad w = \mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda].$$

*Proof.* There exists a nonnegative Radon measure  $\lambda$  in  $\Omega_\rho$ , such that  $-\mathcal{L}_\mu w = -\lambda$  in  $\Omega_\rho$ . Since  $w$  is dominated by an  $\mathcal{L}_\mu$ -superharmonic function in  $\Omega_\rho$  one shows, as in the proof of [10, Proposition 2.14], that  $\lambda \in \mathfrak{M}_{\delta^+}(\Omega_\rho)$ . Then  $v := w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$  is a nonnegative  $\mathcal{L}_\mu$ -harmonic function in  $\Omega_\rho$ . By the Representation Theorem,  $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$  for some  $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ .  $\square$

**Definition 2.9.** A Borel function  $u : \Omega \rightarrow \mathbb{R}$  possesses a *normalised boundary trace*  $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$  if, for some  $\rho \in (0, \bar{\rho}]$ ,

$$(2.25) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]| dS = 0.$$

The normalised boundary trace on  $\partial\Omega$  will be denoted by  $\text{tr}_{\partial\Omega}^*(u)$ .

*Remark.* Since  $u$  is a Borel function  $u|_{\Sigma_\rho}$  is well defined and (2.25) implies that this function is in  $L^1(\Sigma_\varepsilon)$  for all sufficiently small  $\varepsilon$ .

We say that  $u$  has a *measure boundary trace* on  $\Sigma_\rho$  if there exists  $\nu_1 \in \mathfrak{M}^+(\Sigma_\rho)$  such that

$$\lim_{a \rightarrow \rho^-} \int_{\Sigma_a} u \phi dS \rightarrow \int_{\Sigma_\rho} \phi d\nu_1 \quad \forall \phi \in C_0(\bar{\Omega}_\rho).$$

This trace is denoted by  $\text{tr}_{\Sigma_\rho}(u)$ . If both  $\text{tr}_{\Sigma_\rho}(u)$  and  $\text{tr}_{\partial\Omega}^*(u)$  exist then the measure  $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$  given by  $\nu \mathbf{1}_{\partial\Omega} = \text{tr}_{\partial\Omega}^*(u)$  and  $\nu \mathbf{1}_{\Sigma_\rho} = \text{tr}_{\Sigma_\rho}(u)$  is denoted by  $\text{tr}_{\partial\Omega_\rho}^\mu(u)$ .

**Lemma 2.10.** *The normalised boundary trace  $\nu_0$  is uniquely defined, independently of  $\rho$ .*

*Proof.* First we note that (2.25) remains valid if  $\nu_0$  is replaced by any measure  $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$  such that  $\nu_0 = \nu \mathbf{1}_{\partial\Omega}$ . This follows from the fact that, for every measure  $\nu_\rho \in \mathfrak{M}_+(\Sigma_\rho)$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho] dS = 0.$$

This implies that if (2.25) holds with respect to some  $\rho \in (0, \bar{\rho})$  then it is valid for any  $\rho'$  in this range. Suppose for instance that  $\rho < \rho' < \bar{\rho}$  and put  $v = \mathbb{K}_\mu^{\Omega_{\rho'}}[\nu_0]$ . Let  $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$  be the measure equal to  $\nu_0$  on  $\partial\Omega$  and to  $h = v|_{\Sigma_\rho} d\omega_\rho$  on  $\Sigma_\rho$ . (Here  $\omega_\rho$  is the  $\mathcal{L}_\mu$ -harmonic measure on  $\Sigma_\rho$  relative to  $\Omega_{\rho'}$ . Since  $\Sigma_\rho$  is 'smooth'  $\omega_\rho$  is absolutely continuous with respect to surface measure.) Then  $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$  in  $\Omega_\rho$  and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |\mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]| dS = 0.$$

It remains to verify that, if (2.25) holds then  $\nu_0$  is uniquely determined by  $u$  in a fixed domain  $\Omega_\rho$ .

Suppose, by negation, that there exist  $\nu_1, \nu_2 \in \mathfrak{M}_+(\partial\Omega)$  such that (2.25) holds for both  $v_1 = K_\mu^{\Omega_\rho}[\nu_1]$  and  $v_2 = K_\mu^{\Omega_\rho}[\nu_2]$ . Then  $w := |v_1 - v_2|$  is  $\mathcal{L}_\mu$ -subharmonic and  $\text{tr}_{\partial\Omega}^*(w) = 0$ .

Clearly  $w$  is dominated by the  $\mathcal{L}_\mu$ -superharmonic function  $v_1 + v_2$ . Therefore, by Theorem 2.8 there exist  $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho)$  and  $\chi \in \mathfrak{M}^+(\partial\Omega_\rho)$  such that,

$$w = \mathbb{K}_\mu^{\Omega_\rho}[\chi] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda].$$

Thus  $w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$  is  $\mathcal{L}_\mu$ -harmonic. By (2.16) and the fact that  $\text{tr}_{\partial\Omega}^* w = 0$  we have  $\text{tr}_{\partial\Omega}^*(w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]) = 0$ . Hence  $w = 0$  and therefore  $\nu_1 = \nu_2$ .  $\square$

**Theorem 2.11.** *Let  $w$  be a nonnegative  $\mathcal{L}_\mu$ -subharmonic function in  $\Omega_\rho$  dominated by an  $\mathcal{L}_\mu$ -superharmonic function in this domain. Then the boundary trace  $\nu = \text{tr}_{\partial\Omega_\rho}^\mu(w)$  is well-defined and*

$$(2.26) \quad w \leq \mathbb{K}_\mu^{\Omega_\rho}[\nu].$$

If  $\nu_0 := \nu \mathbf{1}_{\partial\Omega}$  then,

$$(2.27) \quad \lim_{x \rightarrow \partial\Omega} \frac{w(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)} = 1 \quad \text{non-tangentially, } \nu_0\text{-a.e. on } \partial\Omega.$$

If  $\nu_0 = 0$  then,

$$(2.28) \quad \limsup_{x \rightarrow \partial\Omega} \frac{w(x)}{\delta^{\alpha_+}(x)} < \infty.$$

*Proof.* The first statement (2.26) follows from (2.24) and Theorem 2.4 (ii).

The second statement (2.27) follows from (2.24) and the fact that  $\mathbb{G}_\mu^{\Omega_\rho}[\lambda]$  is an  $\mathcal{L}_\mu$ -potential (i.e. a positive superharmonic function that does not dominate any positive  $\mathcal{L}_\mu$ -harmonic function). This fact implies (see, e.g. [3]):

$$\lim_{x \rightarrow \partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\lambda](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} \rightarrow 0 \quad \nu\text{-a.e. on } \partial\Omega.$$

By Fatou's limit theorem

$$\lim_{x \rightarrow \partial\Omega} \frac{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \nu\text{-a.e. on } \partial\Omega.$$

Therefore (2.24) implies (2.27).

The third statement (2.28) follows from (2.26) and Corollary 2.6.  $\square$

**Corollary 2.12.** *Let  $w$  be a nonnegative  $\mathcal{L}_\mu$ -subharmonic function in  $\Omega_\rho$  for some  $\rho \in (0, \bar{\rho})$ . Then  $w$  possesses a normalised boundary trace in  $\mathfrak{M}^+(\partial\Omega)$  if and only if  $w$  is dominated by a positive  $\mathcal{L}_\mu$ -superharmonic function  $v$  in a strip around  $\partial\Omega$ .*

*Proof.* If  $w$  is dominated by a positive  $\mathcal{L}_\mu$ -superharmonic function in  $\Omega_\rho$  then the existence of  $\text{tr}_{\partial\Omega}^*(w)$  follows from (2.16) and Theorem 2.8.

Next suppose that  $w$  has a normalized boundary trace  $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$ . Without loss of generality we may assume that it also has a measure boundary trace  $\nu_\rho$  on  $\Sigma_\rho$ . Since  $u$  is  $\mathcal{L}_\mu$ -subharmonic, there exists a positive Radon measure  $\tau$  in  $\Omega$  such that

$$-\mathcal{L}_\mu u = -\tau.$$

Let  $\tau_\beta := \tau \mathbf{1}_{D_\beta \setminus \bar{D}_\rho}$ ,  $w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho]$  and  $\nu_\beta = w|_{\Sigma_\beta}$ .

Let  $u_\beta$  be the solution of the boundary value problem,

$$\begin{aligned} -\mathcal{L}_\mu v &= -\tau_\beta \text{ in } D_\beta \setminus \bar{D}_\rho, \\ v &= \nu_\rho \text{ on } \Sigma_\rho, \quad v = \nu_\beta \text{ on } \Sigma_\beta. \end{aligned}$$

Then

$$u_\beta + \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] = w.$$

It follows that

$$G_\mu^{\Omega_\rho}[\tau] = \lim_{\beta \rightarrow 0} \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] < \infty,$$

which in turn implies that  $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha+})$  and finally

$$u + G_\mu^{\Omega_\rho}[\tau] = w.$$

In particular,

$$(2.29) \quad u \leq w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho].$$

□

**Corollary 2.13.** (i) Suppose that  $u$  is positive and  $\mathcal{L}_\mu$ -subharmonic in  $\Omega_{\bar{\rho}}$ . Then  $\text{tr}_{\partial\Omega}^* u = 0$  if and only if, for every  $\rho \in (0, \bar{\rho})$ , there exists a constant  $c_\rho$  such that

$$(2.30) \quad u(x) \leq c_\rho \delta(x)^{\alpha+} \quad \forall x \in \Omega_\rho.$$

(ii) Suppose that  $u$  is positive and  $\mathcal{L}_\mu$ -superharmonic in  $\Omega_{\bar{\rho}}$ . Then  $u$  has a normalized boundary trace  $\nu \in \mathfrak{M}_+(\partial\Omega)$  and consequently there exists  $c_\rho$  such that

$$(2.31) \quad \int_{\Sigma_\beta} u dS \leq c_\rho \beta^{\alpha-} \quad \forall \beta \in (0, \rho).$$

*Proof.* (i) Obviously (2.30) implies that  $\text{tr}_{\partial\Omega}^*(u) = 0$ . Conversely assume that  $\text{tr}_{\partial\Omega}^*(u) = 0$ .

By the previous corollary  $u$  is dominated by an  $\mathcal{L}_\mu$ -harmonic function. Therefore, by Theorem 2.8, there exist  $\lambda \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$  and  $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$  such that  $u = \mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ . Since  $\text{tr}_{\partial\Omega}^*(u) = 0$ ,  $\nu_0 = \nu \mathbf{1}_{\partial\Omega} = 0$ . Hence  $u < \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho]$  where  $\nu_\rho = \nu \mathbf{1}_{\Sigma_\rho}$ . Therefore the result follows from Corollary 2.6.

(ii) By the Riesz decomposition theorem (see [3]),  $u = u_p + u_h$  where  $u_p$  is an  $\mathcal{L}_\mu$ -potential and  $u_h$  is a nonnegative  $\mathcal{L}_\mu$ -harmonic function in  $\Omega_\rho$ . It is known that every  $\mathcal{L}_\mu$ -potential is the Green potential of a positive measure. Thus there exists  $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha+})$  such that  $u_p = \mathbb{G}_\mu^{\Omega_\rho}[\tau]$ . By the Representation Theorem  $u_h = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$  for some  $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ . Thus

$$(2.32) \quad u = \mathbb{G}_\mu^{\Omega_\rho}[\tau] + \mathbb{K}_\mu^{\Omega_\rho}[\nu].$$

The required result follows from Theorem 2.4.  $\square$

### 3. $\mathcal{L}_\mu$ -MODERATE SOLUTIONS OF NONLINEAR EQUATION

In this section we study the nonlinear equation

$$(P_\mu) \quad -\mathcal{L}_\mu u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded smooth domain,  $\mu < 1/4$  and  $q > 1$ .

**3.1. Preliminaries.** Suppose that  $u \in L^q_{\text{loc}}(\Omega)$  is either a subsolution or a supersolution of  $(P_\mu)$ , in the distribution sense. Then,  $u \in W^{1,p}_{\text{loc}}(\Omega)$  for  $1 \leq p < N/(N-1)$ . If, in addition,  $u$  is a distributional solution of  $(P_\mu)$  then it is also a classical solution.

Consequently, if  $u \in L^q_{\text{loc}}(\Omega)$  is a distributional subsolution in  $\Omega$  then

$$(3.1) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - \int_{\Omega} \frac{\mu}{\delta^2} u \varphi \, dx + \int_{\Omega} |u|^{q-1} u \varphi \, dx \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega).$$

If, in addition,  $u \in H^1_{\text{loc}}(\Omega)$  then (3.1) holds for every  $\varphi \in H^1_c(\Omega)$ .

A similar statement holds for supersolutions, in which case the inequality sign in (3.1) is inverted. Of course these statements remain valid for local subsolutions and supersolutions (in a subdomain  $G \subset \Omega$ ).

We state below two results from [5] that will be used in the sequel.

**Lemma 3.1.** (COMPARISON PRINCIPLE [5, Lemma 3.2])

- (i) Let  $G$  be open with  $G \subset \Omega$ . Let  $0 \leq \underline{u}, \bar{u} \in H^1_{\text{loc}}(G) \cap C(G)$  be a pair of sub and supersolutions to  $(P_\mu)$  in  $G$  such that

$$\limsup_{x \rightarrow \partial G} [\underline{u}(x) - \bar{u}(x)] < 0.$$

Then  $\underline{u} \leq \bar{u}$  in  $G$ .

- (ii) Let  $G$  be open with  $\bar{G} \subset \Omega$ . Let  $\underline{u}, \bar{u} \in H^1(G) \cap C(\bar{G})$  be a pair of sub and supersolutions to  $(P_\mu)$  in  $G$  and  $\underline{u} \leq \bar{u}$  on  $\partial G$ . Then  $\underline{u} \leq \bar{u}$  in  $G$ .

**Lemma 3.2.** ([5, Lemma 4.10]) Assume that  $(P_\mu)$  admits a subsolution  $\underline{u}$  and a supersolution  $\bar{u}$  in  $\Omega$  so that  $0 \leq \underline{u} \leq \bar{u}$  in  $\Omega$ . Then  $(P_\mu)$  has a solution  $U$  in  $\Omega$  such that  $\underline{u} \leq U \leq \bar{u}$  in  $\Omega$ .

In [5, Proposition 3.5] the Keller–Osserman estimate has been extended to equation  $(P_\mu)$ . Specifically it was proved that every subsolution  $u$  of  $(P_\mu)$  in  $\Omega$  satisfies,

$$(3.2) \quad u(x) \leq \gamma_* \delta^{-\frac{2}{q-1}}(x) \quad \text{in } \Omega,$$

where  $\gamma_*$  is a constant independent of  $u$ . In addition it was shown that, if  $u$  is a local subsolution in  $\Omega_\rho$ , continuous at  $\Sigma_\rho$ , then  $u$  satisfies (3.2) in  $\Omega_\rho$ , but  $\gamma_*$  may depend on  $u$ . We prove below a stronger version that is used later on.

**Lemma 3.3.** (KELLER–OSSERMAN ESTIMATE) If  $u$  is a subsolution of  $(P_\mu)$  in  $\Omega$  then it satisfies (3.2) with a constant depending only on  $q, N, \mu$ . If  $u$  is a subsolution of  $(P_\mu)$  in  $\Omega_\rho$  then (3.2) holds with a constant depending only on  $q, N, \mu, \rho$  and  $\delta(x)$  replaced by  $\delta_\rho(x) := \text{dist}(x, \partial\Omega_\rho)$ .



*Proof.* Without loss of generality we may assume that  $u \geq 0$  because  $u_+$  is a subsolution. If  $\mu \leq 0$  then  $u$  is also a subsolution of the equation  $-\Delta u + u^q = 0$ . Therefore in this case (3.2) is a direct consequence of the classical Keller–Osserman inequality.

Now assume that  $\mu > 0$ . Let  $y \in \Omega$  and  $R = \delta(y)/2$ . Then,

$$-\Delta u - \frac{\mu}{R^2}u + u^q \leq 0 \quad \text{in } B_R(y).$$

Therefore in  $B_R(y)$  either  $u \leq (8\mu/R^2)^{\frac{1}{q-1}}$  or  $-\Delta u + u^q/2 \leq 0$ . Hence, by Kato's inequality, the function  $v := (u - (8\mu/R^2)^{\frac{1}{q-1}})_+$  satisfies

$$-\Delta v + v^q/2 \leq 0 \quad \text{in } B_R(y).$$

By the classical Keller–Osserman inequality,

$$v(y) \leq c(q, N)R^{-\frac{2}{q-1}}.$$

Since  $u(y) \leq v(y) + (8\mu/R^2)^{\frac{1}{q-1}}$  we conclude that

$$(3.3) \quad u(y) \leq c(\mu, q, N)\delta_\Omega(y)^{-\frac{2}{q-1}} \quad \forall y \in \Omega.$$

Next, let  $u$  be a subsolution in  $\Omega_\rho$ . As before we may assume that  $u \geq 0$  and that  $\mu > 0$ . By the first part of the proof, (3.3) holds in  $\Omega_{3\rho/4}$ . Further,

$$-\Delta u - (4\mu/\rho^2)u + u^q \leq 0 \quad \text{in } \Omega'_\rho = \{x : \rho/2 \leq \delta(x) < \rho\}.$$

Therefore, either  $u \leq (8\mu/\rho^2)^{\frac{1}{q-1}}$  or  $-\Delta u + u^q/2 \leq 0$ . By the same argument as before, the function  $v := (u - (8\mu/\rho^2)^{\frac{1}{q-1}})_+$  satisfies

$$v(x) \leq c(q, N)\text{dist}(x, \Sigma_\rho)^{-\frac{2}{q-1}} \quad \forall x : 3\rho/4 \leq \delta(x) < \rho.$$

Consequently,

$$(3.4) \quad u(x) \leq c(\mu, q, N, \rho)\text{dist}(x, \partial\Omega_\rho)^{-\frac{2}{q-1}} \quad \forall x \in \Omega_\rho. \quad \square$$

**3.2. Moderate solutions.** We study the generalised boundary trace problem  $(P_\mu^\nu)$  where  $\mu < 1/4$ ,  $q > 1$  and  $\nu \in \mathcal{M}^+(\partial\Omega)$ . First we prove,

**Lemma 3.4.** *Let  $D$  be a  $C^2$  domain such that  $D \Subset \Omega$ . If  $0 \leq f \in C(\partial D)$  then there exists a unique solution of the problem*

$$(3.5) \quad \begin{cases} -\mathcal{L}_\mu u + u^q = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

*Proof.* For  $u \in H^1(D)$ , let

$$J_D(u) = \int_D \left( \frac{1}{2}|\nabla u|^2 - \frac{\mu}{2\delta_\Omega^2}u^2 + \frac{1}{q+1}|u|^{q+1} \right) dx.$$

Since  $\mu\delta_\Omega^{-2} \in L^\infty(D)$ , it is standard to see that  $J_D$  is coercive and weakly l.s.c. on

$$H_f^1(D) = \{u \in H^1(D) : u = f \text{ on } \partial D\}.$$

Therefore there exists a minimizer  $u_f \in H_f^1(D)$ . We may assume that  $u_f > 0$  because  $|u_f|$  too is a minimizer. The minimizer is a solution of (3.5). The uniqueness is a consequence of the comparison principle.  $\square$

Next consider the problem,

$$(P_\mu^\nu(\rho)) \quad \begin{cases} -\mathcal{L}_\mu u + u^q = 0 & \text{in } \Omega_\rho, \\ \text{tr}_{\partial\Omega}^*(u) = \nu \mathbf{1}_{\partial\Omega} =: \nu_0, \\ \text{tr}_{\Sigma_\rho}(u) = \nu \mathbf{1}_{\Sigma_\rho} =: \nu_\rho. \end{cases}$$

where  $\mu < 1/4$ ,  $q > 1$ ,  $\nu \in \mathcal{M}^+(\partial\Omega_\rho)$  and  $\rho \in (0, \bar{\rho}]$ .

The following result is an adaptation of [10, Theorem C] to problem  $(P_\mu^\nu(\rho))$ . Since  $C_H(\Omega_{\bar{\rho}}) = 1/4$  the result applies to every  $\mu < 1/4$ . The proof follows the argument in [10]; for the convenience of the reader it is presented below.

**Proposition 3.5.** *Let  $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$  and assume that  $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$  for some  $\rho \in (0, \bar{\rho}]$ . Then  $(P_\mu^\nu(\rho))$  admits a unique solution  $U_\nu$ .*

*Proof.* Let  $\{D_n\}$  be a sequence of  $C^2$  domains such that  $\bar{D}_n \subset D_{n+1}$  and  $D_n \uparrow \Omega_\rho$ . Let  $u_n$  be the solution of (3.5) with  $D = D_n$  and  $f = f_n := \mathbb{K}_\mu^{\Omega_\rho}[\nu]|_{\partial D_n}$ . Since  $\mathbb{K}_\mu^{\Omega_\rho}[\nu]$  is a supersolution of the equation  $\mathcal{L}_\mu v + v^q = 0$  in  $\Omega_\rho$  it follows that  $u_n$  decreases and  $u = \lim u_n$  is a solution of this equation. We claim that  $u$  is a solution of  $(P_\mu^\nu(\rho))$ . Indeed,

$$(3.6) \quad u_n + \mathbb{G}_\mu^{D_n}[u_n^q] = \mathbb{P}_\mu^{D_n}[f_n] = \mathbb{K}_\mu^{\Omega_\rho}[\nu] \quad \text{in } D_n,$$

where  $\mathbb{P}_\mu^{D_n}$  denotes the Poisson kernel of  $\mathcal{L}_\mu$  in  $D_n$ .

Since  $u_n \leq \mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega)$  it follows that

$$\mathbb{G}_\mu^{D_n}[u_n^q] \rightarrow \mathbb{G}_\mu^{\Omega_\rho}[u^q].$$

Hence, by (3.6),

$$u + \mathbb{G}_\mu^{\Omega_\rho}[u^q] = \mathbb{K}_\mu^{\Omega_\rho}[\nu] \quad \text{in } \Omega_\rho.$$

By Theorem 2.4,  $\text{tr}_{\partial\Omega}^*(u) = \nu \mathbf{1}_{\partial\Omega}$  and (by (2.7))  $\text{tr}_{\Sigma_\rho}(u) = \nu \mathbf{1}_{\Sigma_\rho}$ .  $\square$

The next result is an adaptation of [10, Theorem D]. We omit the proof which – except for obvious modifications – is the same as in [10].

**Proposition 3.6.** *Assume that  $u$  is a positive solution of  $(P_\mu^\nu(\rho))$ . Then*

$$(3.7) \quad \lim_{x \rightarrow \partial\Omega} \frac{u(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)} = 1 \quad \text{non-tangentially, } \nu\text{-a.e. on } \partial\Omega,$$

where  $\nu_0 = \nu \mathbf{1}_{\partial\Omega}$ .

**Theorem 3.7.** *Let  $\nu \in \mathfrak{M}^+(\partial\Omega)$  and  $\rho \in (0, \bar{\rho}]$ . Let  $\nu' \in \mathfrak{M}^+(\partial\Omega_\rho)$  be defined by  $\nu' = \nu$  on  $\partial\Omega$  and  $\nu' = 0$  on  $\Sigma_\rho$ . Assume that, for some  $\rho$  as above,  $\mathbb{K}_\mu^{\Omega_\rho}[\nu'] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$ . Then the boundary value problem  $(P_\mu^{\nu'})$  admits a solution in  $\Omega$ .*

*Proof.* By Proposition 3.5 there exists a (unique) solution  $U_{\nu,0}$  of problem  $(P_\mu^{\nu'}(\rho))$ . For every  $k \geq 0$ , let  $\nu_k \in \mathfrak{M}^+(\partial\Omega_\rho)$  be the measure given by,  $\nu_k \mathbf{1}_{\partial\Omega} = \nu$  and  $\nu_k \mathbf{1}_{\Sigma_\rho} = kdS_{\Sigma_\rho}$ . By the same proposition there exists a (unique) solution  $U_{\nu,k}$  of  $(P_\mu^{\nu_k}(\rho))$ . Put

$$U_{\nu,\infty} = \lim_{k \rightarrow \infty} U_{\nu,k}.$$

Let  $R \in (0, \rho)$ . By Lemma 3.4 there exists a unique solution  $v_R$  of (3.5) in  $D_R$  with  $f = U_{\nu,0}|_{\Sigma_R}$ . By the comparison principle,

$$U_{\nu,0} \leq v_R \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho \cap D_R.$$

By Proposition 3.3 the family  $\{v_R : 0 < R < \rho\}$  is bounded in compact subsets of  $\Omega$ . Therefore there exists a sequence  $\{R_j\}$  converging to zero such that  $v_{R_j}$  converges to a solution  $v$  of the nonlinear equation in  $\Omega$ . By construction,

$$U_{\nu,0} \leq v \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho.$$

Therefore  $\text{tr}_{\partial\Omega}^*(v) = \nu$ . □

*Remark 3.8.* If  $\mu < C_H(\Omega)$  then problem  $(P_\mu^\nu)$  has at most one solution, [10, Theorem B]. However uniqueness fails when  $C_H(\Omega) < \mu < 1/4$ . It was proved in [5, Theorem 5.3] that in this case there exists a positive solution of  $(P_\mu^\nu)$  with  $\nu = 0$ . An alternative, more direct proof, is presented in Appendix A.

**Proposition 3.9.** *Assume that  $u \in L_{loc}^q(\Omega)$  is a positive solution of  $(P_\mu)$ . Then the following assertions are equivalent:*

- (i)  $u$  has a normalized boundary trace,
- (ii)  $u$  is a moderate solution in the sense of Definition 1.3,
- (iii)  $u \in L^q(\Omega; \delta^{\alpha+})$ .

*Proof.* The assumption implies that  $\mathcal{L}_\mu u \leq 0$  in  $\Omega$ . If  $\rho \in (0, \bar{\rho}]$  then, by Lemma 2.12, (i) holds if and only if  $u$  is dominated by an  $\mathcal{L}_\mu$ -superharmonic function in  $\Omega_\rho$ . Consequently, by Lemma 3.2, (i) holds if and only if  $u$  is dominated by an  $\mathcal{L}_\mu$ -harmonic function in  $\Omega_\rho$ . Thus (i) and (ii) are equivalent.

If (iii) holds then  $v := u + \mathbb{G}_\mu^{\Omega_\rho}[u^q]$  is  $\mathcal{L}_\mu$ -harmonic. By the representation theorem there exists  $\nu \in \mathfrak{M}(\partial\Omega_\rho)$  such that  $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ . Since  $\text{tr}_{\partial\Omega}^* \mathbb{G}_\mu^{\Omega_\rho}[u^q] = 0$  it follows that  $\nu \mathbf{1}_{\partial\Omega}$  is the normalized boundary trace of  $u$ . Conversely if (ii) holds then by Theorem 2.8  $\mathcal{L}_\mu u = u^q \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$  which is the same as (iii). □

**3.3. Critical exponents.** The next result provides necessary and sufficient conditions in order that a positive measures  $\nu \in \mathfrak{M}^+(\partial\Omega)$  satisfy,

$$(3.8) \quad \mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha+}}^q(\Omega_\rho) \quad \text{for some } \rho > 0.$$

Let  $\Gamma_a(x-y) = |x-y|^{-(N-a)}$  denote the Riesz kernel of order  $0 < a < N$  in  $\mathbb{R}^N$ .

**Proposition 3.10.** *Let  $\nu \in \mathfrak{M}^+(\partial\Omega)$ .*

- (i) *If  $\Gamma_1 * \nu \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$  then  $\nu$  satisfies (3.8).*
  - (ii) *Assume  $\mu \geq 0$ . If  $\nu$  satisfies (3.8) then  $\mathbb{P}_0^\Omega[\nu] \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$ .*
- Here  $P_0^\Omega$  is the Poisson kernel of  $-\Delta$  in  $\Omega$ :  $P_0^\Omega(x, y) = \delta(x)|x-y|^{-N}$ .*

*Proof.* By (2.13),

$$(3.9) \quad \begin{aligned} K_\mu^{\Omega_\rho}(x, y) &\sim \frac{\delta(x)^{\alpha_+}}{|x-y|^{N-2\alpha_-}} \sim \delta(x)^{\alpha_-} P_0^\Omega(x, y) (|x-y|/\delta(x))^{2\alpha_-} \\ &\sim \delta(x)^{\alpha_-} \Gamma_1(x-y) (|x-y|/\delta(x))^{-1+2\alpha_-}, \end{aligned}$$

for every  $(x, y) \in \Omega_{\rho/2} \times \partial\Omega$ .

For every  $\mu < 1/4$  we have  $-1 + 2\alpha_- < 0$ . Consequently,

$$(3.10) \quad K_\mu^{\Omega_\rho}(x, y) \leq c\delta(x)^{\alpha_-} \Gamma_1(x-y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial\Omega.$$

Hence,

$$\|\mathbb{K}_\mu^{\Omega_\rho} \nu\|_{L_{\delta^{\alpha_+}}^q(\Omega_{\rho/2})}^q \leq c \int_{\Omega_{\rho/2}} \left( \int_{\partial\Omega} \Gamma_1(x-y) d\nu(y) \right)^q \delta(x)^{q\alpha_- + \alpha_+} dx.$$

This proves (i).

If  $\mu \geq 0$ , so that  $\alpha_- \geq 0$  then, by (3.9),

$$(3.11) \quad K_\mu^{\Omega_\rho}(x, y) \geq c\delta(x)^{\alpha_-} P_0^\Omega(x, y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial\Omega.$$

Therefore

$$\|\mathbb{K}_\mu^{\Omega_\rho}[\nu]\|_{L_{\delta^{\alpha_+}}^q(\Omega_{\rho/2})}^q \geq c \int_{\Omega_{\rho/2}} \left( \int_{\partial\Omega} P_0^\Omega(x, y) d\nu(y) \right)^q \delta(x)^{q\alpha_- + \alpha_+} dx.$$

This proves (ii).  $\square$

Using this result we provide a necessary and sufficient condition for the existence of positive moderate solutions of  $(P_\mu)$ .

**Proposition 3.11.** *Let  $\nu \in \mathcal{M}^+(\partial\Omega)$ .*

(i) *If  $\alpha_- > -\frac{2}{q-1}$  then the boundary value problem  $(P_\mu^\nu)$  has a solution for every measure  $\nu = f dS_{\partial\Omega}$  such that  $f \in L^1(\partial\Omega)$ .*

(ii) *If  $\alpha_- \leq -\frac{2}{q-1}$  then, for every  $\nu \gneq 0$ ,  $(P_\mu^\nu)$  has no solution.*

*Remark.* When  $\mu > 0$  and consequently  $\alpha_- > 0$ , the condition in (i) holds for every  $q > 1$ .

*Proof.* Let  $\nu = f dS_{\partial\Omega}$  and  $f \in L^\infty(\partial\Omega)^+$ . Let  $x \in \Omega_{\beta_0}$  and pick  $x' \in \partial\Omega$  such that  $|x' - x| = \delta(x)$ . Then,

$$(3.12) \quad \begin{aligned} \int_{\partial\Omega} |x-y|^{1-N} f(y) dS(y) &\leq c\|f\|_{L^\infty} \left( \int_{\substack{y \in \partial\Omega \\ |x'-y| \geq \delta(x)}} |x'-y|^{1-N} dS(y) + 1 \right) \\ &\leq c\|f\|_{L^\infty} (1 + |\ln \delta(x)|) \leq c'\|f\|_{L^\infty} |\ln \delta(x)|, \end{aligned}$$

where  $c'$  is independent of  $x$ . Therefore, if  $(q-1)\alpha_- + 1 > -1$  then  $\Gamma_1 * \nu \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$ . Consequently, by Proposition 3.10 (i) and Theorem 3.7, problem  $(P_\mu^\nu)$  has a solution.

Next, let  $f \in L^1(\partial\Omega)^+$  and  $\nu = f dS_{\partial\Omega}$ . If  $\nu_n = \min(f, n) dS_{\partial\Omega}$  then problem  $(P_\mu^{\nu_n})$  has a solution  $u_n$  and the sequence  $\{u_n\}$  is non-decreasing. In view of the Keller–Osserman estimate (3.2),  $\{u_n\}$  converges to a solution  $u$  of  $(P_\mu^\nu)$ . This proves (i).

We turn to part (ii). Suppose that  $\alpha_- \leq -\frac{2}{q-1}$  and that there exists  $\nu \in \mathcal{M}^+(\partial\Omega) \setminus \{0\}$  such that problem  $(P_\mu^\nu)$  has a solution  $u$ . Then, there exists  $c > 0$  such that

$$c\beta^{-\frac{2}{q-1}} \leq c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} \mathbb{K}_\mu^{\Omega_\rho}[\nu] dS \quad \forall \beta \in (0, \beta_0).$$

Since  $u = -\mathbb{G}_\mu[u^q] + \mathbb{K}_\mu[\nu]$  and  $\text{tr}_{\partial\Omega}^*(\mathbb{G}_\mu[u^q]) = 0$  it follows that, for sufficiently small  $\beta_1$ ,

$$(3.13) \quad c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} u dS \quad \forall \beta \in (0, \beta_1).$$

But, by the Keller-Osserman estimate,  $u(x) \leq c_1\delta(x)^{-\frac{2}{q-1}}$  so that

$$(3.14) \quad c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} u dS \leq c_2\beta^{-\frac{2}{q-1}} \quad \forall \beta \in (0, \beta_1).$$

If  $\alpha_- < -2/(q-1)$  we reached a contradiction. If  $\alpha_- = -2/(q-1)$  then, in view of the Keller-Osserman estimate (3.2) we conclude that  $u(x) \sim \delta(x)^{-\frac{2}{q-1}}$ . This implies that  $u \sim U_{max}$  (= the maximal solution of  $-\mathcal{L}_\mu v + v^q = 0$ ). Thus  $\sup U_{max}/u := c < \infty$ . Now  $cu$  is a supersolution and, if  $v$  is the largest solution dominated by  $cu$  then  $\text{tr}^*(v) = c\text{tr}^*(u) = cv$ . It follows that  $U_{max} \leq v$  which is impossible.  $\square$

*Remark 3.12.* When  $\mu > 0$  – and consequently  $\alpha_- > 0$  – the condition in (i) holds trivially for every  $q > 1$ . However, if  $\mu < 0$  and

$$q \geq q_\mu^* := 1 - \frac{2}{\alpha_-}$$

then equation  $(P_\mu)$  has no moderate solution except for the trivial solution.

**Lemma 3.13.** *Let  $\mu < C_H(\Omega)$  and put*

$$q_{\mu,c} = \frac{N+1-\alpha_-}{N-1-\alpha_-}.$$

*Then, for  $y \in \partial\Omega$ ,*

$$K_\mu^\Omega(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+}) \iff q < q_{\mu,c}.$$

*For every  $q \in (1, q_{\mu,c})$  there exists a number  $c = c(q, N, \mu)$  such that*

$$(3.15) \quad \|K_\mu^\Omega[\nu]\|_{L^{\frac{N+\alpha_+}{N-1-\alpha_-}}(\Omega, \delta^{\alpha_+})} \leq c\|\nu\| \quad \forall \nu \in \mathfrak{M}(\partial\Omega).$$

*Proof.* Recall that

$$(3.16) \quad K_\mu^\Omega(x, y) \sim |x-y|^{2-N-\alpha_+} (\delta(x)/|x-y|)^{\alpha_+} = \delta(x)^{\alpha_+} |x-y|^{2\alpha_- - N},$$

(see [10, Section 2.2]). Therefore,

$$c' \left( \frac{\delta(x)}{|x-y|} \right)^{\alpha_+} |x-y|^{1+\alpha_- - N} \leq K_\mu(x, y) \leq c |x-y|^{1+\alpha_- - N}.$$

It follows that  $K_\mu(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+})$  if and only if

$$I := \int_0^1 t^{q(1+\alpha_- - N)} t^{\alpha_+} t^{N-1} dt < \infty$$

and

$$\|K_\mu(\cdot, y)\|_{L^q(\Omega, \delta^{\alpha_+})} \sim I.$$

A simple computation shows that  $I < \infty$  if and only if

$$q < q_{\mu, c} = \frac{N+1-\alpha_-}{N-1-\alpha_-}.$$

Finally,

$$\|K_\mu^\Omega[\nu]\|_{L^q(\Omega, \delta^{\alpha_+})} \leq \int_{\partial\Omega} \|K_\mu(\cdot, y)\|_{L^q(\Omega, \delta^{\alpha_+})} d|\nu|(y) \leq c\|\nu\|.$$

□

**Corollary 3.14.** *Let  $\mu < 1/4$ . If  $1 < q < q_{\mu, c}$  then the boundary value problem  $(P_\mu^\nu)$  has a solution for every Borel measure  $\nu$ . Moreover, if  $q \geq q_{\mu, c}$  then problem  $(P_\mu^\nu)$  has no solution when  $\nu$  is the Dirac measure.*

*Proof.* In view of Lemma 3.13, the first assertion follows from Theorem 3.7. The second assertion follows from Proposition 3.6. □

#### APPENDIX A. NON-UNIQUENESS FOR $C_H(\Omega) < \mu < 1/4$

We are going to show that for  $C_H(\Omega) < \mu < 1/4$  the problem

$$(P_\mu^0) \quad \begin{cases} -\mathcal{L}_\mu u + u^q = 0 & \text{in } \Omega, \\ \text{tr}_\mu^*(u) = 0, \end{cases}$$

admits a nontrivial solution. This was proved in [5, Theorem 5.3]. Here we provide a more direct argument.

Recall that if  $C_H(\Omega) < 1/4$  then the operator  $-\mathcal{L}_{C_H(\Omega)}$  admits a positive ground state solution  $\phi_H \in H_0^1(\Omega)$  such that  $-\mathcal{L}_{C_H(\Omega)}\phi_H = 0$  in  $\Omega$ , see [9].

**Proposition A.1.** *Assume that  $C_H(\Omega) < \mu < 1/4$  and  $q > 1$ . Then  $(P_\mu^0)$  admits a positive solution  $U_0$  such that*

$$\liminf_{x \rightarrow \partial\Omega} \frac{U_0(x)}{\phi_H(x)} > 0.$$

*Proof.* Since  $-\mathcal{L}_{C_H(\Omega)}\phi_H = 0$  in  $\Omega$ , for a small  $\tau > 0$  we obtain

$$-\mathcal{L}_\mu(\tau\phi_H) + (\tau\phi_H)^q = -\frac{\mu - C_H(\Omega)}{\delta^2}(\tau\phi_H) + (\tau\phi_H)^q \leq 0 \quad \text{in } \Omega,$$

so that  $\tau\phi_H$  is a subsolution for  $(P_\mu^0)$  in  $\Omega$ .

Fix  $\rho \in (0, \bar{\rho}]$ . Similarly to the proof of Theorem 3.7, for every  $k \geq 0$  denote  $\nu_{\rho, k} = kdS_{\Sigma_\rho}$  and let  $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$  be the measure such that  $\nu\mathbf{1}_{\partial\Omega} = 0$  and  $\nu\mathbf{1}_{\Sigma_\rho} = \nu_{\rho, k}$ . By Proposition 3.5 there exists a (unique) solution of  $(P_\mu^\nu(\rho))$  with this boundary data. Denote this solution by  $U_{0, k}$  and put

$$U_{0, \infty} = \lim_{k \rightarrow \infty} U_{0, k}.$$

Let  $R \in (0, \rho)$ . By Lemma 3.4 there exists a unique solution  $v_R$  of (3.5) in  $D_R$  with  $f = 2U_{0, \infty}$  on  $\Sigma_R$ . We define,

$$\bar{u} := \min\{U_{0, \infty}, u_R\} \quad \text{in } D_R \cap \Omega_\rho.$$

Then  $\bar{u}$  is a supersolution of  $(P_\mu)$  in  $D_R \cap \Omega_\rho$ ,  $\bar{u} = U_{0,\infty}$  in  $D_R \cap \Omega_{\rho'}$  for some  $\rho' \in (R, \rho)$  and  $\bar{u} = u_R$  in  $D_{R'} \cap \Omega_\rho$  for some  $R' \in (R, \rho')$ . Therefore setting  $\bar{u} = u_R$  in  $\Omega \setminus \Omega_\rho$  and  $\bar{u} = U_{0,\infty}$  in  $\Omega \setminus D_R$  provides an extension (still denoted by  $\bar{u}$ ) that is a supersolution of  $(P_\mu)$  in  $\Omega$ . As  $\bar{u} = U_{0,\infty}$  in a neighborhood of  $\partial\Omega$  it follows that  $\bar{u} \sim \delta^{\alpha_+}$  in such a neighborhood. On the other hand  $\phi_H \sim \delta^{a_+}$  where  $a_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - C_H(\Omega)}$ . As  $C_H(\Omega) < \mu$  it follows that  $\alpha_+ < a_+$  so that  $\delta^{\alpha_+} > \delta^{a_+}$ . Therefore  $\tau\phi_H < \bar{u}$  near  $\partial\Omega$  and therefore, by Lemma 3.1, everywhere in  $\Omega$ . Finally by Lemma 3.2 we conclude that there exists a solution  $U_0$  of  $(P_\mu)$  in  $\Omega$  such that  $\tau\phi_H < U_0 < \bar{u}$ . Thus  $U_0$  is a positive solution such that  $\text{tr}^*(U_0) = 0$ .  $\square$

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