Paper:
Zhang, D., Lei, Y., Wang, C. & Shen, Z. (2017). Vibration analysis of viscoelastic single-walled carbon nanotubes resting on a viscoelastic foundation. *Journal of Mechanical Science and Technology, 31*(1), 87-98.
http://dx.doi.org/10.1007/s12206-016-1007-7

Journal of Mechanical Science and Technology 00 (2010) 0000~0000
www.springerlink.com/content/1738-494x
submitted manuscript under review

Vibration analysis of viscoelastic single-walled carbon nanotubes resting on a viscoelastic foundation†

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(Manuscript Received 000 0, 2009; Revised 000 0, 2009; Accepted 000 0, 2009)

Abstract

Vibration responses are investigated for a viscoelastic single-walled carbon nanotube (visco-SWCNT) resting on a viscoelastic foundation. Based on the nonlocal Euler-Bernoulli beam model, velocity-dependent external damping and Kelvin viscoelastic foundation model, the governing equations are derived. The transfer function method (TFM) is then used to compute the natural frequencies for general boundary conditions and foundations. In particular, the exact analytical expressions of both complex natural frequencies and critical viscoelastic parameters are obtained for the Kelvin-Voigt visco-SWCNTs with full foundations and certain boundary conditions, and several physically intuitive special cases are discussed. Substantial nonlocal effect, the influence of geometric and physical parameters of the SWCNT and the viscoelastic foundation are observed for the natural frequencies of the supported SWCNTs. The study demonstrates the efficiency and robustness of the developed model for the vibration of the visco-SWCNT-viscoelastic foundation coupling system.

Keywords: Carbon nanotubes; Nonlocal viscoelastic theory; Vibration; Viscoelastic foundation

1. Introduction

Carbon nanotubes (CNTs) were discovered in 1991 by Iijima [1]. Their novel mechanical, thermal and chemical properties [2] make them promising in the fields of nanoelectromechanical systems (NEMS) [3], nanocomposites [4, 5], fluid storages [6], and nano/biosensors [7-9]. It is thus important to study the mechanical properties of CNTs [10]. Specially, the vibration characteristics of CNTs resting on a foundation [11-15] plays a crucial role in facilitating the development of nano/micro-sensors [16], fluid storages, and nanocomposites [17], etc. It thus becomes a major topic of great interest in recent research of nanomechanics. Indeed, the study of the dynamic response of CNTs on a foundation may provide valuable information for the above mentioned potential applications of CNTs.

In general, an experimental measurement still remains challenging for nanomaterials [18]. The computational simulation thus has been employed as an alternative way. For example molecular dynamic (MD) simulations are applied for the vibration analysis of CNTs. These techniques [3] however are found to be computationally expensive for a large nanostructure of many atoms [19]. To further enlarge the scope of the research, the modified continuum mechanics theories [20-22] accounting for the nanoscale effects have been used as cost-effective modeling tools. The nonlocal mechanics theory is one of the typical examples, which was established by Eringen [23-25] and first used for the nanomechanics of SWCNTs by Peddieson et.al. [26] and Sudak [27]. The last ten years has witnessed the fast growing of the research in the specific area. The nonlocal theory states that the stress at a reference point is affected not only by the strain at that point but also by the strains at every point of the domain. Incorporating the nonlocal theory into classical beam models leads to nonlocal Euler-Bernoulli beam and nonlocal Timoshenko beam theories [28, 29], which have been efficiently used to study the static deformation, buckling, wave propagation and vibration of CNTs. Specifically, in the study of dynamic characteristics for damped visco-CNTs Lei et al.[30] derived the governing equations and associated boundary conditions for CNTs based on the nonlocal Euler-Bernoulli beam theory and TFM. Similar issue was also investigated for a nanobeam by Lei and his coworkers [20] based on nonlocal Timoshenko beam theory. Ghasemi et al. [31] applied both nonlocal Euler-Bernoulli and Timoshenko beam theories for the

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buckling and post-buckling analyses of fluid conveying multi-walled CNTs (MWCNTs). The results showed large influence of the small-scale effects on the buckling flow velocities and post-buckled deflection. Ansari and Ramezannezhad [32] studied the large-amplitude vibrations of MWCNTs embedded in an elastic medium using nonlocal Timoshenko beam model. In addition to the beam theories, the nonlocal shell theory is also developed to study the mechanical responses of CNTs when their length-to-diameter ratio becomes small or local deformation is of major interest. Based on this theory Hoseinzadeh and Khadem [33] studied the dynamic behavior of double-walled CNTs (DWCNTs) where thermo-elastic damping was considered. The numerical results were obtained by using the Donnell-Mushtari-Vlasov approach and Galerkin method. The free vibration was also studied by Ansari et al. [34] for the DWCNTs with arbitrary boundary conditions by combining the nonlocal shell theory and MD simulations.

It is noted that CNTs are embedded in a medium [12, 13, 35], in many of their applications in nanotechnology, to name a few, fluid storages, nanosensors and nanocomposites. Such a system can be reasonably treated as a nanobeam resting on a foundation. To characterize the foundation, Winkler model was developed [36]. Mustapha and Zhong [17] investigated the thermo-mechanical vibration of SWCNTs embedded in a Winkler-type elastic medium. Different boundary conditions were considered for the SWCNTs and the governing equations were solved via Bubnov-Galerkin method. Based on Love's shell assumptions, Torkaman-Asadi et al. [11] derived the governing equations for the SWCNs resting on Winkler elastic foundations. In addition, Kiani [14] studied the free transverse vibration of a stocky SWCNT resting on an elastic foundation. The models used included the nonlocal Rayleigh, Timoshenko and other higher-order beam models. More complicated foundation models [37, 38] have also been developed, such as Pasternak foundation model and Kerr foundation model. Soltani and Farshidianfar [35] presented an energy balance method to investigate the nonlinear vibration of a SWCNT conveying fluid on Pasternak-type elastic foundation. Wu and Lai [39] employed the nonlocal Timoshenko beam theory and differential reproducing kernel (DRK) method for the SWCNTs embedded in a Pasternak-type elastic medium. Mehdipour et al. [13] investigated the nonlinear vibration characteristics for a curved SWCNT lying on a Pasternak-type elastic foundation. The calculation was done by using He's Energy Balance Method (HEBM). The vibration of a nanocone embedded in Winkler and Pasternak medium was investigated in [40] by Fotouhi et al. using a nonlocal shell model. Furthermore, Ghavanloo et al. [15] examined the effects of the internal moving fluid on the vibration and instability of SWCNTs on a linear viscoelastic Winkler foundation. Zeighampour and Beni [41] developed a Donnell's shell model to analyze the vibration and instability of a fluid-conveying DWCNT embedded in a visco-Pasternak foundation. A modified couple stress theory was employed in the study.

As reviewed above, in the analysis of dynamic properties of CNTs the foundations supporting or hosting the CNTs are usually modeled as Winkler-type or Pasternak-type elastic foundation model. Nevertheless, most of the materials seen in the applications of CNTs normally demonstrate viscoelastic behavior. The typical examples are polymer matrix in nanocomposites [12], PMMA substrate in the mechanical property test [42], and biological soft tissue to which biosensors are attached [43]. In these cases, Winkler or Pasternak model may lead to substantial errors due to the viscoelastic properties of the foundation. Accordingly, the new modeling technique is urgently required to accurately characterize and correctly understand the dynamical behavior of the CNTs resting on viscoelastic foundation.

In the present paper, the transverse vibration is studied for a SWCNT resting on a viscoelastic foundation. The analyses are based on the nonlocal Euler-Bernoulli beam model, general Maxwell model and Kelvin viscoelastic foundation model (Fig. 1). In section 3 the governing equations are derived based on nonlocal Euler-Bernoulli beam theory. The exact analytical expressions of the natural frequencies for certain boundary conditions and viscoelastic foundations are shown in section 4. By using the TFM, the natural frequencies in closed form for SWCNTs with arbitrary boundary conditions are presented in section 5. A detailed parametric study is conducted in section 6 and finally the conclusions are drawn in section 7.

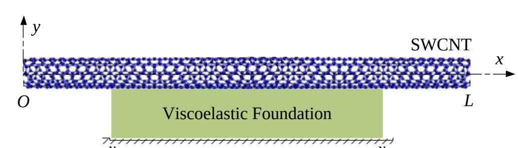


Fig. 1 A SWCNT resting on partial viscoelastic foundation and modeled as nonlocal Euler-Bernoulli beam.

2. Nonlocal viscoelastic model

For a linear and homogeneous elastic solid, the relationship between the local elastic stress tensor σ_{kl} and the nonlocal elastic stress tensor t_{kl} can be written as [23-25]

$$t_{kl} = \int_V K\left(\left|x - x'\right|, \phi\right)\sigma_{kl}\mathrm{d}V \tag{1}$$

where $K\left(\left|x - x'\right|, \phi\right)$ is the nonlocal kernel function. The non-dimensional term ϕ is $\alpha = e_0 a / l_e$, in which e_0 is a constant for individual materials, a and l_e are the internal and external characteristic lengths, respectively. The local elastic stress tensor σ_{kl} is given by

$$\sigma_{kl} = \lambda\varepsilon_{rr}\delta_{kl} + 2\mu\varepsilon_{kl},\ \varepsilon_{kl} = \left(u_{k,l} + u_{l,k}\right)/2 \tag{2}$$

where ε_{kl} is the strain tensor, λ and μ are the Lame's constants. For the special kernel function introduced by Eringen [23], the differential form of Eq. (1) for the nonlocal elastic materials is as follows

$$\left[1-(e_0a)^2\nabla^2\right]t_{kl}=\sigma_{kl} \tag{3}$$

The above constitutive equation for nonlocal elastic materials can be directly extended to a nonlocal viscoelastic material as [20, 44]

$$\left[1-(e_0a)^2\nabla^2\right]t^*_{kl}=\sigma^*_{kl} \tag{4}$$

where t^*_{kl} and σ^*_{kl} are nonlocal and local viscoelastic stress tensor, respectively.

In order to study the dynamic responses of visco-CNTs, the general Maxwell model (Fig. 2) is used for vibration analysis of visco-CNTs [20, 30], and the corresponding extensional relaxation modulus is

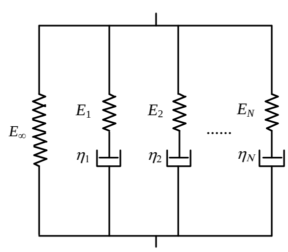


Fig. 2 The general Maxwell model for linear viscoelasticity.

$$E(t)=E_\infty+\sum_{m=1}^{N}E_m e^{-\frac{t}{\tau_m}} \tag{5}$$

where $\tau_m=\eta_m/E_m$. It is noted that the above equation for general Maxwell model can be reduced to a Kelvin-Voigt viscoelasticity model by taking N=1 and $E_1=\infty$ as well as to a three-parameter standard viscoelasticity model [30] when N=1. The general Maxwell model considered here is therefore a generalization of some conventional viscoelastic models. The integral constitutive equation for SWCNTs can be obtained via the Boltzmann superposition principle as

$$\sigma^*_{xx}=E_\infty\varepsilon_{xx}(t)+\int_0^t\sum_{m=1}^{N}E_m e^{\frac{\tau-t}{\tau_m}}\frac{\partial\varepsilon_{xx}(\tau)}{\partial\tau}\mathrm{d}\tau \tag{6}$$

Using this constitutive equation, we shall derive the governing equation for the SWCNT in the following section.

3. Governing equation of motion

Here let us consider a SWCNT of length L along the x direction resting on a viscoelastic foundation, whose extent is denoted by x_1 and x_2, respectively (Fig. 1). The SWCNT is modeled as nonlocal Euler-Bernoulli beam with a section area A and a mass density ρ. The foundation is described by a Kelvin viscoelastic foundation model [37] whose damping considered here is non-viscous. Based on nonlocal Euler-Bernoulli beam theory [30] and variational principle [45], the governing equation can be derived as

$$\begin{aligned}&E_\infty I\frac{\partial^4 w}{\partial x^4}+\int_0^t\sum_{m=1}^{N}E_m I e^{\frac{\tau-t}{\tau_m}}\frac{\partial^5 w}{\partial x^4\partial\tau}\mathrm{d}\tau+\rho A\left[1-(e_0a)^2\nabla^2\right]\frac{\partial^2 w}{\partial t^2}\\&+\left[1-(e_0a)^2\nabla^2\right]\left[kw(x,t)+C_0\int_0^t\mu_0 e^{-\mu_0(t-\tau)}\frac{\partial w}{\partial\tau}\mathrm{d}\tau\right]H(x)\\&+C_2\left[1-(e_0a)^2\nabla^2\right]\int_0^t\mu_2 e^{-\mu_2(t-\tau)}\frac{\partial w}{\partial\tau}\mathrm{d}\tau=\left[1-(e_0a)^2\nabla^2\right]p\end{aligned} \tag{7}$$

where I is the second moment of the cross section, w is the transverse deflection, C_2 is the damping coefficient of the external damping, p is an external distributed force, and the Heaviside step function $H(\cdot)$ denotes the presence of the foundation. In addition, k and C_0 are the stiffness and damping coefficients of the viscoelastic foundation, respectively. It is noted that Eq. (7) is the governing equation in general case, which can be easily reduced to the following equation for a damped nonlocal viscoelastic Euler-Bernoulli beam by assuming $H(x)$=0, i.e., no foundation support [30].

$$\begin{aligned}&E_\infty I\frac{\partial^4 w}{\partial x^4}+\int_0^t\sum_{m=1}^{N}E_m I e^{\frac{\tau-t}{\tau_m}}\frac{\partial^5 w}{\partial x^4\partial\tau}\mathrm{d}\tau+\rho A\left[1-(e_0a)^2\nabla^2\right]\frac{\partial^2 w}{\partial t^2}\\&+C_2\left[1-(e_0a)^2\nabla^2\right]\int_0^t\mu_2 e^{-\mu_2(t-\tau)}\frac{\partial w}{\partial\tau}\mathrm{d}\tau=\left[1-(e_0a)^2\nabla^2\right]p\end{aligned} \tag{8}$$

Eq. (7) can be further reduced to the equation for undamped nonlocal Euler-Bernoulli beams at C_2=0 and $\tau_m\to 0$ [28]

$$E_\infty I\frac{\partial^4 w}{\partial x^4}+\rho A\left[1-(e_0a)^2\nabla^2\right]\frac{\partial^2 w}{\partial t^2}=\left[1-(e_0a)^2\nabla^2\right]p \tag{9}$$

The solution of the Eq. (7) is taken as $w=W(x)\mathrm{e}^{i\omega t}$, based on which the natural frequencies can be obtained for free vibration of the SWCNT. Here, ω and $W(x)$ are, respectively, the complex natural frequency and the corresponding mode shape. Substituting w into Eq. (7) yields

$$\begin{aligned}&c^2\left(1+i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{\mathrm{d}^4\overline{W}}{\mathrm{d}\overline{x}^4}\\&+\left[\omega^2-i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}-\left(k^*+i\omega\zeta_0\frac{\mu_0}{i\omega+\mu_0}\right)H(\overline{x})\right]\alpha^2\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\overline{x}^2}\\&+\left[i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}+\left(k^*+i\omega\zeta_0\frac{\mu_0}{i\omega+\mu_0}\right)H(\overline{x})-\omega^2\right]\overline{W}=0\end{aligned} \tag{10}$$

where $\zeta_0 = C_0 / \rho A$, $\zeta_2 = C_2 / \rho A$, $c = \sqrt{EI / \rho A} / L^2$, $k^* = k / \rho A$, $\alpha = e_0 a / L$, $\bar{x} = x / L$ and $\bar{W} = W / L$. It can be seen from Eq. (10) that the governing equation is a fourth-order ordinary differential equation for $\bar{W}(\bar{x})$. And the coefficients in front of $\bar{W}(\bar{x})$ and its derivatives are the functions of ω. Accordingly, the bending moment M and the shear force Q are given by

$$\begin{aligned} M(\bar{x},t) = \rho A L^3 \Bigg\{ & c^2 \left(1 + i\omega \sum_{m=1}^{N} \frac{E_m}{E_\infty} \frac{\tau_m}{1 + i\omega\tau_m}\right) \frac{\mathrm{d}^2 \bar{W}}{\mathrm{d}\bar{x}^2} \\ & + \alpha^2 \left[\omega^2 - i\omega\zeta_2 \frac{\mu_2}{i\omega + \mu_2} - \left(k^* + i\omega\zeta_0 \frac{\mu_0}{i\omega + \mu_0}\right) H(\bar{x}) \right] \bar{W} \Bigg\} \end{aligned} \quad (11)$$

$$\begin{aligned} Q(\bar{x},t) = \rho A L^2 \Bigg\{ & c^2 \left(1 + i\omega \sum_{m=1}^{N} \frac{E_m}{E_\infty} \frac{\tau_m}{1 + i\omega\tau_m}\right) \frac{\mathrm{d}^3 \bar{W}}{\mathrm{d}\bar{x}^3} \\ & + \alpha^2 \left[\omega^2 - i\omega\zeta_2 \frac{\mu_2}{i\omega + \mu_2} - \left(k^* + i\omega\zeta_0 \frac{\mu_0}{i\omega + \mu_0}\right) H(\bar{x}) \right] \frac{\mathrm{d}\bar{W}}{\mathrm{d}\bar{x}} \Bigg\} \end{aligned} \quad (12)$$

It is noted that the mode shape $\bar{W}(\bar{x})$ can be derived by solving Eq. (10) in conjunction with the required boundary conditions. For a fourth-order ordinary differential equation with constant coefficients, the solution is taken as a sum of four terms in the form $\bar{W} = \bar{W}_n e^{-i\beta\bar{x}}$. Substituting it into Eq. (10), one can obtain

$$\begin{aligned} & c^2 \left(1 + i\omega \sum_{m=1}^{N} \frac{E_m}{E_\infty} \frac{\tau_m}{1 + i\omega\tau_m}\right) \beta^4 \\ & + \left[i\omega\zeta_2 \frac{\mu_2}{i\omega + \mu_2} - \omega^2 + \left(k^* + i\omega\zeta_0 \frac{\mu_0}{i\omega + \mu_0}\right) H(\bar{x}) \right] \alpha^2 \beta^2 \\ & + \left[i\omega\zeta_2 \frac{\mu_2}{i\omega + \mu_2} - \omega^2 + \left(k^* + i\omega\zeta_0 \frac{\mu_0}{i\omega + \mu_0}\right) H(\bar{x}) \right] = 0 \end{aligned} \quad (13)$$

When there is a full foundation or no foundation in the system, the required solution for β can be determined for a few specific boundary conditions [30] (e.g., clamped, hinged or free). For example hinged-hinged boundary conditions give $\beta = k\pi$ for the kth mode, which is independent of nonlocal parameter α. However, for other specific boundary conditions it should be noted that the value of β depends not only on the boundary conditions but also on the nonlocal parameter α. For a given β, Eq. (13) becomes a polynomial of order N+2 for ω, which can be solved to obtain the natural frequencies of the SWCNTs. The natural frequencies for general boundary conditions and foundations are obtained by using the TFM described later.

4. Natural frequency analysis for special cases

In this section, let us consider the damping of a viscous system characterized by $\mu_0 \to \infty$ and $\mu_2 \to \infty$. The general Maxwell model can also be simplified to the Kelvin-Voigt viscoelastic model by taking N=1 and $E_1 \to \infty$. Then Eq. (13) becomes

$$\begin{aligned} & -\left(1 + \alpha^2\beta^2\right)\omega^2 + \left[\left(\zeta_0 H(\bar{x}) + \zeta_2\right)\left(\alpha^2\beta^2 + 1\right) + \tau_d c^2 \beta^4 \right] i\omega \\ & + \left[c^2\beta^4 + k^* \left(\alpha^2\beta^2 + 1\right) H(\bar{x}) \right] = 0 \end{aligned} \quad (14)$$

where $\tau_d = \eta_1 / E_\infty$. In the following subsections, two particular cases will be studied in the presence of a full foundation or the absence of a foundation.

4.1 SWCNTs without foundations

In this section, the SWCNT will be studied without the support of a foundation. The characteristic equation can be obtained by putting $H(\bar{x}) = 0$ into Eq. (14).

$$-\left(1 + \alpha^2\beta^2\right)\omega^2 + \left(\zeta_2 + \alpha^2\beta^2\zeta_2 + \tau_d c^2\beta^4\right) i\omega + c^2\beta^4 = 0 \quad (15)$$

Based on this quadratic equation, the explicit formula below can be derived for ω:

$$\begin{aligned} \omega_n = & \left[\frac{\zeta_2}{2} + \frac{c^2\beta^4}{2\left(1 + \alpha^2\beta^2\right)} \tau_d \right] i \\ & \mp \sqrt{ -\left[\frac{\zeta_2}{2} + \frac{c^2\beta^4}{2\left(1 + \alpha^2\beta^2\right)} \tau_d \right]^2 + \frac{c^2\beta^4}{1 + \alpha^2\beta^2} } \end{aligned} \quad (16)$$

Eq. (16) turns out to be the same as the one obtained in [30] for a damped viscoelastic nonlocal Euler-Bernoulli beam.

4.2 SWCNTs supported by a full foundation

This section is focused on a SWCNT with a full viscoelastic foundation. With $H(\bar{x}) = 1$ Eq. (14) becomes

$$\begin{aligned} & -\left(1 + \alpha^2\beta^2\right)\omega^2 + \left[\left(\zeta_0 + \zeta_2\right)\left(\alpha^2\beta^2 + 1\right) + \tau_d c^2\beta^4 \right] i\omega \\ & + \left[c^2\beta^4 + k^*\left(\alpha^2\beta^2 + 1\right) \right] = 0 \end{aligned} \quad (17)$$

The equation leads to the analytical solutions shown in the following equation

$$\begin{aligned} \omega_n = & \left[\frac{\zeta_0 + \zeta_2}{2} + \frac{c^2\beta^4}{2\left(1 + \alpha^2\beta^2\right)} \tau_d \right] i \\ & \mp \sqrt{ -\left[\frac{\zeta_0 + \zeta_2}{2} + \frac{c^2\beta^4}{2\left(1 + \alpha^2\beta^2\right)} \tau_d \right]^2 + \frac{c^2\beta^4}{1 + \alpha^2\beta^2} + k^* } \end{aligned} \quad (18)$$

Eq. (18) gives the complex natural frequencies in general case, which enables one to discuss some typical special cases below.

(i) *Undamped local elastic SWCNT with a full elastic foundation*: This leads to $\alpha=\tau_d=\zeta_0=\zeta_2=0$ in Eq. (18). One can thus arrive at

$$\omega_n = c\beta^2\sqrt{1+\frac{k^*}{c^2\beta^4}} \tag{19}$$

where the effect of an elastic foundation is taken into consideration as compared with the classical Euler-Bernoulli beam theory, i.e. $\omega_n = c\beta^2$.

(ii) *Damped local elastic SWCNT with a full viscoelastic foundation*: This system is associated with $\alpha=\tau_d=0$ which transforms Eq. (18) into the following equation

$$\omega_n = \frac{\zeta_0+\zeta_2}{2}i + c\beta^2\sqrt{-\left(\frac{\zeta_0+\zeta_2}{2c\beta^2}\right)^2+1+k^*} \tag{20}$$

At $k^*=\zeta_0=0$, Eq. (20) reduces to the natural frequencies for a classical viscously damped Euler-Bernoulli beam.

(iii) *Undamped local viscoelastic SWCNT with a full elastic foundation*: Substitute $\alpha=\zeta_0=\zeta_2=0$ into Eq. (18), the complex natural frequencies read

$$\omega_n = \frac{c^2\beta^4}{2}\tau_d i + c\beta^2\sqrt{-\left(\frac{c\beta^2}{2}\tau_d\right)^2+1+\frac{k^*}{c^2\beta^4}} \tag{21}$$

The critical value of the parameter τ_d for non-oscillatory solutions can be obtained as

$$(\tau_d)_{\text{crit}} = \frac{2}{c\beta^2}\sqrt{1+\frac{k^*}{c^2\beta^4}} \tag{22}$$

(iv) *Undamped nonlocal elastic SWCNT with a full elastic foundation*: The formula for this special case can be obtained by substituting $\tau_d=\zeta_0=\zeta_2=0$ into Eq. (18).

$$\omega_n = c\beta^2\sqrt{\frac{1}{1+\alpha^2\beta^2}+\frac{k^*}{c^2\beta^4}} \tag{23}$$

At $k^*=0$, Eq. (23) shows the natural frequencies for an undamped nonlocal elastic SWCNT without foundations as also shown in [46]. It should be noted that the analytical solutions for the natural frequencies mentioned above is only available as β is known for different boundary conditions and nonlocal parameters.

5. Transfer function method for the SWCNT

In the presence of the viscoelastic foundation, the SWCNT can be divided into two components. The first one in $[\bar{x}_1,\bar{x}_2]$ is supported by the foundation, where Eq. (13) becomes

$$\begin{aligned}&c^2\left(1+i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{d^4\bar{W}}{d\bar{x}^4}\\&+\left[\omega^2-i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}-\left(k^*+i\omega\zeta_0\frac{\mu_0}{i\omega+\mu_0}\right)\right]\alpha^2\frac{d^2\bar{W}}{d\bar{x}^2}\\&+\left[i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}+\left(k^*+i\omega\zeta_0\frac{\mu_0}{i\omega+\mu_0}\right)-\omega^2\right]\bar{W}=0\end{aligned} \tag{24}$$

The second component in $[0,\bar{x}_1)\cup(\bar{x}_2,1]$ stands freely. Its equation for ω is

$$\begin{aligned}&c^2\left(1+i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m}\right)\frac{d^4\bar{W}}{d\bar{x}^4}+\left[i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}-\omega^2\right]\bar{W}\\&+\left[\omega^2-i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2}\right]\alpha^2\frac{d^2\bar{W}}{d\bar{x}^2}=0\end{aligned} \tag{25}$$

Generally, the analytical solutions are not available. Thus, the TFM is used to calculate the eigenfrequencies of the SWCNT with arbitrary boundary conditions.

5.1 The component of the damped SWCNT in touch with foundations

To achieve the eigenvalues and frequency response functions, one can define the state vector $\boldsymbol{\eta}(x,\omega)$ as

$$\boldsymbol{\eta}(x,\omega)=\left[\bar{W},\frac{d\bar{W}}{d\bar{x}},\frac{d^2\bar{W}}{d\bar{x}^2},\frac{d^3\bar{W}}{d\bar{x}^3}\right]^T \tag{26}$$

Thus Eq. (24) can be rewritten in matrix form as

$$\frac{d\boldsymbol{\eta}(x,\omega)}{d\bar{x}}=\boldsymbol{\Phi}(\omega)\boldsymbol{\eta}(x,\omega) \tag{27}$$

where

$$\boldsymbol{\Phi}(\omega)=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\frac{A_2-A_3}{c^2A_1}&0&\frac{\alpha^2[-A_2+A_3]}{c^2A_1}&0\end{bmatrix} \tag{28}$$

$$\begin{aligned}&A_1=1+i\omega\sum_{m=1}^{N}\frac{E_m}{E_\infty}\frac{\tau_m}{1+i\omega\tau_m},\ A_2=\omega^2-i\omega\zeta_2\frac{\mu_2}{i\omega+\mu_2},\\&A_3=k^*+i\omega\zeta_0\frac{\mu_0}{i\omega+\mu_0}\end{aligned} \tag{29}$$

The solution of Eq. (27) can be calculated as

$$\boldsymbol{\eta}(x,\omega)=\mathrm{e}^{\boldsymbol{\Phi}(\omega)(\bar{x}-\bar{x}_1)}\boldsymbol{\eta}_1(\omega) \tag{30}$$

where $\boldsymbol{\eta}_1(\omega)=\boldsymbol{\eta}(\bar{x}_1,\omega)$. Setting $\bar{x}=\bar{x}_2$, we have

$$\boldsymbol{\eta}_2(\omega)=\boldsymbol{\Psi}(\omega)\boldsymbol{\eta}_1(\omega) \tag{31}$$

where $\boldsymbol{\eta}_2(\omega)=\boldsymbol{\eta}(\bar{x}_2,\omega)$ and $\boldsymbol{\Psi}(\omega)=\mathrm{e}^{\boldsymbol{\Phi}(\omega)(\bar{x}_2-\bar{x}_1)}$.

5.2 Solutions of the complete SWCNT

Similarly to Eq. (27), Eq. (25) can be rewritten by using the state vector $\boldsymbol{\eta}(x,\omega)$

$$\frac{\mathrm{d}\boldsymbol{\eta}(x,\omega)}{\mathrm{d}\bar{x}}=\boldsymbol{\Phi}^*(\omega)\boldsymbol{\eta}(x,\omega) \tag{32}$$

where

$$\boldsymbol{\Phi}^*(\omega)=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\ \dfrac{A_2}{c^2A_1}&0&\dfrac{-\alpha^2A_2}{c^2A_1}&0\end{bmatrix} \tag{33}$$

The solution of Eq. (32) can be also obtained as

$$\boldsymbol{\eta}(x,\omega)=\begin{cases}\mathrm{e}^{\boldsymbol{\Phi}^*(\omega)\bar{x}}\boldsymbol{\eta}_L(\omega)\ ,\ \bar{x}\in[0,\bar{x}_1)\\ \mathrm{e}^{\boldsymbol{\Phi}^*(\omega)(\bar{x}-\bar{x}_2)}\boldsymbol{\eta}_2(\omega)\ ,\ \bar{x}\in(\bar{x}_2,1]\end{cases} \tag{34}$$

where $\boldsymbol{\eta}_L(\omega)=\boldsymbol{\eta}(0,\omega)$. Eq. (34) finally gives

$$\boldsymbol{\eta}_1(\omega)=\boldsymbol{T}_L(\omega)\boldsymbol{\eta}_L(\omega),\boldsymbol{\eta}_R(\omega)=\boldsymbol{T}_R(\omega)\boldsymbol{\eta}_2(\omega) \tag{35}$$

where $\boldsymbol{T}_L(\omega)=\mathrm{e}^{\boldsymbol{\Phi}^*(\omega)\bar{x}_1}$ and $\boldsymbol{T}_R(\omega)=\mathrm{e}^{\boldsymbol{\Phi}^*(\omega)(\bar{x}_R-\bar{x}_2)}$. Substituting Eq. (31) into Eq. (35) yields

$$\boldsymbol{\eta}_R(\omega)=\boldsymbol{T}_R(\omega)\boldsymbol{\Psi}(\omega)\boldsymbol{T}_L(\omega)\boldsymbol{\eta}_L(\omega) \tag{36}$$

The boundary conditions of the SWCNT are expressed as

$$\boldsymbol{M}(\omega)\boldsymbol{\eta}_L(\omega)+\boldsymbol{N}(\omega)\boldsymbol{\eta}_R(\omega)=\mathbf{0} \tag{37}$$

where $\boldsymbol{M}(\omega)$ and $\boldsymbol{N}(\omega)$ are boundary condition set matrices at the two ends of the SWCNT, i.e., $\bar{x}=0$ and $\bar{x}=1$. To demonstrate the technique, several typical boundary conditions at $\bar{x}=0$ are shown below, from which the matrices of other boundary conditions can be easily obtained.

For the clamped boundary condition, $\boldsymbol{M}(\omega)$ is given by

$$\boldsymbol{M}(\omega)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{38}$$

For the simply supported boundary condition, one has

$$\boldsymbol{M}(\omega)=\begin{bmatrix}1&0&0&0\\ \alpha^2\left[A_2-A_3H(\bar{x})\right]&0&c^2A_1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{39}$$

For the free end, $\boldsymbol{M}(\omega)$ is

$$\boldsymbol{M}(\omega)=\begin{bmatrix}\alpha^2\left[A_2-A_3H(\bar{x})\right]&0&c^2A_1&0\\0&\alpha^2\left[A_2-A_3H(\bar{x})\right]&0&c^2A_1\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{40}$$

Substituting Eq. (36) into Eq. (37) leads to

$$\left[\boldsymbol{M}(\omega)+\boldsymbol{N}(\omega)\boldsymbol{T}_R(\omega)\boldsymbol{\Psi}(\omega)\boldsymbol{T}_L(\omega)\right]\boldsymbol{\eta}_L(\omega)=\mathbf{0} \tag{41}$$

The natural frequencies of the visco-SWCNT can then be obtained by solving the following transcendental characteristic equation

$$\det\left[\boldsymbol{M}(\omega)+\boldsymbol{N}(\omega)\boldsymbol{T}_R(\omega)\boldsymbol{\Psi}(\omega)\boldsymbol{T}_L(\omega)\right]=\mathbf{0} \tag{42}$$

6. Numerical results and discussion

Here to validate the formulation achieved in prior section we shall first carry out an analysis for a benchmark case, i.e., a free standing elastic SWCNT. This is followed by a study of a SWCNT resting on a viscoelastic foundation, where the effect of both geometrical and material properties of the SWCNT and foundations is examined for the natural frequencies of the SWCNT with different boundary conditions. The values of some parameters used are as follows: length of the SWCNT L=11nm, diameter d=1.1nm, effective thickness δ_{th}=0.342nm, Young's modulus E=1TPa, mass density ρ=2.24g/cm^3, nonlocal parameter $\alpha\in[0,\ 0.3]$ and structural damping coefficient $\tau_d\in[0,\ 2\times10^{-4}]$. Moreover, stiffness coefficient and damping coefficient of the foundation are set respectively to $k^*\in[0,\ 5\times10^5]$ and $\zeta_0\in[0,\ 200]$. To simplify our analysis, the external damping coefficient is taken as ζ_2=0. These geometrical and material properties of the SWCNT are adopted from the papers [7, 20, 30].

Based on the nonlocal Euler-Bernoulli beam theory and TFM, the transverse vibration of a free standing SWCNT is investigated by Lei et al. [30]. To validate the proposed method, the obtained results are compared with those of Lei et al. [30]. From Table 1 it can be seen that the results of present study agree with those in Ref. [30], which verifies the accuracy of the proposed method for vibration analysis.

Also, there are some other analytical approaches to study the free vibration of SWCNTs. For example, Lu et al. [46] derived the governing equations of motion for nanobeams based on nonlocal Euler-Bernoulli beam model. In the study, the governing equations were solved to obtain the natural frequencies and corresponding mode shapes by an analytic method. In addition, a nonlocal Timoshenko beam model was applied by Wang and his coworkers [47] to study the free vibration of micro/nanobeams. The governing equations and the boundary conditions for the micro/nanobeams were derived using Hamilton's principle and the equations were solved analytically. Using the analytical approaches developed in Refs. [46] and [47] respectively, we calculate the first three frequency parameters $\sqrt{\omega / c}$ for a free standing SWCNT with various boundary conditions, such as clamped-clamped (C-C), simply supported-simply supported (S-S) and clamped-free (C-F). The obtained results are presented in Table 2, where excellent agreement is achieved. Furthermore, from Table 2 we can see that the natural frequencies of SWCNTs are generally decreasing with rising nonlocal parameter. Exception is the first natural frequencies for C-F SWCNTs.

For future comparisons with other investigators, the first three natural frequencies of the visco-SWCNTs with various boundary conditions are listed in Table 3 based on the TFM. Furthermore, the natural frequencies of S-S SWCNTs are also calculated based on analytic method. From Table 3 it is found that the numerical results for S-S SWCNTs are nearly the same as analytical solutions based on Eq. (18). The significant effects of both nonlocal parameter α and boundary conditions are also observed in Table 3. In the following text, a detailed parametric study will be conducted to examine the effect of various internal and external factors.

Table 1. The first three natural frequencies (GHz) of free standing undamped elastic SWCNTs ($k^{*}=0$, $\tau_{d}=0$) with simply supported boundary conditions in comparison with those of Ref. [30].

Mode	α=0		α=0.1		α=0.2	
	Present	Ref. [30]	Present	Ref. [30]	Present	Ref. [30]
1	111.711	111.71	106.575	106.58	94.589	94.589
2	446.843	446.84	378.357	378.36	278.239	278.24
3	1005.397	1005.4	731.647	731.65	471.179	471.18

Table 2. The first three frequency parameters $\sqrt{\omega / c}$ for free standing elastic SWCNTs ($k^{*}=0$, $\tau_{d}=0$) with various boundary conditions in comparison with those of Refs. [46, 47].

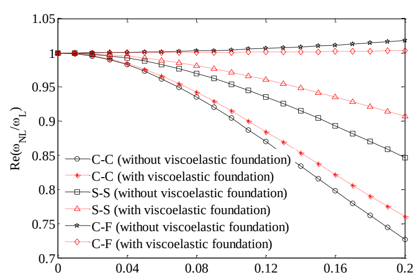
BCs	α=0			α=0.1			α=0.3		
	Present	Ref. [46]	Ref. [47]	Present	Ref. [46]	Ref. [47]	Present	Ref. [46]	Ref. [47]
C-F	1.8751	1.8751	1.8751	1.8792	1.8792	1.8792	1.9154	1.9154	1.9154
	4.6941	4.6941	4.6941	4.5475	4.5475	4.5475	3.7665	3.7665	3.7665
	7.8548	7.8548	7.8548	7.1459	7.1459	7.1459	5.2988	5.2988	5.2988
S-S	3.1416	3.1416	3.1416	3.0685	3.0685	3.0685	2.6800	2.6800	2.6800
	6.2832	6.2832	6.2832	5.7817	5.7817	5.7817	4.3013	4.3013	4.3013
	9.4248	9.4248	9.4248	8.0400	8.0400	8.0400	5.4422	5.4422	5.4422
C-C	4.7300	4.7300	4.7300	4.5945	4.5945	4.5945	3.9184	3.9184	3.9184
	7.8532	7.8532	7.8532	7.1402	7.1402	7.1402	5.1963	5.1963	5.1963
	10.9956	10.9956	10.9956	9.2583	9.2583	9.2583	6.2317	6.2317	6.2317

Table 3. The first three natural frequencies (GHz) of undamped visco-SWCNTs with a full elastic foundation ($k^{*}=3\times10^{5}$, $\tau_{d}=10^{-4}$).

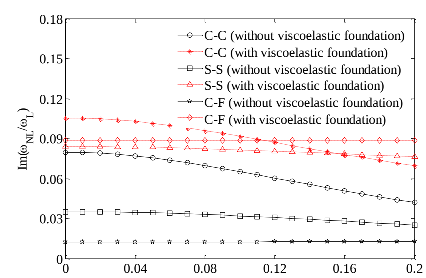
α	C-F	C-C	S-S	
			TFM	Analytical
0.0	95.826+0.498i	265.05+21.15i	141.64+3.920i	141.64+3.920i
	263.47+19.54i	715.1+160.74i	450.92+62.73i	450.92+62.73i
	686.9+153.20i	1297.3+617.74i	957.9+317.56i	957.9+317.56i
0.1	95.898+0.502i	239.74+17.29i	137.64+3.568i	137.64+3.568i
	249.18+17.212i	567.81+99.42i	385.66+44.97i	385.66+44.97i
	575.01+104.95i	925.44+279.64i	717.4+168.18i	717.4+168.18i
0.2	96.126+0.516i	192.83+11.16i	128.60+2.811i	128.60+2.811i
	216.84+12.433i	390.91+46.36i	290.56+24.32i	290.56+24.32i
	422.19+54.54i	585.29+105.85i	474.07+69.75i	474.07+69.75i

Fig. 3 shows the influence of nonlocal parameter α on the vibration of a SWCNT in the presence of a full viscoelastic foundation or the absence of a foundation. The normalized frequency is calculated, which is defined as ω_{NL}/ω_{L}, where ω_{NL} and ω_{L} are the natural frequencies of the nonlocal and local (classical) systems respectively. For numerical calculations here, the basic parameters used for SWCNT remain the same as those motioned above except that ζ_0=100. It is understood that the real and the imaginary parts of the complex natural frequencies are related to damped frequencies and damping ratios, respectively.

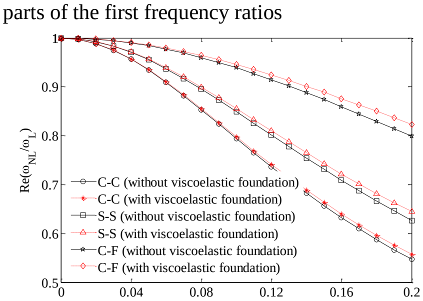
The α-dependence of the real and the imaginary parts of the first frequency ratios are calculated for the SWCNT and the results are plotted in Fig. 3 a) and Fig. 3 b). From the figures it can be observed that both parts of the frequency ratios decrease monotonically with rising nonlocal parameter α for both C-C and S-S boundary conditions. Thus, the rigidity of the system is reduced due to enhanced nonlocal effect as C-C or S-S boundary conditions are considered. Specially, the fundamental frequency ratios of the C-F SWCNTs are found to increase as α increases, which shows hardening effect for the C-F boundary conditions. A similar phenomenon is also described by Lei et al. [30] and Lu et al. [46]. Compared with the imaginary parts, the real parts are more sensitive to the variation of the nonlocal parameter. Furthermore, the influence of the nonlocal effect on damped frequencies and damping ratios turns out to be less pronounced when the softer constraints are imposed on the boundaries. It also can be seen that the effect of α on the real parts of the frequency ratios is more substantial in the presence of viscoelastic foundations. The effects of nonlocal parameter on the higher natural frequencies are also calculated in Fig. 3 c) to Fig. 3 d) where the rising nonlocal parameter again leads to the down-shift of the damped frequencies and damping ratios. In particular, the nonlocal effect on these higher natural frequency ratios turns out to be much larger than its influence on the fundamental ones. It also can be seen that the curves shown in Fig. 3 d) tend to converge slowly with an increase in nonlocal parameter. This suggests that the imaginary parts of natural frequencies or damping ratios are less sensitive to the boundary conditions when nonlocal parameter increases. Thus the damping effect becomes small at nanoscale when the nonlocal atom-atom interaction is significant. In addition, as expected the effect of viscoelastic foundations on both parts of higher frequency ratios becomes smaller.



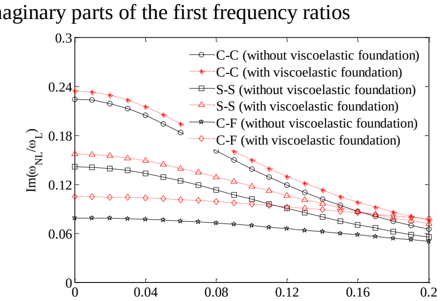
a) The real parts of the first frequency ratios



b) The imaginary parts of the first frequency ratios



c) The real parts of the second frequency ratios



d) The imaginary parts of the second frequency ratios

Fig. 3 Effect of nonlocal parameter on the first two frequency ratios for the SWCNT considering different boundary conditions with and without foundations supporting.

Next let us examine the effect of aspect ratio *L/d* and structural damping coefficient τ_d on the vibration of S-S SWCNTs with a full viscoelastic foundation (Fig. 4 and Fig. 5). In this calculation, the parameters used for the SWCNT and the foundation are the same as shown above. The variations of the real parts of the first three frequency ratios for different values of nonlocal parameter and structural damping coefficient are plotted in Fig. 4. From the figure we can observe that the real parts of the first two frequency ratios increase monotonically as aspect ratio *L/d* increases. This implies that the nonlocal effect on the damped frequencies decrease with rising aspect ratio *L/d* as the nonlocal effect decreases with decreasing strain gradient due to the increasing length. Also, the effect of aspect ratio *L/d* can be enhanced as far as the higher frequency ratios are concerned. For example, as aspect ratio changes from 10 to 50 the third frequency ratios increase about 46.7% but the first frequency ratios increase only 9.2% in the case of $\tau_d=10^{-4}$. Nevertheless, the third frequency ratios of the SWCNT with large values of τ_d (e.g. $\tau_d=2\times10^{-4}$) are having extremal i.e. minimum values for a certain value of aspect ratio. This shows that the strongest influence of the nonlocal parameter on the third damped frequencies can be achieved for the SWCNTs with a certain value of aspect ratio. On the other hand, we see that the effect of structural damping coefficient τ_d on the first two frequency ratios is small and can be nearly neglected for various values of nonlocal parameter. τ_d however has a strong influence on the third frequency ratios for the SWCNTs with small values of aspect ratio (e.g. *L*/*d*<17).

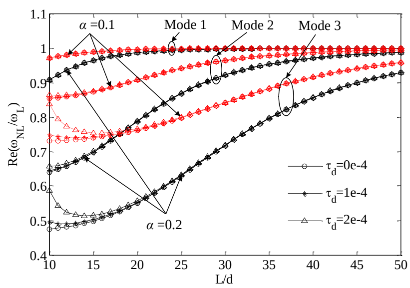
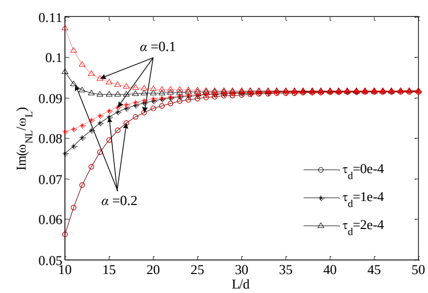


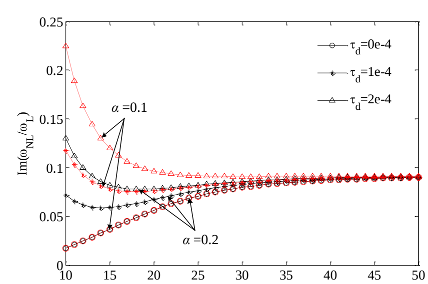
Fig. 4 Effect of aspect ratio *L*/*d* on the real parts of the first third frequency ratios for different values of nonlocal parameter and structural damping coefficient.

Fig. 5 shows the effect of *L*/*d* on the imaginary parts of the first three frequency ratios ω_{NL}/ω_L for the SWCNTs with various values of nonlocal parameter and structural damping coefficient. It is obvious that structural damping coefficient τ_d has strong influence on the imaginary parts of the first three frequency ratios for small values of aspect ratio. This effect however becomes smaller or negligible as far as larger aspect ratios are concerned. Furthermore, the effect of τ_d on the imaginary parts of the frequency ratios is found to be much larger for larger values of α. It also can observed that as aspect ratio *L*/*d* increases from 10 to 50 the frequency ratios increase significantly for small τ_d (e.g. τ_d =0) but decrease for large τ_d (e.g. τ_d=2×10^{-4}). The effect of *L*/*d* on the frequency ratios turn out to be smaller for the SWCNTs of greater *L*/*d*. Specifically, the imaginary parts of the first frequency ratios are approaching unit as *L*/*d* ≥ 30. In other words, the effect of aspect ratio is negligible for the first frequency ratios of the long SWCNT of *L*/*d* ≥ 30. Moreover, the effect of both aspect ratio and structural damping coefficient becomes stronger as the modes of natural frequencies increase.

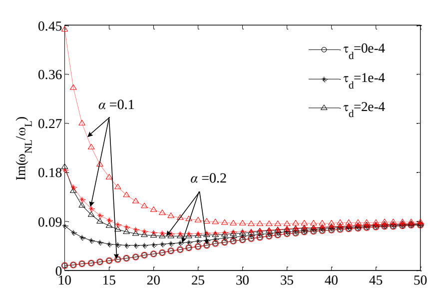
In order to examine the effect of geometrical and material parameters of the foundation on the vibration characteristics of the SWCNTs, the variations of the first two frequency ratios ω_{NL}/ω_L versus the length ratio *l* are shown in Fig. 6 for various values of stiffness coefficient k^* and damping coefficient ζ_0 of the foundation. Here we define the length ratio for the foundation as $l = \bar{x}_2 - \bar{x}_1$, and the support region is in the center of the SWCNT, i.e. $(\bar{x}_2 + \bar{x}_1)/2 = 1/2$. From Fig. 6 it can be seen that as expected the real parts of the first two frequency ratios remain almost constant as damping coefficient changes from 50 to 200, which suggests that the damping coefficient of the foundation has almost no effect on the damped frequencies of the SWCNTs. It is obvious that stiffness coefficient k^* has strong influence on the damped frequencies which increase monotonically with rising k^*. On the other hand, the imaginary parts of the frequency ratios increase significantly for an increase of damping coefficient ζ_0 but decrease slightly for an increase of k^*. The reason for this is that the rigidity and the damping of the system increase, respectively as stiffness coefficient k^* and damping coefficient ζ_0 increase. It also can be observed that the two parts of the first two frequency ratios increase when length ratio *l* increases, which implies that rising the length of the foundation leads to an increase in both the rigidity and the damping of the system. Furthermore, the effects of length ratio *l*, stiffness coefficient k^* and damping coefficient ζ_0 on the two parts of frequency ratios become smaller for higher frequency modes.



a) The imaginary parts of the first frequency ratios

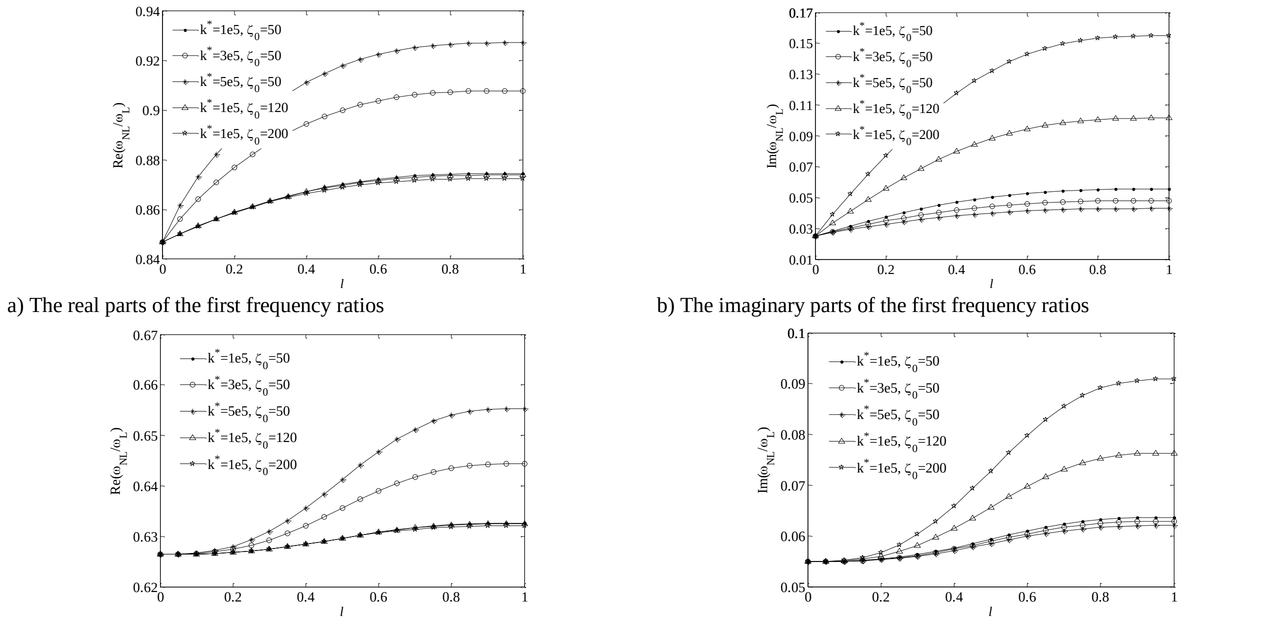


b) The imaginary parts of the second frequency ratios



c) The imaginary parts of the third frequency ratios

Fig. 5 Effect of aspect ratio *L*/*d* on the imaginary parts of the first third frequency ratios for different values of nonlocal parameter and structural damping coefficient.



a) The real parts of the first frequency ratios

b) The imaginary parts of the first frequency ratios

c) The real parts of the second frequency ratios

d) The imaginary parts of the second frequency ratios

Fig. 6 Effect of the length ratio l of the foundation on the first two frequency ratios for different values of stiffness and damping coefficients of the foundation.

7. Conclusions

In this study, the transverse vibration is investigated for a damped viscoelastic single-walled carbon nanotube (visco-SWCNT) resting on a viscoelastic foundation, where different boundary conditions are considered. The analyses are based on nonlocal Euler-Bernoulli beam theory, general Maxwell model and Kelvin viscoelastic foundation model. In general cases, the closed form expression is obtained for the complex natural frequencies of the SWCNT with arbitrary boundary conditions by using the transfer function method (TFM). The exact analytical expressions of the natural frequencies are obtained for certain boundary conditions and viscoelastic foundations. Here the proposed model is validated by comparing the obtained results with those available in the literature, where good agreement has been achieved. Subsequently, the effect of various internal and external factors are examined for the vibration characteristics of the SWCNTs, including the nonlocal parameter, boundary conditions, aspect ratio and structural damping coefficient of the SWCNTs, and the stiffness, damping and length of the viscoelastic foundations. Some of the key contributions made in this study include:

• Novelty of this work includes simultaneous consideration of viscoelastic foundation, velocity-dependent external damping and viscoelasticity of the SWCNT for nanotube dynamics.

• Closed form solutions of the natural frequencies are derived by using the transfer function method for the supported SWCNTs with arbitrary boundary conditions.

• For Kelvin-Voigt viscoelastic model, the exact analytical expressions of the complex natural frequencies for the SWCNTs with full foundations and certain boundary conditions can be expressed as

$$\omega_n = \left[\frac{\zeta_0+\zeta_2}{2}+\frac{c^2\beta^4}{2\left(1+\alpha^2\beta^2\right)}\tau_d\right]i \mp \sqrt{-\left[\frac{\zeta_0+\zeta_2}{2}+\frac{c^2\beta^4}{2\left(1+\alpha^2\beta^2\right)}\tau_d\right]^2+\frac{c^2\beta^4}{1+\alpha^2\beta^2}+k^*}$$

which is a generalization and a set of typical special cases can be identified. The critical value of the structural damping coefficient τ_d to obtain non-oscillatory solution is

$$(\tau_d)_{\text{crit}} = \frac{1+\alpha^2\beta^2}{c^2\beta^4}\left(2\sqrt{\frac{c^2\beta^4}{1+\alpha^2\beta^2}+k^*}-(\zeta_0+\zeta_2)\right)$$

• The nonlocal effects on the natural frequencies are reduced with rising the aspect ratio of the SWCNT or the length of the viscoelastic foundation. Both the rigidity and the damping of the system increase significantly with increasing the length of the viscoelastic foundation.

As shown above this study is focused on CNTs but the technique developed here can be readily extended into the vibration analysis of other beam-like nanostructures supported by a viscoelastic foundation. The numerical results can also provide valuable guidelines for further investigation and applications of the dynamic systems with the coupling between nanotubes and viscoelastic foundations.

Acknowledgment

This research is supported by the National Natural Science Foundation of China (Grant Nos. 11272348 and 11302254).

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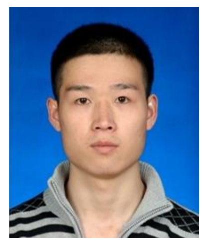

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