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POISSON-RIEMANNIAN GEOMETRY

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Abstract. We study noncommutative bundles and Riemannian geometry at the semiclassical level of first order in a deformation parameter λ, using a functorial approach. This leads us to field equations of 'Poisson-Riemannian geometry' between the classical metric, the Poisson bracket and a certain Poisson-compatible connection needed as initial data for the quantisation of the differential structure. We use such data to define a functor Q to $O(\lambda^2)$ from the monoidal category of all classical vector bundles equipped with connections to the monoidal category of bimodules equipped with bimodule connections over the quantized algebra. This is used to 'semiquantize' the wedge product of the exterior algebra and in the Riemannian case, the metric and the Levi-Civita connection in the sense of constructing a noncommutative geometry to $O(\lambda^2)$. We solve our field equations for the Schwarzschild black-hole metric under the assumption of spherical symmetry and classical dimension, finding a unique solution and the necessity of nonassociativity at order λ^2, which is similar to previous results for quantum groups. The paper also includes a nonassociative hyperboloid, nonassociative fuzzy sphere and our previously algebraic bicrossproduct model.

1. Introduction

Noncommutative geometry has been successful in recent years in extending notions of geometry to situations where the 'coordinate algebra' is noncommutative. Such algebras could arise on quantisation of the phase space in the passage from a classical mechanical system to a quantum one, in which case noncommutative geometry allows us to understand the deeper geometry of such systems. An example here is the quantum Hall effect[13, 31]. It is also now widely accepted that noncommutative Riemannian geometry of some kind should be a more accurate description of spacetime coordinates so as to include the effects of quantum corrections arising out of quantum gravity. The deformation parameter in this case is not expected to be Planck's constant but the Planck scale λ_P. The main evidence for such a *quantum spacetime hypothesis* is by analogy with 3D quantum gravity, see e.g. [36], but the hypothesis has also been extensively explored in specific models such as the bicrossproduct one[35], with key implications such as variable speed of light[2] and frequency dependent gravitational time dilation[33].

These noncommutative models have, however, all been constructed on a case by case basis using algebraic methods and there has so far been no fully systematic 'quantisation method' that takes wider geometrical semiclassical data and quantizes it in the same manner as we are used to for the algebra alone, although there have been some early steps in this direction notably concerning quantizing vector bundles[29, 15, 23, 27] as well as later works including our own[25, 26, 5, 6, 7] and recent works such as [9, 10, 4]. We recall that for the noncommutative algebra alone the semiclassical data is well-known to be a Poisson bracket and in this case the quantisation problem was famously solved to all orders in deformation theory by Kontsevich[30] and more explicitly in the symplectic case by Fedosov[22]. The question we address is *what exactly is the semiclassical*

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theory underlying the quantisation of the rest of differential geometry, particularly Riemannian geometry? Our approach to this is that we will know we have succeeded if we have identified suitable classical data and field equations among them so as to be able to 'semiquantize', by which we mean to construct the associated noncommutative geometry to 1st order. We will make this precise by saying that semiquantisation means quantisation but not over $\mathbb{C}$, rather over the ring $\mathbb{C}[\lambda]/(\lambda^2)$ where λ will be our deformation parameter. For technical reasons, we will take this to be imaginary (so we might have $\lambda = \imath\lambda_P$ in the case of quantum gravity effects). Semiquantisation is not actual quantisation (which in deformation theory would be over the ring of formal powerseries $\mathbb{C}[[\lambda]]$), but it usefully organises the Poisson-level semiclassical data into the first step on the road to the quantisation.

Solving this problem amounts to a semiclassicalization of noncommutative differential geometry and is important at a practical level. Although we will only be able to see perturbative effects, it is useful to be able to solve first at 1st order and then proceed to the next order, etc. More importantly, a step by step approach allows us to go beyond the strict assumptions of noncommutative geometry by only imposing them at 1st order, which basically means certain classical Poisson-level data inspired by but not assuming conventional noncommutative geometry at all orders in the deformation parameter. This greater flexibility is needed because, based on experience with model-building, noncommutative geometry has a much higher rigidity than classical geometry, i.e. one encounters obstructions or 'quantum anomalies' to constructing noncommutative geometries with the same dimension as their classical counterparts [5]. These can often be absorbed by adding extra dimensions, which leaves the strict deformation setting, or else we may need to row back from some expected element such as (quantum) symmetry or associativity of the differential forms at higher deformation order. Our semiclassical analysis will allow us for the first time to understand these obstructions systematically as quantisation constraints on the classical geometric data. Understanding these quantisation constraints may also provide insights into the origin and structure of the equations of physics, such as Einstein's equation, if these too could be seen as an imprint of the rigidity constraints of noncommutative geometry and hence forced by the quantum spacetime hypothesis. That remains speculation but one can see in the 2D toy model of [10] that the constraints of noncommutative geometry forces a curved metric on the chosen spacetime algebra and this metric describes either a strongly gravitational source at the origin in space or a toy model of a big-bang cosmology with fluid matter, depending on the sign of a parameter. Similarly, [37] shows how the Bertotti-Robinson solution of the Einstein equations with Maxwell field and/or cosmological constant is forced by the quantisability constraints of Poisson-Riemannian geometry for a different choice of differential calculus. Both models show gravity in some form emerging from the quantisability constraints, motivated by and referring to the present work.

There are also good *physical* reasons for focussing as we do on the semiclassical or Poisson-Riemannain theory. Particularly when we are talking about Planck-scale corrections, it is only 1st order that is physically most relevant since noncommutative geometry is an effective description of first quantum gravity corrections (it is tempting to suppose an absolute significance to the full noncommutative geometry but that is a further assumption). In distance units the value of λ_P at around 10^{-35}m is also extremely small, making these effects only just now beginning to be measurable in principle, in which case $O(\lambda_P^2)$ effects can be expected to be so much even smaller as to be beyond any possible relevance. This motivates a deeper analysis of the semiclassical level where we work to order λ_P and what is remarkable is that such a paradigm of semiclassical quantum gravity effects exists and has its own status (which we call Poisson-Riemannian geometry) much as does classical mechanics where we take quantum mechanics to first order in $\hbar$.

We now turn to our specific formulation of the problem. In fact there are different approaches to noncommutative geometry and in order for our analysis to have significance we want to make only the most common and minimal assumptions among them, and analyse this as semiclassical level. Aside from a possibly noncommutative algebra A, all main approaches make use (even if it is not the starting point) of differential forms expressed as a differential graded algebra $(\Omega(A), \mathrm{d})$ with $\mathrm{d}^2 = 0$, so we will assume this also at least at order λ. We assume that all operations, notably the quantum wedge product $\wedge_1$, can be built on the same vector spaces as their classical counterparts except now depending on and expandable order by order in a parameter λ. Of the various approaches to noncommutative geometry, we mention notably the approach of Connes[16] coming out of the Dirac operator and cyclic cohomology, a ring-theoretic projective module approach due to Van den Bergh, Stafford and others, e.g. [41], and a constructive approach coming out of quantum groups, but not limited to them, and within that the comprehensive 'bimodule approach' to noncommutative Riemannian geometry[38, 19, 20, 7, 8, 9, 10, 32, 33]. It is this latter approach which we take as our starting framework for semiclassicalization simply because it is the most explicit and hence most amenable to a layer by layer analysis. After the differential calculus, the next layers we consider are the quantum metric $g_1 \in \Omega^1(A) \otimes_1 \Omega^1(A)$ where $\otimes_1$ means the tensor product over the quantized algebra, with reasonable properties such as 'quantum symmetry' $\wedge_1(g_1) = 0$, and a quantum Levi-Civita connection $\nabla_1 : \Omega^1(A) \to \Omega^1(A) \otimes_1 \Omega^1(A)$ with reasonable properties such as respecting that one can multiply $\Omega^1(A)$ by functions from both the left and the right (a 'bimodule connection'). Note that the first copy of $\Omega^1(A)$ would classically evaluate against a vector field to create a covariant derivative on 1-forms. Similarly, we consider vector bundles via their space of sections E modelled as projective modules over the 'coordinate algebra' A and quantum connections $E \to \Omega^1(A) \otimes_1 E$, etc. In the bimodule approach, E will be a bimodule and the connection will respect, in some form, both left and right products by functions. We have shown in some of the works cited above that such quantum metrics and bimodule connections occur quite widely in noncommutative geometry. This set-up is summarised in the preliminary Section 2.

In this context the quantisation of a vector bundle means a deformed product between functions and sections and it has been proposed already at the end of the 1980's in [29] to achieve this by means of a Lie-Rienhart or 'contravariant' connection to give the new product $a \bullet e = ae + \frac{\lambda}{2}\nabla_{\mathrm{d}a} e$, or in the bimodule setting the commutator $[a, e] = \lambda \nabla_{\mathrm{d}a} e$. Such a contravariant derivative 'along 1-forms' was later used in [15] to explore among other results the quantisation of line bundles inducing Morita equivalence of star products. Contravariant connections were also used in [23] to study the global properties of Poisson manifolds and in [27] to revisit geometric quantisation previously exhibited in [29] introducing additional results. We will use such a deformed product but we want to go much further and the key to our approach will be a step backwards: we focus on contravariant connections which are the pull-back along the map from 1-forms to vector fields provided by the Poisson tensor ω, of ordinary connections. Thus we will quantise pairs (E, ∇_E) where E is (the sections of) a vector bundle and ∇_E is a usual connection. The commutation relations between functions and sections now have ω in them, see (3.7). Having the stronger data of a full connection will allow us in Section 3 to construct a monoidal functor $Q : \mathcal{D}_0 \to \mathcal{D}_1$ that 'semiquantizes' (i.e. quantizes to 1st order) the category $\mathcal{D}_0$ of classical bundles E with connection ∇_E, to an A-bimodule $Q(E)$ and bimodule connection to errors in $O(\lambda^2)$ (Theorem 3.5). This means we not only semiquantise objects but also their tensor product. As soon as we semiquantise a tensor product, one of the bundle connections in our approach has to be a full one but the other could still be a contravariant one.

In Section 4 we apply our monoidal functor and further analysis to semiquantise the entire exterior algebra of differential $(\Omega(M), \mathrm{d})$ of a Poisson manifold (M, ω) i.e. with a deformed wedge product $\wedge_1$ forming a differential graded algebra at semiclassical order

(Theorem 4.4). Deformations of $(\Omega^1(M), \mathrm{d})$ were already analysed in [25, 5] as controlled at 1st order by a 'preconnection' operator $\gamma(a,\ ) : \Omega^1(M) \to \Omega^1(M)$ for each $a \in C^\infty(M)$ subject to certain axioms. We thought of them as covariant derivatives partially defined along hamiltonian vector fields but one can also see them as contravariant connection $\nabla_{\mathrm{d}a} = \gamma(a,\ )$. In the present paper we assume an actual linear connection ∇ on $\Omega^1(M)$ with $\gamma(a,\ ) = \nabla_{\hat a}$ where $\hat a = \{a,\ \}$. In a coordinate basis, this means that we take as data an ordinary linear connection ∇_i, but only the combinations $\omega^{ij}\nabla_j$ are actually used in the semi-quantisation of $\Omega^1(M)$ itself. This means that if we take full connections on other bundles E then we strictly need only a contravariant ∇^i for most of Section 3, but from Lemma 3.9 onwards to the end of the paper we will not have this luxury. For noncommutative geometry we should ideally like ∇ to also be flat as its curvature implies breakdown of the bimodule property at order λ^2 (i.e. failure of the associativity of left and right multiplications of functions on differential forms)[25, 5], but in our theory at order λ we do not require this. The Leibniz rule on the deformed algebra requires Poisson-compatibility between $\gamma(a,\ )$ and ω as in [25, 5], which can also be seen as vanishing of contravariant-torsion in that language. For us this now appears in terms of the torsion T of ∇ (see Lemma 3.1) as

$$\omega^{ij}_{;m} = \omega^{ki} T^j_{km} + \omega^{jk} T^i_{km} \tag{1.1}$$

where the semicolon is ∇. The semi-quantization of $\Omega^1(M)$ is needed for the semi-quantisation of connections in the functor Q in Section 3, which in turn is used on the semi-quantisation of all of $\Omega(M)$. We also quantise ∇ itself by the functor and in Theorem 4.11 we extend this to semi-quantise any real linear connection $\nabla_S = \nabla + S$ on $\Omega^1(M)$ to a quantum $*$-preserving connection ∇_1 and sends a zero torsion connection to one of zero quantum torsion. Here $*$-preserving is a notion of 'unitarity' for bimodule connections in [8] which corresponds to reality of the Christoffel symbols classically. Remarkably, all of this is achieved at order λ for free without further conditions on the connection ∇ beyond (1.1). At higher order we have mentioned nonassociativity when ∇ has curvature and there are issues interpreting the deformed integral as a trace [26] relevant to other aspects of noncommutative geometry. We also note that a quick way to achieve examples of quantisation of connections to bimodule connections is by means of a Drinfeld twist as explained in [9, Prop. 6.1] and later in [4]. These require a classical or quantum symmetry which induces the quantization through an equivalence of categories as used for differential structures in [39] in the associative case and [5, 6] in the nonassociative case. Our constructions are compatible with such twisting examples but we do not want to be limited to them.

Section 5 assumes that M is now a Riemannian or pseudo-Riemannian manifold with metric g as a 1-1-form. After some work, we find eventually that the only condition for quantisability of the metric at 1st order is

$$\nabla g = 0 \tag{1.2}$$

We need this at least along hamiltonian vector fields but for simplicity we suppose it fully. In this case (1.2) is solved by writing $\nabla + S = \widehat\nabla$ (the Levi-Civita connection) where S is determined by T. We then solve the problem of constructing a quantum metric g_1. Aside from use of the functor Q, a key ingredient that we need for is a certain 2-form 'generalised Ricci 2-form',

$$\mathcal{R}_{nm} = \tfrac{1}{2}\, g_{ij}\omega^{is}(T^j_{nm;s} - R^j{}_{nms} + R^j{}_{mns})$$

where R is the curvature of ∇, see Proposition 5.2. This also puts $\widehat\nabla$ into the form needed for the quantisation by Theorem 4.11 and applying we obtain a torsion free quantum connection ∇_1 on $\Omega^1(A)$ which may or may not be quantum metric compatible. We show that the condition for this is

$$\widehat\nabla\mathcal{R} + \omega^{ij}\, g_{rs}\, S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k \otimes \mathrm{d}x^m \wedge \mathrm{d}x^n = 0, \tag{1.3}$$

which is our third condition for Poisson-Riemannian geometry, namely for existence of a quantum Levi-Civita connection at order λ (Corollary 5.9). If this condition does not hold then we show (Theorem 5.7) that there is still a 'best possible' quantum connection ∇_1 which is quantum torsion free, for which the symmetric part of $\nabla_1 g_1$ vanishes and for which the antisymmetric part is order λ (namely given by the left hand side of (1.3)). This is a potential order λ quantum effect that has no classical analogue. The general pattern of our constructions will be to use the semiquantisation functor Q to obtain first approximations $\wedge_Q$ quantising the wedge product, ∇_Q quantising ∇, ∇_{QS} quantising $\widehat{\nabla}$ and g_Q quantising the metric, then make unique order λ adjustments to obtain the desired properties.

Section 6 turns to examples in the simplest case where $S = 0$, i.e. where the background Poisson-compatible connection ∇ and the Levi-Civita connection $\widehat{\nabla}$ coincide. Their quantisation $\nabla_1 = \nabla_Q = \nabla_{QS}$ is automatically star-preserving and 'best possible' in terms of metric compatibility. The constraint (1.1) simplifies to $\widehat{\nabla}\omega = 0$, (1.2) is automatic, while the condition (1.3) for full metric-compatibility simplifies to $\widehat{\nabla}\mathcal{R} = 0$. The latter is automatically solved for example if (ω, g) is Kähler-Einstein. It is similarly solved for any surface of constant curvature and we give hyperbolic space and the sphere in detail. This section thus provides the simplest class of solutions. Our sequel [11] similarly gives $\mathbb{C}P^n$, the classical Riemannian geometry of which is linked to Berry phase and higher uncertainty relations in quantum mechanics[14]. The downside is that the Levi-Civita connection typically has significant curvature in examples of interest and if we take this for our background connection then the quantum differential calculus will be significantly nonassociative at $O(\lambda^2)$.

Finally, Section 7 provides two examples where (1.1)-(1.3) again hold but where *necessarily* $\nabla \neq \widehat{\nabla}$ (because ω is not covariantly constant). The first is the 2D bicrossproduct model quantum spacetime with curved metric in [10] but analysed now at the semiclassical level. Here ∇ has zero curvature but a lot of torsion, and application of our general machinery yields ∇_1 in agreement with one of the two quantum Levi-Civita connections in [10] (the other is non-perturbative with no $\lambda \to 0$ limit). This provides a nontrivial check on our analysis. We then turn to our main example, the Schwarzschild black-hole metric g with a rotationally invariant ω. We are led under the assumption of rotational invariance to a 4-functional parameter moduli of ∇, all of them with torsion. The parameters are killed when we contract with ω^{ij}, so that we find a *unique* rotationally invariant black-hole differential calculus in this context. This time there is curvature and hence necessarily nonassociativity at $O(\lambda^2)$. There is again a unique quantum Levi-Civita connection ∇_1 for the black hole. Note that it was claimed in [40] to quantize the black hole within a star-product approach; our analysis tells us that this must have a hidden nonassociativity when cast into the more conventional setting of noncommutative differentials not considered there. The spin-off sequel [24] extends key features of the black hole case such as the uniqueness to generic spherically symmetric metrics. It also studies the quantum wave operator $\square_1 = (\ ,\ )_1\nabla_1 \mathrm{d}$, where $(\ ,\ )_1$ is inverse to g_1, and the quantum Ricci tensor defined as in [10] by lifting the curvature of ∇_1 and tracing.

Both Sections 6 and 7 show the existence of interesting solutions of the Poisson-Riemannian equations (1.1)-(1.3) and hence construct quantum geometries to $O(\lambda^2)$; their fuller development and applications are directions for further work. It would also be important to understand better the physical meaning of ∇ following its role as semiclassical data for quantisation of the differential structure prior to any metric (reading (1.2) as a quantisability condition on the metric when this is introduced later). ∇ could also have further geometric meaning, as would then its quantisation ∇_Q. For example, in teleparallel gravity[1] on a parallelisable manifold one can take ∇ to be the Weitzenböck connection, which has torsion but zero curvature and working with it (instead of the Levi-Civita one) is equivalent to General Relativity but interpreted differently, with the contorsion S

above now taking a primary role and our results similarly viewed as its quantisation. The Weitzenböck ∇ with its zero curvature corresponds to an associative quantum differential calculus at $O(\lambda^2)$, in contrast to the black-hole example where we saw that we need to have some amount of curvature to have a compatible ω, i.e. to be quantisable. Secondly, we note that equation (1.1) has a striking similarity to weak-metric-compatibility [34]

$$g^{ij}_{;m} = g^{ki}T^j_{km} + g^{jk}T^i_{km}.$$

which applies to metric-connection pairs arising from cleft central extensions within noncommutative geometry of the classical exterior algebra by an extra closed 1-form θ' with $\theta'^2 = 0$ and θ' graded-commutative. This resembles a super-version of the present paper where we extend by λ a central scalar with $\lambda^2 = 0$. The θ' approach was used to associatively quantize the Schwarzschild black-hole in [33] in contrast to our approach now, so both versions have now been constructed, which is similar to the story for quantum groups where there are two routes to deal with the quantum anomaly for differentials [5]. Moreover, we see that these two different ideas might be unified into a single construction.

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2. Preliminaries

2.1. **Classical differential geometry.** We assume that the reader is comfortable with classical differential geometry and recall its noncommutative algebraic generalisation in a bimodule approach. For classical geometry suffice it to say that we assume M is a smooth manifold with further smooth structures notably the exterior algebra $(\Omega(M), \mathrm{d})$ but more generally we could start with any graded-commutative classical differential graded algebra with further structure (i.e. the graded-commutative case of the next section). One small generalisation: we allow complexifications. However, one could work with real values and a trivial $*$-operation or one could have a complex version and, in the classical case, pick out the real part. We use the following categories based on vector bundles on M and bundle maps that preserve the base:

Name	Objects	Morphisms
$\mathcal{E}_0$	vector bundles over M	bundle maps
$\tilde{\mathcal{D}}_0$	(E, ∇_E) bundle and connection	bundle maps
$\mathcal{D}_0$	(E, ∇_E) bundle and connection	bundle maps intertwining the connections

To fit the viewpoint of noncommutative geometry we will always work with a bundle through its space of sections as a (finitely generated projective) module over the algebra of smooth functions $C^\infty(M)$, which takes a little getting used as the more familiar set maps are in the opposite direction. Thus a morphism in $\tilde{\mathcal{D}}_0$ means a 'bundle map' in the sense of map $\theta : E \to F$ of modules over $C^\infty(M)$, and to be a morphism in $\mathcal{D}_0$ this also has to intertwine the connections in the sense $\theta(\nabla_{Ei} e) = \nabla_{Fi}(\theta e)$, where $e \in E$ is a section.

In principle, $E \otimes F$ denotes tensor product over $\mathbb{C}$ and $E \otimes_0 F$ denotes the tensor product over $C^\infty(M)$. Thus $E \otimes_0 F$ obeys the relation $e.a \otimes_0 f = e \otimes_0 a.f$ for all $a \in C^\infty(M)$, and this corresponds to the usual tensor product of vector bundles, whereas $E \otimes F$ is much larger. We can use the tensor product to rewrite a connection ∇_E as a map $\nabla_E : E \to \Omega^1(M) \otimes_0 E$ by using the formula $\nabla_E(e) = \mathrm{d}x^i \otimes_0 \nabla_{Ei}(e)$, and the Leibniz rule becomes

$$\nabla_E(a.e) = \mathrm{d}a \otimes_0 e + a.\nabla_E(e)\ .$$

In keeping with common practice, we will sometimes drop the subscript in $\otimes_0$ when this is clear from context.

As most readers will be more familiar with tensor calculus on manifolds than with the commutative case of the algebraic version above, we use the former throughout for computations in the classical case. We adopt here standard conventions for curvature and torsion tensors as well as Christoffel symbols for a linear connection ∇. On forms and in a local coordinate system we have

$$\nabla_j \mathrm{d}x^i = -\Gamma^i_{jk}\mathrm{d}x^k$$

while $T_\nabla = \wedge\nabla - \mathrm{d} : \Omega^1(M) \to \Omega^2(M)$ is the torsion tensor

$$T_\nabla(\mathrm{d}x^i) = -\Gamma^i_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k = \tfrac{1}{2}T^i_{jk}\mathrm{d}x^k \wedge \mathrm{d}x^j, \quad T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}\ . \tag{2.1}$$

Similarly, for the curvature tensor

$$R_\nabla(\mathrm{d}x^k) = \tfrac{1}{2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes_0 [\nabla_i, \nabla_j]\mathrm{d}x^k = \tfrac{1}{2}\,R^k{}_{mij}\mathrm{d}x^j \wedge \mathrm{d}x^i \otimes_0 \mathrm{d}x^m.$$

The summation convention is understood unless specified otherwise.

We also recall the interior product $\lrcorner : \mathrm{Vec}(M) \otimes \Omega^n(M) \to \Omega^{n-1}(M)$ defined by $v \lrcorner \eta$ being the evaluation for $\eta \in \Omega^1(M)$, or in terms of indices $v^i\eta_i$, extended recursively to higher degrees by

$$v \lrcorner (\xi \wedge \eta) = (v \lrcorner \xi) \wedge \eta + (-1)^{|\xi|}\xi \wedge (v \lrcorner \eta)\ .$$

2.2. **Noncommutative bundles and connections.** Here we briefly summarise the elements of noncommutative differential geometry that we will be concerned with in our bimodule approach[19, 20, 8, 10]. The following picture can be generalised at various places, but for readability we will not refer to this further. The associative algebra A plays the role of 'functions' on our noncommutative space and need not be commutative. We work over $\mathbb{C}$ but other than when we consider $*$-structures, we could as well work over a general field.

A differential calculus on A consists of n forms $\Omega^n(A)$ for $n \geq 0$, an associative product $\wedge : \Omega^n(A) \otimes \Omega^m(A) \to \Omega^{n+m}(A)$ and an *exterior derivative* $\mathrm{d} : \Omega^n(A) \to \Omega^{n+1}(A)$ satisfying the rules

(1) $\Omega^0(A) = A$ (i.e. the zero forms are just the 'functions')
(2) $\mathrm{d}^2 = 0$
(3) $\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^{|\xi|}\,\xi \wedge \mathrm{d}\eta$ where $|\xi| = n$ if $\xi \in \Omega^n(A)$
(4) Ω is generated by $A, \mathrm{d}A$.

These are the rules for a standard (surjective) differential graded algebra. Note that we *do not* assume graded commutativity, which would be $\xi \wedge \eta = (-1)^{|\xi||\eta|}\,\eta \wedge \xi$.

A vector bundle is expressed as a (projective) A-module. If E is a left A-module we define a left connection ∇_E on E to be a map $\nabla_E : E \to \Omega^1(A) \otimes_A E$ obeying the left Leibniz rule

$$\nabla_E(ae) = \mathrm{d}a \otimes_A e + a\nabla_E(e).$$

We say that we have a bimodule connection if E is an A-A-bimodule and there is a bimodule map

$$\sigma_E : E \otimes_A \Omega^1(A) \to \Omega^1(A) \otimes_A E, \quad \nabla_E(ea) = (\nabla_E e)a + \sigma_E(e \otimes_A \mathrm{d}a).$$

If σ_E is well-defined as a bimodule map then it is uniquely determined, so its existence is a property of a left connection on a bimodule. In this case one can also deduce a useful formula

$$\sigma_E(e \otimes \mathrm{d}a) = \mathrm{d}a \otimes e + \nabla_E[e, a] + [a, \nabla_E e]\ . \tag{2.2}$$

There is a natural tensor product of bimodule connections $(E, \nabla_E) \otimes (F, \nabla_F)$ built on the tensor product $E \otimes_A F$ and

$$\nabla_{E\otimes_A F}(e \otimes_A f) = \nabla_E e \otimes_A f + (\sigma_E \otimes \mathrm{id})(e \otimes_A \nabla_F f). \tag{2.3}$$

There is necessarily an associated $\sigma_{E\otimes_A F}$. We denote by $\mathcal{E}$ the monoidal category of A-bimodules with $\otimes_A$. We denote by $\mathcal{D}$ the monoidal category of pairs (E, ∇_E) of bimodules and bimodule connections over A, with morphisms bimodule maps intertwining the connections.

For any left connection on $E = \Omega^n(A)$, we define the torsion by $T_\nabla = \wedge\nabla - \mathrm{d}$ as an extension of the $n = 1$ case. In the case $E = \Omega^1(A)$ we define a metric as $g \in \Omega^1(A) \otimes_A \Omega^1(A)$ and for a bimodule connection the metric-compatibility tensor $\nabla g \in \Omega^1(A)^{\otimes_A 3}$. We require g to have an inverse $(\ ,\ ) : \Omega^1(A) \otimes_A \Omega^1(A) \to A$ with the usual bimodule map properties implying that g is central[10].

2.3. **Conjugates and star operations.** Here we suppose that A is a star algebra, i.e. that there is a conjugate linear map $a \mapsto a^*$ so that $(ab)^* = b^* a^*$ and $a^{**} = a$. Also suppose that this extends to a star operation on the differential forms $\Omega(A)$, so that $\mathrm{d}(\xi^*) = (\mathrm{d}\xi)^*$ and $(\xi \wedge \eta)^* = (-1)^{|\xi||\eta|}\eta^* \wedge \xi^*$.

Next, for any A-bimodule E, we consider its conjugate bimodule $\overline{E}$ with elements denoted by $\overline{e} \in \overline{E}$, where $e \in E$ and new right and left actions of A, $\overline{e}a = \overline{a^* e}$ and $a\overline{e} = \overline{ea^*}$. There is a canonical bimodule map $\Upsilon : \overline{E \otimes_A F} \to \overline{F} \otimes_A \overline{E}$ given by $\Upsilon(\overline{e \otimes f}) = \overline{f} \otimes \overline{e}$. There is also a natural equivalence given by functorial isomorphisms $\mathrm{bb}_E : E \to \overline{\overline{E}}$, which in our case are the identification $e \mapsto \overline{\overline{e}}$. Also, if $\phi : E \to F$ is a bimodule map we have a bimodule map $\bar{\phi} : \bar{E} \to \bar{F}$ by $\bar{\phi}(\bar{e}) = \overline{\phi(e)}$. These constructions are examples of a general notion of a bar category[8] but for our purposes the reader should view the conjugate notation as a useful way to keep track of conjugates for noncommutative geometry, and as a book-keeping device to avoid problems. It allows, for example, conjugate linear functions to be viewed as linear functions to the conjugate of the original map's codomain. Bimodules form a bar category as explained and so does the category of pairs (E, ∇_E). Here $\bar{E}$ acquires a right handed connection $\bar{\nabla}_{\bar{E}}(\bar{e}) = (\mathrm{id} \otimes \star^{-1})\Upsilon\overline{\nabla_E e}$ which we convert to a left connection

$$\nabla_{\bar{E}}\bar{e} = (\star^{-1} \otimes \mathrm{id})\Upsilon\overline{\sigma^{-1}\nabla_E e}.$$

Here $\star : \Omega^1(A) \to \overline{\Omega^1(A)}$ is the $*$-operation viewed formally as a linear map.

In general we say that E is a $\star$-object if there is a linear operation $\star : E \to \overline{E}$ (which we can also write as $\star(e) = \overline{e^*}$ where $e \mapsto e^*$ is antilinear) such that $\bar{\star}\star(e) = \overline{\overline{e}}$ for all $e \in E$. Here $\bar{\star}$ means to apply the bar functor to $\star$, so $\bar{\star} : \bar{E} \to \overline{\overline{E}}$ but note that in our case we canonically identify the latter with E and then the condition is a categorical way of saying that $\star$ is an anitlinear involution. Also given $\star$-objects E, F we say a morphism $\phi : E \to F$ is $*$-preserving $\bar{\phi}$ commutes with $\star$. If E is a star-object then we define a connection as $*$-preserving if

$$(\mathrm{id} \otimes \star)\nabla_E(\star^{-1}\overline{e}) = (\star^{-1} \otimes \mathrm{id})\Upsilon\overline{\sigma^{-1}\nabla_E e}$$

and in this case (E, ∇_E) becomes a $\star$-object in this bar category. Clearly $\Omega^1(A)$ itself is an example of a star-object and so is $\Omega(A)$ in every degree. The product $\wedge$ is an example of an anti-$*$-preserving map (i.e. with a minus sign) on products of degree 1.

In the $*$-algebra case we say a metric $g \in \Omega^1(A) \otimes_A \Omega^1(A)$ is 'real' in the sense

$$\Upsilon^{-1}(\star \otimes \star)\, g = \overline{g}$$

If g_{ij} is real symmetric as a matrix valued function, then as the phrase 'reality property' suggests, this is true classically. We can also work with general metrics equivalently as 'hermitian metrics' $G = (\star \otimes \mathrm{id})g \in \overline{\Omega^1(A)} \otimes_A \Omega^1(A)$ and this is 'real' precisely when

$\Upsilon^{-1}(\mathrm{id}\otimes \mathrm{bb})G = \overline{G}$, where $\mathrm{bb}:\Omega^1(A)\to \overline{\overline{\Omega^1(A)}}$ is the 'identity' map to the double conjugate $\xi\mapsto \bar{\bar{\xi}}$. In this context it is more natural to formulate metric compatibility using the 'hermitian-metric compatibility tensor'

$$(\bar\nabla\otimes \mathrm{id}+\mathrm{id}\otimes\nabla)G\in \overline{\Omega^1(A)}\otimes_A\Omega^1(A)\otimes_A\Omega^1(A)\ . \tag{2.4}$$

If ∇ is $*$-preserving, then vanishing of this coincides with the regular notion of metric compatibility of the corresponding g.

3. Semiquantisation of bundles

This section constructs a monoidal functor Q that quantizes geometric data on a smooth manifold M to first order in a deformation parameter λ. Because it is only first order, this is not really 'quantisation' but construction of the semiclassical theory or *semiquantisation*. Here $C^\infty(M)$ is our initial algebra and its usual deformation quantisation means informally that we extend it so as to include dependence on a formal parameter λ and define a new product $\bullet$ which gives an associative algebra A over formal powerseries $\mathbb{C}[[\lambda]]$. We require that working modulo λ gives our original $C^\infty(M)$ with its original product and we assume that all expressions can be expanded and equated order by order. Formally, classical vector spaces and classical linear maps are extended by $\otimes\mathbb{C}[[\lambda]]$ as change of base field $\mathbb{C}$ to this base ring. We define the semiquantisation in exactly the same way as such quantisation but dropping errors that are $O(\lambda^2)$ by formally changing base from $\mathbb{C}$ to the ring $\mathbb{C}[\lambda]/(\lambda^2)$. In an application where λ was actually a number, the dropping of higher powers would need to be justified by the physics. We use $\otimes_1$ for tensor product over the quantum algebra at first order, i.e. with the noncommutative product; the suffix 1 here and elsewhere is to remind us that we are working at first order in λ and dropping errors $O(\lambda^2)$. Similarly $\mathcal{E}_1$ denotes the category of A-bimodules and $\mathcal{D}_1$ the category of pairs (E,∇_E) as in Section 2.2 but working over $\mathbb{C}[\lambda]/(\lambda^2)$ or equivalently working with a formal deformation parameter up to errors in $O(\lambda^2)$.

3.1. **Quantizing the algebra and modules.** The data we suppose is an antisymmetric bivector ω on M along with a linear connection ∇ subject to the following 'Poisson compatibility' [5]

$$\mathrm{d}(\omega^{ij})-\omega^{kj}\,\nabla_k(\mathrm{d}x^i)-\omega^{ik}\,\nabla_k(\mathrm{d}x^j)\ =\ 0\ . \tag{3.1}$$

The bivector as usual controls the deformation of the algebra product with commutator $[a,b]=\lambda\omega^{ij}a_{,i}b_{,j}+O(\lambda^2)$ for all $a,b\in C^\infty(M)$ while the combination $\nabla^i:=\omega^{ij}\nabla_j=\gamma(x^i,\ )$ in the notation of [5] similarly controls the commutation relations of the deformed differential calculus as below in (3.5). The condition (3.1) arises in this context [25, 5] as the condition for d (which we do not deform) to still obey the Leibniz rule at order λ. One can also think of the latter data as defining a contavariant or Lie-Rinehart connection ∇ along 1-forms with $\nabla_{\mathrm{d}x^i}=\nabla^i$, in which context the Poisson-compatibility appears naturally as zero 'contravariant torsion'. In our case, we assume that ∇^i is given via ω by an actual connection ∇_i. This not essential for the present section but will be important in later sections. Note that we have used local coordinates but all constructions are global.

Lemma 3.1. *Let ω be an antisymmetric bivector and ∇ a linear connection, with torsion tensor T. Then ω obeys (3.1)* if and only if

$$\omega^{ij}_{;m}+\omega^{ik}T^j_{km}-\omega^{jk}T^i_{km}=0.$$

In this case ω is a Poisson tensor if and only if

$$\sum_{cyclic\ (i,j,k)}\omega^{im}\,\omega^{jp}\,T^k_{mp}=0.$$

Proof. The first part is essentially in [5] but given here more generally. For the first part the explicit version of (3.1) in terms of Christoffel symbols is

$$\omega^{ij}{}_{,m} + \omega^{kj}\Gamma^i_{km} + \omega^{ik}\Gamma^j_{km} = 0. \tag{3.2}$$

We write the expression on the left as

$$\omega^{ij}{}_{,m} + \omega^{kj}\Gamma^i_{mk} + \omega^{ik}\Gamma^j_{mk} + \omega^{kj}T^i_{km} + \omega^{ik}T^j_{km}$$

and we recognise the first three terms as the covariant derivative. For the second part, we put (3.2) into the following condition for a Poisson tensor:

$$\sum_{\text{cyclic }(i,j,k)} \omega^{im}\,\omega^{jk}{}_{,m} = 0\ . \quad \square \tag{3.3}$$

For example, any manifold with a torsion free connection and ω a covariantly constant antisymmetric bivector will do. This happens for example in the case of a Kähler manifold, so our results include these.

The action of the bivector on a pair of functions will be denoted $\{\ ,\ \}$ as usual. If ω is a Poisson tensor then this is a Poisson bracket and from Fedosov[22] and Kontsevich[30] there is an associative multiplication for functions

$$a \bullet b = ab + \tfrac{\lambda}{2}\{a, b\} + O(\lambda^2). \tag{3.4}$$

We take this formula in any case and denote by A any (possibly not associative) quantisation of $C^\infty(M)$ with this leading order part, which means we fix our associative algebra over $\mathbb{C}[\lambda]/(\lambda^2)$ and the leave higher order unspecified. We will normally assume that ω is a Poisson tensor because that will be desirable at higher order, but strictly speaking the results in the present paper do not require this.

Similarly, from [5] the commutator of a function a and a 1-form $\xi \in \Omega^1(M)$ has the form

$$[a, \xi]_\bullet = \lambda\,\omega^{ij}\,a_{,i}\,(\nabla_j\xi) + O(\lambda^2)\ , \tag{3.5}$$

which we can realise by defining the deformed product of a function a and a 1-form ξ as

$$\begin{aligned} a \bullet \xi &= a\,\xi + \tfrac{\lambda}{2}\omega^{ij}\,a_{,i}\,(\nabla_j\xi) + O(\lambda^2)\ , \\ \xi \bullet a &= a\,\xi - \tfrac{\lambda}{2}\omega^{ij}\,a_{,i}\,(\nabla_j\xi) + O(\lambda^2)\ . \end{aligned} \tag{3.6}$$

This need not be the only way to realise the commutation relations but is the natural choice extending (3.4) on functions. One can check that the Leibniz rule holds to $O(\lambda^2)$ in view of (3.1). We define $\Omega^1(A)$ as built on the vector space $\Omega^1(M)$ extended over λ and taken with these $\bullet$ actions $\Omega^1(A)\otimes_1 A \to \Omega^1(A)$ and $A\otimes_1 \Omega^1(A) \to \Omega^1(A)$ forming an A-bimodule over $\mathbb{C}[\lambda]/(\lambda^2)$ or a bimodule up to $O(\lambda^2)$ in a deformation point of view. The exterior derivative d is not deformed and gives us a differential calculus on A as in Section 2.2 up to $O(\lambda^2)$.

Now let (E, ∇_E) be a classical bundle and covariant derivative on it, and define, for $e \in E$,

$$\begin{aligned} a \bullet e &= a\,\xi + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{Ej}e) + O(\lambda^2)\ , \\ e \bullet a &= a\,\xi - \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{Ej}e) + O(\lambda^2)\ . \end{aligned} \tag{3.7}$$

We only need the combination $\nabla^i_E := \omega^{ij}\nabla_{Ej}$, i.e. a contravariant or Lie-Rinehart conneciton, but as before we focus on the case where this is the pullback via ω of a usual connection. A brief check reveals that the following associative laws hold up to errors in $O(\lambda^2)$:

$$(a \bullet b) \bullet e = a \bullet (b \bullet e)\ ,\ (a \bullet e) \bullet b = a \bullet (e \bullet b)\ ,\ (e \bullet a) \bullet b = e \bullet (a \bullet b)\ . \tag{3.8}$$

We define E_A as the vector space of E extended over λ and with actions $E_A \otimes_1 A \to E_A$ and $A \otimes_1 E_A \to E_A$ forming an A-bimodule over $\mathbb{C}[\lambda]/(\lambda^2)$ or a bimodule up to $O(\lambda^2)$ in deformation setting. We similarly consider the following categories with these two points of view:

Name	Objects	Morphisms
$\tilde{\mathcal{E}}_1$	A-bimodules	left module maps
$\mathcal{E}_1$	A-bimodules	bimodule maps
$\mathcal{D}_1$	bimodules with connection	bimodule maps intertwining the connections

where all properties are stated over $\mathbb{C}[\lambda]/(\lambda^2)$, i.e. are only required to hold to $O(\lambda^2)$ if we take a deformation point of view. Here we introduce the notation

$$\mathbb{V}_j(T) \coloneqq \nabla_{Fj} \circ T - T \circ \nabla_{Ej}$$

for a bundle map $T : E \to F$, which is just the usual covariant derivative of tensors.

Lemma 3.2. *We define a map $Q : \tilde{\mathcal{D}}_0 \to \tilde{\mathcal{E}}_1$ sending a bundle with connection to a deformed bimodule to $O(\lambda^2)$ with actions* (3.7) *and sending a bundle map $T : E \to F$ to a left module map to $O(\lambda^2)$ defined by*

$$Q(T) = T + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi} \circ \mathbb{V}_j(T).$$

If $S : G \to E$ is another bundle map then

$$Q(T \circ S) = Q(T) \circ Q(S) + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{V}_i(T) \circ \mathbb{V}_j(S)\ .$$

In particular, $Q(E, \nabla_E) = E$ with $\bullet$ product and $Q(T) = T$ provides a functor $Q : \mathcal{D}_0 \to \mathcal{E}_1$.

Proof. Take $T_0 : E \to F$ a bundle map and write $Q(T_0) = T_0 + \lambda T_1$ for some linear map $T_1 : E \to F$ to be determined. In particular, we aim for the bimodule properties

$$\begin{aligned}(T_0 + \lambda T_1)(a \bullet e) &= a \bullet (T_0 + \lambda T_1)(e)\ ,\\ (T_0 + \lambda T_1)(e \bullet a) &= (T_0 + \lambda T_1)(e) \bullet a\ ,\end{aligned}$$

which to errors in $O(\lambda^2)$ requires

$$\begin{aligned}T_0(a \bullet e) + \lambda T_1(a\, e) &= a \bullet T_0(e) + \lambda\, a\, T_1(e)\ ,\\ T_0(e \bullet a) + \lambda T_1(e\, a) &= T_0(e) \bullet a + \lambda T_1(e)\, a\ .\end{aligned}$$

Using the formula (3.7) for the deformed product gives our conditions as

$$\begin{aligned}T_0(\omega^{ij}\, a_{,i}\,(\nabla_{Ej} e)) + 2T_1(a\, e) &= \omega^{ij}\, a_{,i}\,(\nabla_{Fj} T_0(e)) + 2a\, T_1(e)\ ,\\ -T_0(\omega^{ij}\, a_{,i}\,(\nabla_{Ej} e)) + 2T_1(e\, a) &= -\,\omega^{ij}\, a_{,i}\,(\nabla_{Fj} T_0(e)) + 2T_1(e)\, a\ .\end{aligned} \tag{3.9}$$

It is not possible to satisfy both parts of (3.9) unless T_0 preserves the covariant derivatives, i.e.

$$\nabla_{Fj} T_0(e)\ =\ T_0(\nabla_{Ej} e)$$

and in this case we set $T_1 = 0$ as a solution and $Q(T_0) = T_0$, which is the restricted case where we indeed have a bimodule map to $O(\lambda^2)$ as output, i.e. a morphism of $\mathcal{E}_1$.

More generally, we solve only the first part of (3.9), i.e. a left module map to $O(\lambda^2)$ for $(A, \bullet)$, which needs

$$T_1(a\, e) - a\, T_1(e) = \tfrac{1}{2}\omega^{ij}\, a_{,i}\left(\nabla_{Fj} T_0(e) - T_0(\nabla_{Ej} e)\right)\ ,$$

which is solved by

$$T_1\ =\ \tfrac{1}{2}\omega^{ij}\,\nabla_{Fi} \circ \mathbb{V}_j(T_0)$$

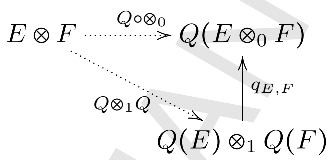
to give $Q(T_0)$. For compositions,

$$\begin{aligned}Q(T \circ S) &= T \circ S + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi} \circ \mathbb{V}_j(T \circ S)\\ &= T \circ S + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi} \circ (\mathbb{V}_j(T) \circ S + T \circ \mathbb{V}_j(S))\\ &= Q(T) \circ S + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi} \circ T \circ \mathbb{V}_j(S)\\ &= Q(T) \circ S + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{V}_i(T) \circ \mathbb{V}_j(S) + \tfrac{\lambda}{2}\,\omega^{ij}\, T \circ \nabla_{Ei} \circ \mathbb{V}_j(S)\end{aligned}$$

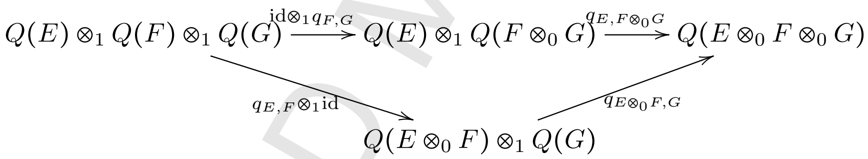
which we then recognise as the expression stated. This implies in particular that we have a functor when we restrict to $\mathcal{D}_0$ i.e. to the subcategory where morphisms are required to intertwine the covariant derivatives. □

Note that in keeping with conventions for functors we use the same symbol Q to map both objects and morphisms, but in our case the map $Q(E,\nabla_E)=E_A$ on objects is just E extended over λ and taken with deformed products $\bullet$. We can therefore view $T:E\to F$ as a linear map $Q(E,\nabla_E)\to Q(F,\nabla_F)$ which we corrected further to obtain the morphism $Q(T):Q(E,\nabla_E)\to Q(F,\nabla_F)$. We will mainly need the functor $Q:\mathcal{D}_0\to\mathcal{E}_1$, with the more general case $Q:\tilde{\mathcal{D}}_0\to\tilde{\mathcal{E}}_1$ being needed in Section 3.4, which in turn will be needed for our construction of the quantum Levi-Civitia connection in later sections.

3.2. **Quantizing the tensor product.** So far we have described how to deform the algebra and bimodules. To avoid too cumbersome a notation we will normally now leave the connection ∇_E as an understood part of our classical data (E,∇_E) and write the quantisation map on objects as the identity map $Q(E)=E_A$ on the underlying vector spaces (with $\bullet$ defined by ∇_E). We now define the "fiberwise" tensor product of two bimodules in the deformed case as $Q(E)\otimes_1 Q(F)$, where similarly to the definition of $\otimes_0$ in Section 2.1, we take $e\otimes_1 a\bullet f=e\bullet a\otimes_1 f$ for all $a\in A$. It is natural to ask how this is related to $Q(E\otimes_0 F)$ where classical bundles with connection have a tensor product bundle with connection. We use $\otimes_0$ for the classical tensor product over $C^\infty(M)$. The nicest case is when Q is a monoidal functor, i.e., there is a natural isomorphism q,

(3.10)


making

(3.11)


commute. Recalling that $Q(E)$ is essentially the identity on vector spaces, we formally write $Q(e)$ to mean $e\in E$ regarded in $Q(E)=E_A$ or when the context is clear we will simply say that we view $e\in Q(E)$. As before, our constructions are over $\mathbb{C}[\lambda]/\langle\lambda^2\rangle$ or to order $O(\lambda^2)$ in a deformation setting.

Proposition 3.3. *The functor $Q:\mathcal{D}_0\to\mathcal{E}_1$ is monoidal to $O(\lambda^2)$ with associated natural transformation $q:Q\otimes_1 Q\implies Q\circ\otimes_0$ given by*

$$q_{V,W}(Q(v)\otimes_1 Q(w))=Q(v\otimes_0 w)+\tfrac{\lambda}{2}Q(\omega^{ij}\,\nabla_{Vi}v\otimes_0\nabla_{Wj}w)\ .$$

More generally the map $Q:\tilde{\mathcal{D}}_0\to\tilde{\mathcal{E}}_1$ together with q obeys (3.11) to $O(\lambda^2)$ while for $T:E\to V$ and $S:F\to W$ and to $O(\lambda^2)$,

$$q_{V,W}\,(T\otimes_1\mathrm{id}_W)=(T\otimes\mathrm{id}_W+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{V}_i(T)\otimes\nabla_{Wj})\,q_{E,W}\ ,$$
$$q_{V,W}\,(\mathrm{id}_V\otimes_1 S)=(\mathrm{id}_V\otimes S+\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Vi}\otimes\mathbb{V}_j(S))\,q_{V,F}\ .$$

Proof. We want a natural morphism $q_{V,W}:Q(V)\otimes_1 Q(W)\to Q(V\otimes_0 W)$ but we suppress writing Q during calculations since it is essentially the identity on objects. For the proposed q to be well-defined we need

$$q_{V,W}(v\bullet a\otimes_1 w)=q_{V,W}(v\otimes_1 a\bullet w)\ ,$$

so from (3.7),

$$q_{V,W}((a\,v-\tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{Vj}v))\otimes_1 w)=q_{V,W}(v\otimes_1(a\,w+\tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{Wj}w)))\ ,$$

which is satisfied by the formula for $q_{V,W}$.

Next, we require each $q_{V,W}$ to be a bimodule map over A. Thus,

$$\begin{aligned} q_{V,W}(v\otimes_1(w\bullet a)) &= v\otimes_0 w\,a - v\otimes_0 \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\nabla_{Wj}w + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Vi}v\otimes_0\nabla_{Wj}(wa)\\ &= v\otimes_0 w\,a + \tfrac{\lambda}{2}\,\omega^{ij}\,(\nabla_{Vi}v\otimes_0\nabla_{Wj}w)\,a + \tfrac{\lambda}{2}\,\omega^{ij}\,(\nabla_{V\otimes_0 W i}(v\otimes_0 w))\,a_{,j}\\ q_{V,W}(v\otimes_1 w)\bullet a &= (v\otimes_0 w + \tfrac{\lambda}{2}\omega^{ij}\nabla_{Vi}v\otimes_0\nabla_{Wj}w)\bullet a\\ &= (v\otimes_0 w)\bullet a + \tfrac{\lambda}{2}\omega^{ij}(\nabla_{Vi}v\otimes_0\nabla_{Wj}w)a\\ &= v\otimes_0 wa - \tfrac{\lambda}{2}\omega^{ij}a_{,i}\nabla_{V\otimes_0 W j}(v\otimes_0 w) + \tfrac{\lambda}{2}\omega^{ij}(\nabla_{Vi}v\otimes_0\nabla_{Wj}w)a\ . \end{aligned} \tag{3.12}$$

using the deformed right module structure on V etc (i.e. regarding it as $Q(V)$) from (3.7). We do not need to use the $\bullet$ product or non-trivial terms in q if an expression already has a λ, as we are working to errors in $O(\lambda^2)$. Our two expressions agree using antisymmetry of ω. Similarly on the other side,

$$\begin{aligned} q_{V,W}((a\bullet v)\otimes_1 w) &= a\,v\otimes_0 w + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{Vj}v)\otimes_0 w + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Vi}(a\,v)\otimes_0\nabla_{Wj}w\\ &= v\otimes_0 w\,a + \tfrac{\lambda}{2}\,a\,\omega^{ij}\,\nabla_{Vi}v\otimes_0\nabla_{Wj}w + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\nabla_{V\otimes_0 W j}(v\otimes_0 w)\\ a\bullet q_{V,W}(v\otimes_1 w) &= a\bullet(v\otimes_0 w + \tfrac{\lambda}{2}\omega^{ij}\nabla_{Vi}v\otimes_0\nabla_{Wj}w)\\ &= av\otimes_0 w + \tfrac{\lambda}{2}\omega^{ij}a_{,i}\nabla_{V\otimes_0 W j}(v\otimes_0 w) + \tfrac{\lambda}{2}\omega^{ij}\nabla_{Vi}v\otimes_0\nabla_{Wj}w\ . \end{aligned} \tag{3.13}$$

Next, we check that $q_{V,W}$ is functorial. Let $T:E\to V$ be a morphism in $\mathcal{D}_0$ (so intertwining the covariant derivatives) and recall that $Q(T)$ is just T as a linear map. Then

$$\begin{aligned} q_{V,W}(Te\otimes_1 w) &= Te\otimes_0 w + \tfrac{\lambda}{2}\omega^{ij}\nabla_{Vi}(Te)\otimes_0\nabla_{Wj}w\\ &= Te\otimes_0 w + \tfrac{\lambda}{2}\omega^{ij}(T\circ\nabla_{Ei}e)\otimes_0\nabla_{Wj}w = (T\otimes\mathrm{id})q_{E,W}(e\otimes_1 w)\ . \end{aligned} \tag{3.14}$$

We used $\nabla\!\!\!\nabla(T)=0$ and have a correction from this in the more general case. Similarly for functoriality on the other side.

Finally, it remains to check that $q_{V\otimes_0 W,Z}\circ(q_{V,W}\otimes\mathrm{id}) = q_{V,W\otimes_0 Z}\circ(\mathrm{id}\otimes q_{W,Z})$ where the associators implicit here are all trivial at order λ. This is immediate from the formulae for q working to $O(\lambda^2)$. Our q are clearly also invertible to this order by the same formula with $-\lambda$. □

Now we discuss conjugate modules and star operations. For vector bundles with connection on real manifolds, we define covariant derivatives of conjugates in the obvious manner, $\nabla_{\bar{E}i}(\bar{e}) = \overline{\nabla_{Ei}e}$. A star operation on a vector bundle will be a conjugate linear bundle map to itself, denoted $e\mapsto e^*$, and is compatible with a connection if $\nabla_{\bar{E}i}(e^*) = (\nabla_{Ei}e)^*$. It will be convenient to use the language of bar categories and view the $*$ as a linear map $\star : E\to\overline{E}$ to the conjugate bundle defined by $\star(e)=\overline{e^*}$.

Proposition 3.4. *With $\lambda^*=-\lambda$, the functor $Q:\mathcal{D}_0\to\mathcal{E}_1$ is a bar functor to $O(\lambda^2)$. Hence if $\star:E\to\overline{E}$ is a star object and compatible with the connection then $Q(\star): Q(E)\to\overline{Q(E)}$ is a star object to $O(\lambda^2)$.*

Proof. To show we have a functor, we begin by identifying $Q(\overline{E})$ and $\overline{Q(E)}$. Recall that $Q(E)$ is simply E as a vector space but with a different module structure and $Q(e)\in Q(E)$ is simply $e\in E$ viewed under this identification. We need to show that, for all $a\in A$ and $e\in E$,

$$a.\overline{Q(e)} = \overline{Q(e).a^*} = \overline{Q(e\bullet a^*)}\ ,\quad a.Q(\bar{e}) = Q(a\bullet\bar{e})\ ,$$

so we need to show that $a\bullet\bar{e} = \overline{e\bullet a^*}$. Now

$$\begin{aligned} a\bullet\bar{e} &= a.\bar{e} + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}.\nabla_{\overline{E}j}(\bar{e}) = a.\bar{e} + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}.\overline{\nabla_{Ej}e}\\ &= \overline{e.a^*} + \overline{\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ej}e.a_{,i}{}^*} = \overline{e.a^*} - \overline{\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ej}e.a_{,i}{}^*}\\ &= \overline{e.a^*} + \overline{\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ei}e.a_{,j}{}^*} = \overline{e\bullet a^*}\ . \end{aligned}$$

Now we have to check the morphisms $T : E \to F$, that $Q(\overline{T}) = \overline{Q(T)}$.

$$\begin{aligned}\overline{Q(T)}(\overline{Q(e)}) &= \overline{Q(T)(Q(e))} = \overline{Q(T(e) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi}(\mathbb{W}_j(T)e))}\,,\\ Q(\overline{T})(Q(\overline{e}) &= \overline{T}(\overline{e}) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\overline{F}i}(\mathbb{W}_j(\overline{T})(\overline{e}))\end{aligned}$$

Now we check what $\mathbb{W}_j(\overline{T})$ is:

$$\begin{aligned}\mathbb{W}_j(\overline{T})(\overline{e}) &= \nabla_{\overline{F}j}(\overline{T(e)}) - \overline{T}(\nabla_{\overline{E}j}(\overline{e}))\\ &= \overline{\nabla_{Fj}(T(e)) - T(\nabla_{Ej}(e))} = \overline{\mathbb{W}_j(T)e}\,.\end{aligned}$$

Then we have, as λ is imaginary,

$$\begin{aligned}Q(\overline{T})(Q(\overline{e}) &= \overline{T(e)} + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\overline{F}i}(\overline{\mathbb{W}_j(T)e})\\ &= \overline{T(e)} + \tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\nabla_{Fi}(\mathbb{W}_j(T)e)}\\ &= \overline{T(e) - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Fi}(\mathbb{W}_j(T)e)}\,.\end{aligned}$$

Thus we have

$$(\overline{Q(T)} - Q(\overline{T}))(\overline{Q(e)}) = \overline{Q(\lambda\,\omega^{ij}\,\nabla_{Fi}(\mathbb{W}_j(T)e))}\,,$$

so if T is a morphism in $\mathcal{D}_0$ we get $\overline{Q(T)} = Q(\overline{T})$.

Now we show that the natural transformation q is compatible with the natural transformation Υ in the bar category. This means that the following diagram commutes:

(3.15)
$$\begin{array}{ccccc}\overline{Q(E)\otimes_1 Q(F)} & \xrightarrow{\overline{q_{E,F}}} & \overline{Q(E\otimes_0 F)} & \xrightarrow{=} & Q(\overline{E\otimes_0 F})\\ \downarrow{\scriptstyle \Upsilon_{Q(E),Q(F)}} & & & & \downarrow{\scriptstyle \Upsilon_{E,F}}\\ \overline{Q(F)}\otimes_1\overline{Q(E)} & \xrightarrow{=} & Q(\overline{F})\otimes_1 Q(\overline{E}) & \xrightarrow{q_{\overline{F},\overline{E}}} & Q(\overline{F}\otimes_0\overline{E})\end{array}$$

Now we have

$$\begin{aligned}\Upsilon_{E,F}(\overline{q_{E,F}}(\overline{Q(e)\otimes_1 Q(f)})) &= \Upsilon_{E,F}(\overline{Q(e\otimes_0 f + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ei}e\otimes_0\nabla_{Fj}f)})\\ &= \Upsilon_{E,F}(Q(\overline{e\otimes_0 f + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ei}e\otimes_0\nabla_{Fj}f}))\\ &= Q(\overline{f}\otimes_0\overline{e} - \overline{\tfrac{\lambda}{2}}\,\omega^{ij}\,\overline{\nabla_{Fj}f}\otimes_0\overline{\nabla_{Ei}e})\\ &= Q(\overline{f}\otimes_0\overline{e} + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\overline{F}i}\overline{f}\otimes_0\nabla_{\overline{E}j}\overline{e})\,,\\ q_{\overline{F},\overline{E}}\Upsilon_{Q(E),Q(F)}(\overline{Q(e)\otimes_1 Q(f)}) &= q_{\overline{F},\overline{E}}(\overline{Q(f)}\otimes_1\overline{Q(e)})\\ &= q_{\overline{F},\overline{E}}(Q(\overline{f})\otimes_1 Q(\overline{e}))\\ &= Q(\overline{f}\otimes_0\overline{e} + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\overline{F}i}\overline{f}\otimes_0\nabla_{\overline{E}j}\overline{e})\,. \quad\square\end{aligned}$$

3.3. **Quantizing the underlying covariant derivative.** Our approach to quantising a bundle E is to equip it with an underlying connection ∇_E which is used to deform the bimodule actions (one could call it the 'quantising connection' for this reason). We now want to quantise this connection itself, which in categorical terms means to extend the above to a functor $Q : \mathcal{D}_0 \to \mathcal{D}_1$ working to $O(\lambda^2)$. Here $Q(E) = E_A$ meaning E with a $\bullet$ bimodule structure and we seek to define $Q(\nabla_E) = \nabla_{Q(E)} : E_A \to \Omega^1(A)\otimes_1 E_A$ as a bimodule connection on this. We assume for this that we have also quantised $\Omega^1(M)$ by a Poisson-compatible connection ∇_i or more generally a contravariant connection $\nabla^i = \omega^{ij}\nabla_j$ obeying (1.1) to a differential calculus $\Omega^1(A)$ to $O(\lambda^2)$. Also note that although we will continue to write expressions in local coordinates, all our constructions are global. For example,

$$\omega^{ij}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei} = (\omega^\sharp\otimes\mathrm{id})\circ(\mathrm{id}\otimes R_E)\circ\nabla_E : E \to \Omega^1(M)\otimes_0 E$$

where R_E is the curvature as a 2-form valued operator and $\omega^\sharp : \Omega^1(M)\otimes_0\Omega^2(M) \to \Omega^1(M)$ means to map $\Omega^1(M)$ to vector fields using the bivector ω and then apply interior product. Unless we need to be more explicit for clarity, we now normally give formulae on the underlying classical elements such as $e \in E$ viewed in $Q(E) = E_A$, etc.

Theorem 3.5. *Let (E,∇_E) be a classical bundle and connection. Then E with the bimodule structure $\bullet$ over A has bimodule covariant derivative*

$$\nabla_{Q(E)} = q^{-1}_{\Omega^1,E}\nabla_E - \tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k \otimes_1 [\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}$$
$$\sigma_{Q(E)}(e\otimes_1 \xi) = \xi\otimes_1 e + \lambda\,\omega^{ij}\,\nabla_j\xi\otimes_1\nabla_{Ei}e + \lambda\,\omega^{ij}\,\xi_j\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ei}]e$$

to $O(\lambda^2)$. Moreover, $Q(E,\nabla_E) = (Q(E),\nabla_{Q(E)})$ is a monoidal functor $Q:\mathcal{D}_0\to\mathcal{D}_1$ to $O(\lambda^2)$ via q from Proposition 3.3.

Proof. We start by considering the following, where $q^{-1} = q^{-1}_{\Omega^1,E}$,

$$\begin{aligned}
q^{-1}\nabla_E(a\bullet e) &= q^{-1}\nabla_E\big(a\,e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\nabla_{Ej}e\big)\\
&= q^{-1}\big(\mathrm{d}a\otimes_0 e + a\,\mathrm{d}x^k\otimes_0\nabla_{Ek}e + \tfrac{\lambda}{2}\,\mathrm{d}(a_{,i}\,\omega^{ij})\otimes_0\nabla_E e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_0\nabla_{Ek}\nabla_{Ej}e\big)\\
&= \mathrm{d}a\otimes_1 e + a\,\mathrm{d}x^k\otimes_1\nabla_{Ek}e + \tfrac{\lambda}{2}\,\mathrm{d}(a_{,i}\,\omega^{ij})\otimes_1\nabla_{Ej}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1\nabla_{Ek}\nabla_{Ej}e\\
&\quad - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\mathrm{d}a\otimes_1\nabla_{Ej}e - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(a\,\mathrm{d}x^k)\otimes_1\nabla_{Ej}\nabla_{Ek}e\\
&= \mathrm{d}a\otimes_1 e + a\,\mathrm{d}x^k\otimes_1\nabla_{Ek}e + \tfrac{\lambda}{2}\,\mathrm{d}(a_{,i}\,\omega^{ij})\otimes_1\nabla_{Ej}e - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\mathrm{d}a\otimes_1\nabla_{Ej}e\\
&\quad - \tfrac{\lambda}{2}\,\omega^{ij}\,a\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{Ej}\nabla_{Ek}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e\ ,
\end{aligned}$$

and

$$\begin{aligned}
a\bullet q^{-1}\nabla_E(e) &= a\bullet q^{-1}\big(\mathrm{d}x^k\otimes_0\nabla_k e\big)\\
&= a\bullet\big(\mathrm{d}x^k\otimes_1\nabla_{Ek}e - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{Ej}\nabla_{Ek}e\big)\\
&= a\,\mathrm{d}x^k\otimes_1\nabla_{Ek}e - \tfrac{\lambda}{2}\,\omega^{ij}\,a\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{Ej}\nabla_{Ek}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ek}e\ ,
\end{aligned}$$

and then

$$\begin{aligned}
&q^{-1}\nabla_E(a\bullet e) - a\bullet q^{-1}\nabla_E(e)\\
&= \mathrm{d}a\otimes_1 e + \tfrac{\lambda}{2}\,\mathrm{d}(a_{,i}\,\omega^{ij})\otimes_1\nabla_{Ej}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e\\
&\quad - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\mathrm{d}a\otimes_1\nabla_j e - \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ek}e\\
&= \mathrm{d}a\otimes_1 e + \tfrac{\lambda}{2}\,\mathrm{d}(a_{,i}\,\omega^{ij})\otimes_1\nabla_{Ej}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e)\\
&\quad - \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(a_{,k}\,\mathrm{d}x^k)\otimes_1\nabla_{Ej}e - \tfrac{\lambda}{2}\,\omega^{ik}\,a_{,i}\,\nabla_k(\mathrm{d}x^j)\otimes_1\nabla_{Ej}e\\
&= \mathrm{d}a\otimes_1 e + \tfrac{\lambda}{2}\,a_{,i}\,\mathrm{d}(\omega^{ij})\otimes_1\nabla_{Ej}e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e\\
&\quad - \tfrac{\lambda}{2}\,\omega^{kj}\,a_{,i}\,\nabla_k(\mathrm{d}x^i)\otimes_1\nabla_{Ej}e - \tfrac{\lambda}{2}\,\omega^{ik}\,a_{,i}\,\nabla_k(\mathrm{d}x^j)\otimes_1\nabla_{Ej}e\\
&= \mathrm{d}a\otimes_1 e + \tfrac{\lambda}{2}\,a_{,i}\,\big(\mathrm{d}(\omega^{ij}) - \omega^{kj}\,\nabla_k(\mathrm{d}x^i) - \omega^{ik}\,\nabla_k(\mathrm{d}x^j)\big)\otimes_1\nabla_{Ej}e\\
&\quad + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e
\end{aligned}$$

where we relabelled some indices for the 3rd equality. Then (3.1) tells us that the outer long bracket in the last expression vanishes, giving

$$q^{-1}\nabla_E(a\bullet e) - a\bullet q^{-1}\nabla_E(e) = \mathrm{d}a\otimes_1 e + \tfrac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]e\ . \tag{3.16}$$

Now we can set the first order quantisation of the left covariant derivative to be

$$Q(\nabla_E)(e) \ = \ q^{-1}_{\Omega^1,E}\nabla_E(e) - \tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}(e)$$

which we can write as stated.

Next, we want to see about a bimodule connection. We compute

$$\begin{aligned}
\sigma_{Q(E)}(e\otimes_1\mathrm{d}a) &= \mathrm{d}a\otimes_1 e + \nabla_E[e,a] + [a,\nabla_E e]\\
&= \mathrm{d}a\otimes_1 e + \lambda\,\nabla(\omega^{ij}\,\nabla_{Ei}e\,a_{,j}) + [a,\mathrm{d}x^k\otimes_1\nabla_{Ek}e]\\
&= \mathrm{d}a\otimes_1 e + \lambda\,\mathrm{d}(\omega^{ij}\,a_{,j})\otimes\nabla_{Ei}e + \lambda\,\omega^{ij}\,a_{,j}\,\mathrm{d}x^k\otimes_1\nabla_{Ek}\nabla_{Ei}e\\
&\quad + \lambda\,\omega^{ij}\,a_{,i}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ek}e + \lambda\,\omega^{ij}\,a_{,i}\,\mathrm{d}x^k\otimes_1\nabla_{Ej}\nabla_{Ek}e\\
&= \mathrm{d}a\otimes_1 e + \lambda\,\mathrm{d}(\omega^{ij})\,a_{,j}\otimes\nabla_{Ei}e + \lambda\,\omega^{ij}\,a_{,jk}\,\mathrm{d}x^k\otimes_1\nabla_{Ei}e\\
&\quad + \lambda\,\omega^{ij}\,a_{,j}\,\mathrm{d}x^k\otimes_1\nabla_{Ek}\nabla_{Ei}e\\
&\quad + \lambda\,\omega^{ij}\,a_{,i}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ek}e + \lambda\,\omega^{ji}\,a_{,j}\,\mathrm{d}x^k\otimes_1\nabla_{Ei}\nabla_{Ek}e\\
&= \mathrm{d}a\otimes_1 e + \lambda\,\mathrm{d}(\omega^{ij})\,a_{,j}\otimes\nabla_{Ei}e + \lambda\,\omega^{ij}\,a_{,jk}\,\mathrm{d}x^k\otimes_1\nabla_{Ei}e\\
&\quad + \lambda\,\omega^{ij}\,a_{,j}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ei}]e + \lambda\,\omega^{jk}\,a_{,j}\,\nabla_k(\mathrm{d}x^i)\otimes_1\nabla_{Ei}e\\
&= \mathrm{d}a\otimes_1 e + \lambda\,\omega^{ij}\,\nabla_j(a_{,k}\,\mathrm{d}x^k)\otimes_1\nabla_{Ei}e - \lambda\,\omega^{ij}\,a_{,k}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ei}e
\end{aligned}$$

$$
\begin{aligned}
&+\lambda\,\omega^{ij}\,a_{,j}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ei}]e+\lambda\,a_{,j}\big(\mathrm{d}(\omega^{ij})-\omega^{kj}\,\nabla_k(\mathrm{d}x^i)\big)\otimes_1\nabla_{Ei}e\\
&=\mathrm{d}a\otimes_1 e+\lambda\,\omega^{ij}\,\nabla_j(a_{,k}\,\mathrm{d}x^k)\otimes_1\nabla_{Ei}e+\lambda\,\omega^{ij}\,a_{,j}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ei}]e\\
&+\lambda\,a_{,j}\big(\mathrm{d}(\omega^{ij})-\omega^{kj}\,\nabla_k(\mathrm{d}x^i)-\omega^{ik}\,\nabla_k(\mathrm{d}x^j)\big)\otimes_1\nabla_{Ei}e\ ,
\end{aligned}
$$

and under condition (3.1) this implies the formula for $\sigma_{Q(E)}$ stated in the theorem. This constructs the quantized covariant derivative. The following two lemmas then verify the desired categorical properties so as to complete the proof. □

To complete the proof of the theorem, our first lemma shows that the quantisation of the covariant derivative respects tensor products – i.e. that the quantisation of the classical tensor product covariant derivative is the tensor product of the quantized covariant derivatives as bimodule connections (using the σ map). This is summarised by

$$
\begin{array}{ccc}
Q(E)\otimes_1 Q(F) & \xrightarrow{\ q_{E,F}\ } & Q(E\otimes_0 F)\\
\Big\downarrow{\scriptstyle \nabla_{Q(E)\otimes_1 Q(F)}} & & \Big\downarrow{\scriptstyle \nabla_{Q(E\otimes_0 F)}}\\
\Omega^1(A)\otimes_1 Q(E)\otimes_1 Q(F) & \xrightarrow[\ \mathrm{id}\otimes q\ ]{} & \Omega^1(A)\otimes_1 Q(E\otimes_0 F)
\end{array}
\tag{3.17}
$$

where the tensor product bimodule connection on the left is defined as in (2.3).

Lemma 3.6. *For all $e\in E$ and $f\in F$ as objects of $\mathcal{D}_0$,*

$$(\mathrm{id}\otimes q_{E,F})\big(\nabla_{Q(E)}e\otimes_1 f+(\sigma_{Q(E)}\otimes\mathrm{id})(e\otimes_1\nabla_{Q(F)}f)\big)=\nabla_{Q(E\otimes_0 F)}q_{E,F}(e\otimes_1 f)$$

holds to $O(\lambda^2)$.

Proof. We begin with

$$
\begin{aligned}
&\nabla_{Q(E\otimes_0 F)}q_{E,F}(e\otimes_1 f)\\
&=\nabla_{Q(E\otimes_0 F)}(e\otimes_0 f)+\tfrac{\lambda}{2}\,\nabla_{Q(E\otimes_0 F)}\big(\omega^{ij}\,\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&=q^{-1}_{\Omega^1,E}\nabla_{E\otimes F}(e\otimes f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{E\otimes Fk},\nabla_{E\otimes Fj}]\nabla_{E\otimes Fi}(e\otimes f)+\tfrac{\lambda}{2}\,\nabla_{Q(E\otimes_0 F)}\big(\omega^{ij}\,\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&=q^{-1}_{\Omega^1,E\otimes F}(\mathrm{d}x^k\otimes(\nabla_{Ek}e\otimes f)+\mathrm{d}x^k\otimes(e\otimes\nabla_{Fk}f))+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{E\otimes Fk},\nabla_{E\otimes Fj}](\nabla_{Ei}e\otimes f+e\otimes\nabla_{Fi}f)+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\nabla_{E\otimes Fk}\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&=\mathrm{d}x^k\otimes_1(\nabla_{Ek}e\otimes f)+\mathrm{d}x^k\otimes_1(e\otimes\nabla_{Fk}f)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{E\otimes Fj}\big((\nabla_{Ek}e\otimes f)+(e\otimes\nabla_{Fk}f)\big)+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\nabla_{E\otimes Fk}\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{E\otimes Fk},\nabla_{E\otimes Fj}](\nabla_{Ei}e\otimes f+e\otimes\nabla_{Fi}f)+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{id}\otimes q_{E,F})(\nabla_{Q(E)}e\otimes_1 f)\\
&=(\mathrm{id}\otimes q_{E,F})\big((q^{-1}_{\Omega^1,E}\nabla_E(e)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e\big)\otimes_1 f\big)\\
&=(\mathrm{id}\otimes q_{E,F})\big((\mathrm{d}x^k\otimes_1\nabla_{Ek}e-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{Ej}\nabla_{Ek}e\big)\otimes_1 f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\big([\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e\otimes_0 f\big)\\
&=\mathrm{d}x^k\otimes_1(\nabla_{Ek}e\otimes_0 f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ej}\nabla_{Ek}(e)\otimes_0 f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\big([\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e\otimes_0 f\big)+\tfrac{\lambda}{2}\,\mathrm{d}x^k\otimes_1\big(\omega^{ij}(\nabla_{Ei}\nabla_{Ek}e\otimes_0\nabla_{Fj}f)\big)\ ,
\end{aligned}
$$

from which

$$
\begin{aligned}
&\nabla_{Q(E\otimes_0 F)}q_{E,F}(e\otimes_1 f)-(\mathrm{id}\otimes q_{E,F})(\nabla_{Q(E)}e\otimes_1 f)\\
&=\mathrm{d}x^k\otimes_1(e\otimes\nabla_{Fk}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ek}e\otimes\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{E\otimes Fj}\big(e\otimes\nabla_{Fk}f\big)+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\nabla_{E\otimes Fk}\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ei}e\otimes[\nabla_{Fk},\nabla_{Fj}]f)+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ek}\nabla_{Ej}e\otimes\nabla_{Fi}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(e\otimes[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}f)\\
&\quad-\lambda\,\mathrm{d}x^k\otimes_1\big(\omega^{ij}(\nabla_{Ei}\nabla_{Ek}e\otimes_0\nabla_{Fj}f)\big)\\
&=\mathrm{d}x^k\otimes_1(e\otimes\nabla_{Fk}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ek}e\otimes\nabla_{Fj}f\big)\\
&\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{E\otimes Fj}\big(e\otimes\nabla_{Fk}f\big)+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\big(\nabla_{Ek}\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ei}e\otimes\nabla_{Fj}\nabla_{Fk}f)+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ek}\nabla_{Ej}e\otimes\nabla_{Fi}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(e\otimes[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}f)\\
&-\lambda\,\mathrm{d}x^k\otimes_1\big(\omega^{ij}(\nabla_{Ei}\nabla_{Ek}e\otimes_0\nabla_{Fj}f)\big)\\
=\;&\mathrm{d}x^k\otimes_1(e\otimes\nabla_{Fk}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ek}e\otimes\nabla_{Fj}f\big)\\
&-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ej}e\otimes\nabla_{Fk}f\big)+\lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes_1\big([\nabla_{Ek},\nabla_{Ei}]e\otimes_0\nabla_{Fj}f\big)\\
&+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ei}e\otimes\nabla_{Fj}\nabla_{Fk}f)+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(e\otimes[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(e\otimes\nabla_{Fj}\nabla_{Fk}f\big)\ .
\end{aligned}
$$

Next

$$
\begin{aligned}
&(\sigma_{Q(E)}\otimes\mathrm{id})(e\otimes_1\nabla_{Q(F)}f)\\
&\quad=(\sigma_{Q(E)}\otimes\mathrm{id})\big(e\otimes_1 q^{-1}_{\Omega^1,E}\nabla_F(f)-\tfrac{\lambda}{2}\,\omega^{ij}\,e\otimes_1(\mathrm{d}x^k\otimes_1[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}(f))\big)\\
&\quad=(\sigma_{Q(E)}\otimes\mathrm{id})\big(e\otimes_1\mathrm{d}x^k\otimes_1\nabla_{Fk}f-\tfrac{\lambda}{2}\,\omega^{ij}\,e\otimes_1\nabla_i(\mathrm{d}x^k)\otimes_1\nabla_{Fj}\nabla_{Fk}f\\
&\qquad-\tfrac{\lambda}{2}\,\omega^{ij}\,e\otimes_1\mathrm{d}x^k\otimes_1[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}(f)\big)\\
&\quad=\mathrm{d}x^k\otimes_1 e\otimes_1\nabla_{Fk}f-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1 e\otimes_1\nabla_{Fj}\nabla_{Fk}f\\
&\qquad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1 e\otimes_1[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}(f)+\lambda\,\omega^{ij}\,\nabla_j(\mathrm{d}x^k)\otimes_1\nabla_{Ei}e\otimes_1\nabla_{Fk}f\\
&\qquad+\lambda\,\omega^{ik}\,\mathrm{d}x^l\otimes_1[\nabla_{El},\nabla_{Ei}]e\otimes_1\nabla_{Fk}f
\end{aligned}
$$

so that

$$
\begin{aligned}
&(\mathrm{id}\otimes q_{E,F})(\sigma_{Q(E)}\otimes\mathrm{id})(e\otimes_1\nabla_{Q(F)}f)\\
&\quad=\mathrm{d}x^k\otimes_1(e\otimes\nabla_{Fk}f)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1(e\otimes\nabla_{Fj}\nabla_{Fk}f)\\
&\qquad-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(e\otimes[\nabla_{Fk},\nabla_{Fj}]\nabla_{Fi}(f))+\lambda\,\omega^{ij}\,\nabla_j(\mathrm{d}x^k)\otimes_1(\nabla_{Ei}e\otimes\nabla_{Fk}f)\\
&\qquad+\lambda\,\omega^{ik}\,\mathrm{d}x^l\otimes_1([\nabla_{El},\nabla_{Ei}]e\otimes\nabla_{Fk}f)+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1(\nabla_{Ei}e\otimes\nabla_{Fj}\nabla_{Fk}f)\ .
\end{aligned}
$$

Putting these calculations together gives us

$$
\begin{aligned}
&\nabla_{Q(E\otimes_0F)}q_{E,F}(e\otimes_1 f)-(\mathrm{id}\otimes q_{E,F})(\nabla_{Q(E)}e\otimes_1 f)-(\mathrm{id}\otimes q_{E,F})(\sigma_{Q(E)}\otimes\mathrm{id})(e\otimes_1\nabla_{Q(F)}f)\\
&=-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ek}e\otimes\nabla_{Fj}f\big)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}x^k)\otimes_1\big(\nabla_{Ej}e\otimes\nabla_{Fk}f\big)\\
&\quad-\lambda\,\omega^{ij}\,\nabla_j(\mathrm{d}x^k)\otimes_1(\nabla_{Ei}e\otimes\nabla_{Fk}f)+\tfrac{\lambda}{2}\,\mathrm{d}(\omega^{ij})\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)\\
&=\tfrac{\lambda}{2}\big(\mathrm{d}(\omega^{ij})-\omega^{kj}\,\nabla_k(\mathrm{d}x^i)+\omega^{ki}\,\nabla_k(\mathrm{d}x^j)\big)\otimes_1\big(\nabla_{Ei}e\otimes_0\nabla_{Fj}f\big)
\end{aligned}
$$

and this vanishes by (3.1). □

Our second lemma checks functoriality under morphisms.

Lemma 3.7. *If $T : E \to F$ is a bundle map intertwining the covariant derivative then $Q(\nabla_F)Q(T) = (\mathrm{id}\otimes_1 Q(T))\,Q(\nabla_E)$ holds to $O(\lambda^2)$.*

Proof. In this case $Q(T) = T$ and

$$
\begin{aligned}
(\mathrm{id}\otimes_1 Q(T))\,\nabla_{Q(E)}&=(\mathrm{id}\otimes_1 Q(T))\,q^{-1}_{\Omega^1,E}\nabla_E(e)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1 T[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}(e)\\
&=q^{-1}_{\Omega^1,E}(\mathrm{id}\otimes T)\,\nabla_E(e)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1 T[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}(e)\ ,\\
Q(\nabla_F)T&=q^{-1}_{\Omega^1,E}\nabla_E T(e)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}T(e)\ .
\end{aligned}
$$

These are equal as T intertwines the covariant derivatives. □

Lemma 3.8. *If $\lambda^* = -\lambda$ then ∇_Q is star preserving to $O(\lambda^2)$, i.e.*

$$
\begin{array}{ccccc}
\overline{Q(E)}=Q(\overline{E}) & \xleftarrow{\;\star\;} & Q(E) & \xrightarrow{\;\nabla_Q\;} & Q(\Omega^1(M))\otimes_1 Q(E)\\
\big\downarrow{\scriptstyle\overline{\nabla_Q}} & & & & \big\downarrow{\scriptstyle\star\otimes_1\star}\\
\overline{Q(\Omega^1(M))\otimes_1 Q(E)} & \xrightarrow{\;\overline{\sigma^{-1}_{QE}}\;} & \overline{Q(E)\otimes_1 Q(\Omega^1(M))} & \xrightarrow{\;\Upsilon\;} & \overline{Q(\Omega^1(M))}\otimes_1\overline{Q(E)}
\end{array}
\tag{3.18}
$$

commutes to $O(\lambda^2)$.

Proof. Begin with, using q a natural transformation and (3.15), labelling the qs where they first occur

$$
\begin{aligned}
&\overline{\sigma_{QE}}\,\Upsilon^{-1}(\star\otimes_1\star)\nabla_Q(Q(e))\\
&\quad=\overline{\sigma_{QE}}\Upsilon^{-1}(\star\otimes_1\star)(q_{\Omega^1,E}{}^{-1}(\mathrm{d}x^p\otimes_0\nabla_{Ep}e)-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e)\\
&\quad=\overline{\sigma_{QE}}\Upsilon^{-1}(q_{\overline{\Omega^1},\overline{E}}{}^{-1}(\overline{\mathrm{d}x^p}\otimes_0\overline{\nabla_{Ep}e^*})-\tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\mathrm{d}x^k}\otimes_1\overline{[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*})\\
&\quad=\overline{\sigma_{QE}\,q_{E,\Omega^1}{}^{-1}}\Upsilon^{-1}(\overline{\mathrm{d}x^p}\otimes_0\overline{\nabla_{Ep}e^*})-\tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\sigma_{QE}}(\overline{[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*\otimes_1\mathrm{d}x^k})\\
&\quad=\overline{\sigma_{QE}\,q^{-1}(\nabla_{Ep}e^*\otimes_0\mathrm{d}x^p)}-\tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\sigma_{QE}}(\overline{[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*\otimes_1\mathrm{d}x^k})\\
&\quad=\overline{q_{\Omega^1,E}{}^{-1}q_{\Omega^1,E}\,\sigma_{QE}\,q_{E,\Omega^1}{}^{-1}(\nabla_{Ep}e^*\otimes_0\mathrm{d}x^p)}-\tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*}\\
&\quad=\overline{q^{-1}(\mathrm{d}x^p\otimes_0\nabla_{Ep}e^*-\lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes_0[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*)}-\tfrac{\lambda}{2}\,\omega^{ij}\,\overline{\mathrm{d}x^k\otimes_1[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*}\\
&\quad=\overline{q^{-1}(\mathrm{d}x^p\otimes_0\nabla_{Ep}e^*-\tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\otimes_0[\nabla_{Ek},\nabla_{Ej}]\nabla_{Ei}e^*)}\\
&\quad=\overline{\nabla_Q(e^*)}
\end{aligned}
$$

where we relabelled indices and used antisymmetry of ω for the 6th equality. □

To conclude this section we note that while our monoidal categories are not normally braided ones, the 'generalised braiding' $\sigma_{Q(E)}:Q(E)\otimes_1\Omega^1(A)\to\Omega^1(A)\otimes_1 Q(E)$ is a step in this direction. To explain this, will need to give special focus to the case $E=\Omega^1(M)$ with the underlying Poisson-compatible connection ∇, where $Q(\Omega^1(M))=\Omega^1(A)$ and where, for brevity, we will write

$$\nabla_Q:=\nabla_{Q(\Omega^1(M))},\quad \sigma_Q:=\sigma_{Q(\Omega^1(M))}.$$

Also note that up until this point we could have taken ∇_E to be full connections and worked with the contravariant ∇^i on $\Omega^1(M)$ but for ∇_Q we need the full connection ∇_i. We will study ∇_Q much more extensively in later sections.

Lemma 3.9. *The generalised braiding obeys the mixed braid relation, in the sense that for any (E,∇_E) giving $\sigma_{Q(E)}$,*

$$(\sigma_Q\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_{Q(E)})(\sigma_{Q(E)}\otimes\mathrm{id})=(\mathrm{id}\otimes\sigma_{Q(E)})(\sigma_{Q(E)}\otimes\mathrm{id})(\mathrm{id}\otimes\sigma_Q)$$

to $O(\lambda^2)$ as a map $Q(E)\otimes_1\Omega^1(A)\otimes_1\Omega^1(A)\to\Omega^1(A)\otimes_1\Omega^1(A)\otimes_1 Q(E)$.

Proof. The generalised braiding is of the following form, with summation implicit,

$$
\begin{aligned}
\sigma_{Q(E)}(e\otimes_1\xi)&=\xi\otimes_1 e+\lambda\,T(\xi,e)\otimes_1 T'(\xi,e)\ ,\\
\sigma_Q(\eta\otimes_1\xi)&=\xi\otimes_1\eta+\lambda\,S(\xi,\eta)\otimes_1 S'(\xi,\eta)\ ,
\end{aligned}
$$

so both sides of the equation in the statement above give the following on being applied to $e\otimes_1\xi\otimes_1\eta$:

$$\eta\otimes_1\xi\otimes_1 e+\lambda\,\eta\otimes_1 T(\xi,e)\otimes_1 T'(\xi,e)+\lambda\,T(\eta,e)\otimes_1\xi\otimes_1 T'(\eta,e)+\lambda\,S(\eta,\xi)\otimes_1 S'(\eta,\xi)\otimes_1 e\ ,$$

the other terms being $O(\lambda^2)$. □

This includes that σ_Q itself always obeys the braid relations to $O(\lambda^2)$. It is then interesting to ask if it is involutive or strictly braided. Using $[\nabla_k,\nabla_i]\mathrm{d}x^s=-R^s{}_{nki}\,\mathrm{d}x^n$, we have

$$\sigma_Q(\eta\otimes_1\xi)\ =\xi\otimes_1\eta+\lambda\,\omega^{ij}\,\nabla_j\xi\otimes_1\nabla_i\eta-\lambda\,\omega^{ij}\xi_j\,\eta_s\,R^s{}_{nki}\,\mathrm{d}x^k\otimes_1\mathrm{d}x^n$$

for 1-forms $\xi=\xi_j\mathrm{d}x^j,\eta=\eta_s\mathrm{d}x^s\in\Omega^1(M)$, and hence

$$
\begin{aligned}
\sigma_Q^2(\eta\otimes_1\xi)\ &=\eta\otimes_1\xi+\lambda\,\omega^{ij}\,\nabla_i\eta\otimes_1\nabla_j\xi+\lambda\,\omega^{ij}\,\nabla_j\eta\otimes_1\nabla_i\xi-\lambda\,\omega^{ij}\xi_j\eta_s\,R^s{}_{nki}\,\mathrm{d}x^n\otimes_1\mathrm{d}x^k\\
&\quad-\lambda\,\omega^{ij}\xi_s\eta_j\,R^s{}_{nki}\,\mathrm{d}x^k\wedge\otimes_1\mathrm{d}x^n\\
&=\eta\otimes_1\xi-\lambda\,\xi_j\eta_s(\omega^{ij}\,R^s{}_{nki}+\omega^{is}\,R^j{}_{kni})\,\mathrm{d}x^n\otimes_1\mathrm{d}x^k\ .
\end{aligned}
$$

In these formulae the classical 1-forms are being viewed in $\Omega^1(A)$ so $\sigma_Q:\Omega^1(A)^{\otimes 2}\to\Omega^1(A)^{\otimes 2}$. We see that the curvature causes σ_Q to be not involutive in keeping with experience in other contexts.

3.4. **Quantizing other connections relative to (E,∇_E).** Our constructions require an underlying connection ∇_E as part of the quantisation data for a bundle E and we have seen how to quantise it. Any other covariant derivative on E is given by $\nabla_S=\nabla_E+S$, where $S:E\to\Omega^1(M)\otimes_0 E$ is a bundle map. This has the form of a left module map added to a left covariant derivative to give another left covariant derivative on the same bundle and we take the same approach for the quantisation using $Q(S)$ from Lemma 3.2.

Corollary 3.10. *For any bundle map $S:E\to\Omega^1(M)\otimes_0 E$,*

$$\nabla_{QS}=\nabla_{Q(E)}+q^{-1}_{\Omega^1,E}Q(S),\quad \sigma_{QS}(e\otimes_1\xi)=\sigma_{Q(E)}(e\otimes_1\xi)+\lambda\,\omega^{ij}\,\xi_i\,\mathbb{\nabla}_j(S)(e)$$

defines a bimodule connection on $Q(E)$ to $O(\lambda^2)$.

Proof. That we have a connection is an immediate corollary of Theorem 3.5 which constructs $\nabla_{Q(E)}$ and Lemma 3.2 which tells us that $Q(S)$ is a left module map to $O(\lambda^2)$. We have explicitly

$$q_{\Omega^1,E}\nabla_{QS}=q_{\Omega^1,E}\,\nabla_{Q(E)}+S+\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes E i}\circ\mathbb{\nabla}_j(S)\ .$$

The new part is that this is a bimodule connection to $O(\lambda^2)$. From (2.2) we require to $O(\lambda^2)$,

$$\begin{aligned}\sigma_{QS}(e\otimes_1 \mathrm{d}a)&=\sigma_{Q(E)}(e\otimes_1\mathrm{d}a)+S([e,a])+[a,S(e)]\\&=\sigma_{Q(E)}(e\otimes_1\mathrm{d}a)-\lambda\,\omega^{ij}\,a_{,i}\,S(\nabla_{Ej}e)+\lambda\,\omega^{ij}\,a_{,i}\,\nabla_{\Omega^1\otimes Ej}S(e)\\&=\sigma_{Q(E)}(e\otimes_1\mathrm{d}a)+\lambda\,\omega^{ij}\,a_{,i}\,\mathbb{\nabla}_j(S)(e)\end{aligned}$$

which is well-defined as the correction term factors through $\mathrm{d}a$. In this way our connection becomes a bimodule connection on E with respect to $\bullet$ at our order. □

Now we look at the tensor products and reality of such quantised connections:

Proposition 3.11. *Given $S:E\to\Omega^1(M)\otimes_0 E$ and $T:F\to\Omega^1(M)\otimes_0 F$, define $H:E\otimes_0 F\to\Omega^1(M)\otimes_0 E\otimes_0 F$ by*

$$H=S\otimes\mathrm{id}_F+(\tau\otimes\mathrm{id}_F)(\mathrm{id}_E\otimes T)\ .$$

where τ is transposition. Then the tensor product ∇_{QST} of ∇_{QS} and ∇_{QT} is given as a connection on $Q(E)\otimes_1 Q(F)$ to $O(\lambda^2)$ by

$$\nabla_{QST}(e\otimes_1 f)=q^{-1}_{E,F}\nabla_{QH}\,q_{E,F}(e\otimes_1 f)+\lambda\omega^{ij}\,(\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]e-\mathbb{\nabla}_i(S)(e))\otimes T_j(f)$$

where $T(f)=\mathrm{d}x^k\otimes T_k(f)$.

Proof. Here ∇_{QH} leads with $q^{-1}_{\Omega^1,E\otimes_0F}$ which we clear along with $q^{-1}_{E,F}$ by multiplying both sides by $q^2:=q_{\Omega^1,E\otimes_0F}(\mathrm{id}\otimes q_{E,F})=q_{\Omega^1\otimes_0E,F}(q_{\Omega^1,E}\otimes\mathrm{id})$ in view of Proposition 3.3. Hence we prove

$$q^2\nabla_{QST}=q_{\Omega^1,E\otimes_0F}\,\nabla_{QH}\,q_{E,F}+\lambda\,\mathrm{rem}:Q(E)\otimes_1Q(F)\to Q(\Omega^1(M)\otimes_0E\otimes_0F)$$

where

$$\mathrm{rem}(e\otimes_0 f)=\omega^{ij}\,(\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]e-\mathbb{\nabla}_i(S)(e))\otimes T_j(f)$$

can be treated classically since we are already at order λ. To this end we compute

$$\begin{aligned}q^2(\nabla_{QS}\otimes_1\mathrm{id}_F)&=q^2((\nabla_{Q(E)}+q^{-1}_{\Omega^1,E}\,Q(S))\otimes_1\mathrm{id}_F)\\&=q^2(\nabla_{Q(E)}\otimes_1\mathrm{id}_F)+q_{\Omega^1\otimes_0E,F}(Q(S)\otimes_1\mathrm{id}_F)\\&=q^2(\nabla_{Q(E)}\otimes_1\mathrm{id}_F)+(Q(S)\otimes\mathrm{id}_F+\tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_{Fj})q_{E,F}\end{aligned}$$

where we used the second form of q^2 for the second equality and the deformed functoriality property of q displayed in Proposition 3.3 for the third equality (applied to the bundle map $S:E\to\Omega^1(M)\otimes_0E$). We next compute

$$q^2(\sigma_{QS}\otimes_1\mathrm{id}_F)(\mathrm{id}_E\otimes_1\nabla_{QT})$$

$$\begin{aligned}
&= q_{\Omega^1\otimes_0 E,F}(q\sigma_{QS}q^{-1}\otimes_1 \mathrm{id}_F)(q\otimes_1 \mathrm{id}_F)(\mathrm{id}_E\otimes_1 \nabla_{QT})\\
&= (q\sigma_{QS}q^{-1}\otimes \mathrm{id}_F + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(q\sigma_{QS}q^{-1})\otimes\nabla_j)q^2(\mathrm{id}_E\otimes_1 \nabla_{QT})\\
&= (q\sigma_{QS}q^{-1}\otimes \mathrm{id}_F + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(q\sigma_{QS}q^{-1})\otimes\nabla_j)q_{E,\Omega^1\otimes F}(\mathrm{id}_E\otimes_1 q\nabla_{QT})\\
&= (q\sigma_{QS}q^{-1}\otimes \mathrm{id}_F + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(q\sigma_{QS}q^{-1})\otimes\nabla_j)q_{E,\Omega^1\otimes F}(\mathrm{id}_E\otimes_1 (q\nabla_{Q(F)} + Q(T)))
\end{aligned}$$

where $q\sigma_{QS}q^{-1}$ is $q_{\Omega^1,E}\sigma_{QS}q_{E,\Omega^1}{}^{-1}$ with labels added. Now $\lambda\,\sigma_{QS} = \lambda\,\tau$ to order λ, where τ is transposition, so $\frac{\lambda}{2}\,\nabla_i(q\sigma_{QS}q^{-1}) = 0$. Then, where we set $q\sigma_{QS}q^{-1} = q\sigma_{Q(E)}q^{-1} + \lambda\,S'$,

$$\begin{aligned}
&q^2(\sigma_{QS}\otimes_1 \mathrm{id}_F)(\mathrm{id}_E\otimes_1 \nabla_{QT})\\
&\quad= (q\sigma_{QS}q^{-1}\otimes \mathrm{id}_F)q\big(\mathrm{id}_E\otimes_1 q\nabla_{Q(F)} + \mathrm{id}_E\otimes_1 Q(T)\big)\\
&\quad= ((q\sigma_{Q(E)}q^{-1} + \lambda\,S')\otimes \mathrm{id}_F)q(\mathrm{id}_E\otimes_1 q\nabla_{Q(F)})\\
&\qquad + ((q\sigma_{Q(E)}q^{-1} + \lambda\,S')\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes Q(T) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ei}\otimes \mathbb{\nabla}_j(Q(T)))\,q_{E,F}\\
&\quad= ((q\sigma_{Q(E)}q^{-1} + \lambda\,S')\otimes \mathrm{id}_F)q(\mathrm{id}_E\otimes_1 q\nabla_{Q(F)})\\
&\qquad + ((q\sigma_{Q(E)}q^{-1} + \lambda\,S')\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes Q(T) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{Ei}\otimes \mathbb{\nabla}_j(T))\,q_{E,F}\ .
\end{aligned}$$

It follows that the contribution of S and T to q^2 of the tensor product derivative is

$$\begin{aligned}
&(Q(S)\otimes \mathrm{id}_F)\,q_{E,F} + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_{Fj} + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_F)\\
&\quad + (q\sigma_{Q(E)}q^{-1}\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes Q(T))\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,(\tau\otimes \mathrm{id}_F)(\nabla_{Ei}\otimes\mathbb{\nabla}_j(T))\\
&\quad + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\\
&\ = (Q(S)\otimes \mathrm{id}_F)\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_j + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_T)\\
&\quad + (q\sigma_{Q(E)}q^{-1}\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes Q(T))\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,(\tau\otimes \mathrm{id}_F)(\nabla_{Ei}\otimes\mathbb{\nabla}_j(T))\ .
\end{aligned}\tag{3.19}$$

From Theorem 3.5 we expand $q\sigma_{Q(E)}q^{-1}$ explicitly as

$$q_{\Omega^1,E}\,\sigma_{Q(E)}q_{E,\Omega^1}{}^{-1}(e\otimes\xi) = \xi\otimes e + \lambda\,\omega^{ij}\,\xi_j\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]e\ ,$$

which is of the form $q\,\sigma_{Q(E)}q^{-1} = \tau + \lambda\,\sigma'$ where τ flips the $\otimes_1$ factors and σ' contains the order λ correction. Then (3.19) becomes

$$\begin{aligned}
&(S\otimes \mathrm{id}_F)\,q_{E,F} + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_{Fj} + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_T)\\
&\quad + (\tau\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes Q(T))\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,(\tau\otimes \mathrm{id}_F)(\nabla_{Ei}\otimes\mathbb{\nabla}_j(T))\\
&\quad + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\circ\mathbb{\nabla}_j(S)\otimes \mathrm{id}_F + \lambda\,(\sigma'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\\
&\ = (S\otimes \mathrm{id}_F)\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_{Fj} + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_T)\\
&\quad + (\tau\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\,q + \tfrac{\lambda}{2}\,\omega^{ij}\,(\tau\otimes \mathrm{id}_F)(\nabla_i\otimes\mathbb{\nabla}_j(T))\\
&\quad + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes E}{}_i\circ\mathbb{\nabla}_j(S)\otimes \mathrm{id}_F + \lambda\,(\sigma'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\\
&\quad + \tfrac{\lambda}{2}\,\omega^{ij}\,(\tau\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_i\circ\mathbb{\nabla}_j(T))\ .
\end{aligned}\tag{3.20}$$

Now we use H given above with

$$\mathbb{\nabla}_j(H) = \mathbb{\nabla}_j(S)\otimes \mathrm{id}_F + (\tau\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\mathbb{\nabla}_j(T))$$

to write (3.20) as

$$Q(H)\,q + \lambda\,\omega^{ij}\,\mathbb{\nabla}_i(S)\otimes\nabla_j + \lambda\,(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_T) + \lambda\,(\sigma'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\ .$$

Now from Corollary 3.10

$$\begin{aligned}
(S'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes\nabla_T)(e\otimes f) &= (S'\otimes \mathrm{id}_F)(e\otimes \mathrm{d}x^k\otimes(\nabla_{Fk}f + T_k(f)))\\
&= \omega^{ij}\,\mathbb{\nabla}_j(S)(e)\otimes(\nabla_{Fi}f + T_i(f))
\end{aligned}$$

so we rewrite (3.20) as

$$Q(H)\,q + \lambda\,\omega^{ij}\,\mathbb{\nabla}_j(S)\otimes T_i + \lambda\,(\sigma'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)\ .$$

Finally, writing $T(f) = \mathrm{d}x^i\otimes T_i(f)$,

$$\begin{aligned}
(\sigma'\otimes \mathrm{id}_F)(\mathrm{id}_E\otimes T)(e\otimes f) &= (\sigma'\otimes \mathrm{id}_F)(e\otimes \mathrm{d}x^p\otimes T_p(f))\\
&= \lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]e\otimes T_j(f)\ . \qquad\square
\end{aligned}$$

Lemma 3.12. *If $\lambda^* = -\lambda$ and S is real, the difference in going clockwise minus anti-clockwise round the diagram*

$$
(3.21)\qquad
\begin{array}{ccccc}
\overline{Q(E)} = Q(\overline{E}) & \xleftarrow{\;\star\;} & Q(E) & \xrightarrow{\;\nabla_{QS}\;} & Q(\Omega^1(M))\otimes_1 Q(E) \\
\downarrow{\scriptstyle \overline{\nabla_Q S}} & & & & \downarrow{\scriptstyle \star\otimes_1\star} \\
\overline{Q(\Omega^1(M))\otimes_1 Q(E)} & \xleftarrow{\;\overline{\sigma_{QS}}\;} & \overline{Q(E)}\otimes_1 \overline{Q(\Omega^1(M))} & \xleftarrow{\;\Upsilon^{-1}\;} & \overline{Q(\Omega^1(M))}\otimes_1 \overline{Q(E)}
\end{array}
$$

starting at $Q(e)\in Q(E)$ is

$$
\overline{\lambda\,\omega^{ij}\,\mathbb{W}_j(S)(S_i(e^*))} - \overline{\lambda\,\omega^{ij}\,\mathbb{W}_i(\mathbb{W}_j(S))(e^*)} + \overline{\lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_j(e^*)}
$$

to $O(\lambda^2)$.

Proof. From lemma 3.8 the diagram commutes for $S=0$. We look only at the difference from the $S=0$ to the general S case. Going anticlockwise from $Q(E)$ we get

$$
\overline{q_{\Omega^1,E}{}^{-1}Q(S)}\star(e) = \overline{q_{\Omega^1,E}{}^{-1}Q(S)(e^*)}
$$

Going clockwise is more complicated, as two of the arrows involve S. If we set $q\sigma_{QS}q^{-1} = q\sigma_{Q(E)}q^{-1}+\lambda\,S'$ as in the proof of Proposition 3.11, then to order λ we get the clockwise contributions, omitting repeats of indices on q once we have given them in full,

$$
\begin{aligned}
&\overline{\sigma_{Q(E)}}\,\Upsilon^{-1}(\star\otimes_1\star)q_{\Omega^1,E}{}^{-1}Q(S)(Q(e)) + \overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,\nabla_S(e)\\
&\quad= \overline{\sigma_{Q(E)}}\,\Upsilon^{-1}\,q_{\overline{\Omega^1},\overline{E}}{}^{-1}(\star\otimes_0\star)Q(S)(Q(e)) + \overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,\nabla_S(e)\\
&\quad= \overline{\sigma_{Q(E)}\,q_{E,\Omega^1}{}^{-1}}\,\Upsilon^{-1}(\star\otimes_0\star)Q(S)(Q(e)) + \overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,\nabla_S(e)\\
&\quad= \overline{q_{\Omega^1,E}{}^{-1}\,q_{\Omega^1,E}\,\sigma_{Q(E)}\,q_{E,\Omega^1}{}^{-1}}\,\Upsilon^{-1}(\star\otimes_0\star)Q(S)(Q(e)) + \overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,\nabla_S(e)\\
&\quad= \overline{q^{-1}\,q\,\sigma_{Q(E)}\,q^{-1}}\,\Upsilon^{-1}(\star\otimes_0\star)(S(e)+\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e)))\\
&\qquad+\overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,(\mathrm{d}x^p\otimes\nabla_{Ep}e+S(e))\\
&\quad= \overline{q^{-1}\,\tau}\,\Upsilon^{-1}(\star\otimes_0\star)(S(e)+\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e)))\\
&\qquad+\overline{\lambda\,q^{-1}\,\sigma'}\,\Upsilon^{-1}(\star\otimes_0\star)(S(e)) + \overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,(\mathrm{d}x^p\otimes\nabla_{Ep}e+S(e))\ ,
\end{aligned}
$$

where we have put $q\,\sigma_{Q(E)}q^{-1}=\tau+\lambda\,\sigma'$. As the classical connections preserve $\star$ and λ is imaginary, we get the following for the clockwise contributions, where $S(e)=\mathrm{d}x^p\otimes S_p(e)$,

$$
\begin{aligned}
&\overline{q_{\Omega^1,E}{}^{-1}\,(S(e^*)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*)))} + \overline{\lambda\,q^{-1}\,\sigma'}\,\Upsilon^{-1}(\star\otimes_0\star)(\mathrm{d}x^p\otimes S_p(e))\\
&\quad+\overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,(\mathrm{d}x^p\otimes\nabla_{Ep}e+S(e))\\
&= \overline{q^{-1}\,(S(e^*)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*)))} + \overline{\lambda\,q^{-1}\,\sigma'}\,\Upsilon^{-1}(\overline{\mathrm{d}x^p}\otimes\overline{S_p(e^*)})\\
&\quad+\overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,(\mathrm{d}x^p\otimes\nabla_{Ep}e+S(e))\\
&= \overline{q^{-1}\,(S(e^*)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*)))} + \overline{\lambda\,q^{-1}\,\sigma'(S_p(e^*)\otimes\mathrm{d}x^p)}\\
&\quad+\overline{\lambda\,S'}\,\Upsilon^{-1}(\star\otimes\star)\,(\mathrm{d}x^p\otimes(\nabla_{Ep}e+S_p(e)))\\
&= \overline{q^{-1}\,(S(e^*)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*)))} + \overline{\lambda\,q^{-1}\,(\omega^{ip}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_p(e^*))}\\
&\quad+\overline{\lambda\,S'((\nabla_{Ep}e^*+S_p(e^*))\otimes\mathrm{d}x^p)}\\
&= \overline{q^{-1}\,(S(e^*)-\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*)))} + \overline{\lambda\,q^{-1}\,(\omega^{ip}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_p(e^*))}\\
&\quad+\overline{\lambda\,\omega^{pj}\,\mathbb{W}_j(S)(\nabla_{Ep}e^*+S_p(e^*))}\ .
\end{aligned}
$$

Then the difference, clockwise minus anticlockwise, is to order λ,

$$
\begin{aligned}
&-\overline{\lambda\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*))} + \overline{\lambda\,\omega^{ip}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_p(e^*)} + \overline{\lambda\,\omega^{pj}\,\mathbb{W}_j(S)(\nabla_{Ep}e^*+S_p(e^*))}\\
&= -\overline{\lambda\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S)(e^*))} + \overline{\lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_j(e^*)} + \overline{\lambda\,\omega^{ij}\,\mathbb{W}_j(S)(\nabla_{Ei}e^*+S_i(e^*))}\\
&= \overline{\lambda\,\omega^{ij}\,\mathbb{W}_j(S)(S_i(e^*))} - \overline{\lambda\,\omega^{ij}\,\nabla_{\Omega^1\otimes Ei}(\mathbb{W}_j(S))(e^*)} + \overline{\lambda\,\omega^{ij}\,\mathrm{d}x^k\otimes[\nabla_{Ek},\nabla_{Ei}]S_j(e^*)}\ . \qquad\square
\end{aligned}
$$

Note that Lemma 3.12 shows that $\nabla_{QS}(\star)$ is λ times a module map (i.e. it involves no derivatives of e). This means that $\nabla_{QS}(\star)$ is also a right module map, and thus it is automatically star-compatible at order λ in the sense described in [8]. We will also need the following observation.

Lemma 3.13. *Let (E, ∇_E) be a bundle with connection and $e \in E$ such that $\nabla_E(e) = 0$. Then to $O(\lambda^2)$, e is central in the quantized bimodule and $\nabla_{Q(E)}(e) = 0$. If in addition $S(e) = 0$ for some $S : E \to \Omega^1 \otimes_0 E$ then $\nabla_{QS}(e) = 0$ to $O(\lambda^2)$.*

Proof. This is immediate from the definitions of $\nabla_{Q(E)}$, $Q(S)$ and ∇_{QS}. Here classically $\nabla_S(e) = \nabla_E(e) + S(e) = 0$ so that $\mathbb{\nabla}_j(S)(e) = \nabla_{\Omega^1\otimes Ej}(S(e)) - S(\nabla_{Ej}(e)) = 0$ from which $Q(S)(e) = 0$. □

4. Semiquantisation of the exterior algebra

In noncommutative geometry the notion of 'differential structure' is largely encoded as a differential graded algebra extending the quantisation of functions to differential forms. The main result in this section adapts the semiquantisation functor of Section 3 to show that the same data that allows us to semiquantise $\Omega^1(M)$ (namely a Poisson bivector ω and a Poisson-compatible connection ∇) provides in fact a canonical semiquantisation of the wedge product of forms of all degree. This is Theorem 4.4. We consider the differential calculus as the backdrop to the Riemannian geometry and hence will refer to the Poisson-compatible connection underlying its quantisation as the *background connection* on $\Omega^1(M)$.

4.1. **Quantizing the wedge product.** Our starting point is a functorial quantum wedge product $\wedge_Q$ obtained as an application of the semiquantisation monoidal functor $Q : \mathcal{D}_0 \to \mathcal{E}_1$ in Proposition 3.3. Now any classical linear covariant derivative extends to forms of all degree as a derivation because the usual tensor product covariant derivative on $\Omega^1(M) \otimes_0 \Omega^1(M)$ preserves symmetry, so anything in the kernel of $\wedge$ stays in the kernel. So our background connection ∇ automatically extends to forms of all degrees and the classical wedge product $\Omega^m(M) \otimes_0 \Omega^n(M) \to \Omega^{m+n}(M)$ intertwines these covariant derivatives. Thus all $\Omega^n(M)$ become objects in $\mathcal{D}_0$ and the classical wedge products between them are morphisms. We now apply Q to these objects and morphisms to define $\Omega^n(A) = Q(\Omega^n(M))$ and

$$\wedge_Q : Q(\Omega^m(M)) \otimes_1 Q(\Omega^n(M)) \xrightarrow{q} Q(\Omega^m(M) \otimes_0 \Omega^1(M)) \xrightarrow{Q(\wedge)} Q(\Omega^{m+n}(M))$$

$$\xi \wedge_Q \eta = \xi \wedge \eta + \tfrac{\lambda}{2}\omega^{ij}\, \nabla_i\xi \wedge \nabla_j\eta \; . \tag{4.1}$$

This $\wedge_Q$ is associative to $O(\lambda^2)$ since Q is monoidal to this order. We now look at the Leibniz rule for d with respect to it.

Lemma 4.1.

$$\mathrm{d}(\xi \wedge_Q \eta) - (\mathrm{d}\xi) \wedge_Q \eta - (-1)^{|\xi|}\xi \wedge_Q \mathrm{d}\eta = -\lambda H^{ji} \wedge (\partial_i \lrcorner\, \xi) \wedge \nabla_j\eta + \lambda(-1)^{|\xi|}H^{ij} \wedge \nabla_i\xi \wedge (\partial_j \lrcorner\, \eta)$$

where

$$H^{ij} := \tfrac{1}{4}\omega^{is}\left(T^j_{nm;s} - 2R^j{}_{nms}\right)\mathrm{d}x^m \wedge \mathrm{d}x^n.$$

Proof. Using (3.1) in the following form

$$\mathrm{d}(\omega^{ij}) - \omega^{kj}\,\nabla_k(\mathrm{d}x^i) - \omega^{ik}\,\nabla_k(\mathrm{d}x^j) \;=\; 0\;,$$

and also using

$$\mathrm{d}\zeta \;=\; \mathrm{d}x^k \wedge \nabla_k\zeta + \tfrac{1}{2}\,T^s_{kn}\,\mathrm{d}x^k \wedge \mathrm{d}x^n \wedge (\partial_s \lrcorner\, \zeta)$$

and relabeling indices, we find

$$\begin{aligned}\mathrm{d}(\omega^{ij}\,\nabla_i\xi \wedge \nabla_j\eta) = {}& \omega^{ij}\,\nabla_i(\mathrm{d}x^k) \wedge \nabla_k\xi \wedge \nabla_j\eta + \omega^{ij}\,\nabla_j(\mathrm{d}x^k) \wedge \nabla_i\xi \wedge \nabla_k\eta \\ &+ \omega^{ij}\,\mathrm{d}x^k \wedge \nabla_k\nabla_i\xi \wedge \nabla_j\eta + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi \wedge \mathrm{d}x^k \wedge \nabla_k\nabla_j\eta \\ &+ \omega^{ij}\,\tfrac{1}{2}\,T^s_{kn}\,\mathrm{d}x^k \wedge \mathrm{d}x^n \wedge (\partial_s \lrcorner\, \nabla_i\xi) \wedge \nabla_j\eta \\ &+ (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi \wedge \tfrac{1}{2}\,T^s_{kn}\,\mathrm{d}x^k \wedge \mathrm{d}x^n \wedge (\partial_s \lrcorner\, \nabla_j\eta)\end{aligned}$$

$$
\begin{aligned}
&= \omega^{ij}\,\nabla_i(\mathrm{d}x^k\wedge\nabla_k\xi)\wedge\nabla_j\eta + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\nabla_j(\mathrm{d}x^k\wedge\nabla_k\eta)\\
&\quad + \omega^{ij}\,\mathrm{d}x^k\wedge[\nabla_k,\nabla_i]\xi\wedge\nabla_j\eta + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\mathrm{d}x^k\wedge[\nabla_k,\nabla_j]\eta\\
&\quad + \omega^{ij}\,\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\nabla_i\xi)\wedge\nabla_j\eta\\
&\quad + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\nabla_j\eta)\ .
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
&\mathrm{d}(\omega^{ij}\,\nabla_i\xi\wedge\nabla_j\eta) - \omega^{ij}\,\nabla_i\mathrm{d}\xi\wedge\nabla_j\eta - (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\nabla_j\mathrm{d}\eta\\
&\quad = -\,\omega^{ij}\,\nabla_i(\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\xi))\wedge\nabla_j\eta\\
&\qquad - (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\nabla_j(\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\eta))\\
&\qquad + \omega^{ij}\,\mathrm{d}x^k\wedge[\nabla_k,\nabla_i]\xi\wedge\nabla_j\eta + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\mathrm{d}x^k\wedge[\nabla_k,\nabla_j]\eta\\
&\qquad + \omega^{ij}\,\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\nabla_i\xi)\wedge\nabla_j\eta\\
&\qquad + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\tfrac12\,T^s_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\nabla_j\eta)\\
&\quad = -\,\omega^{ij}\,\tfrac12\,T^s_{kn;i}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\xi)\wedge\nabla_j\eta\\
&\qquad - (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\tfrac12\,T^s_{kn;j}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\eta)\\
&\qquad + \omega^{ij}\,\mathrm{d}x^k\wedge[\nabla_k,\nabla_i]\xi\wedge\nabla_j\eta + (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\mathrm{d}x^k\wedge[\nabla_k,\nabla_j]\eta\\
&\quad = -\,\omega^{ij}\,\tfrac12\,T^s_{kn;i}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\xi)\wedge\nabla_j\eta\\
&\qquad - (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\tfrac12\,T^s_{kn;j}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_s\lrcorner\eta)\\
&\qquad - \omega^{ij}\,\mathrm{d}x^k\wedge R^s{}_{nki}\,\mathrm{d}x^n\wedge(\partial_s\lrcorner\xi)\wedge\nabla_j\eta\\
&\qquad - (-1)^{|\xi|}\,\omega^{ij}\,\nabla_i\xi\wedge\mathrm{d}x^k\wedge R^s{}_{nkj}\,\mathrm{d}x^n\wedge(\partial_s\lrcorner\eta)\ .
\end{aligned}
$$

using $\nabla_i(v\lrcorner\xi)=\nabla_i(v)\lrcorner\xi+v\lrcorner\nabla_i\xi$ for the second equality and $[\nabla_k,\nabla_i]\xi=-R^s{}_{nki}\,\mathrm{d}x^n\wedge(\partial_s\lrcorner\xi)$ for the third. This gives the stated result on recognising the expressions in terms of the stated H. □

We see that $\wedge_Q$ will not in general obey the Leibniz rule for the undeformed d. We have a choice of persisting with a modified Leibniz rule perhaps linking up to examples such as [18, 21], or modifying the wedge product, or modifying d. We choose the second option:

Lemma 4.2. *For vector field v, $\xi\in\Omega^n(M)$ and covariant derivative ∇,*

$$
v\lrcorner\mathrm{d}\xi+\mathrm{d}(v\lrcorner\xi)=v\lrcorner\nabla\xi+\wedge(\nabla(v)\lrcorner\xi)+v^j\,T^k_{ji}\,\mathrm{d}x^i\wedge(\partial_k\lrcorner\xi)\ .
$$

(The left hand side here is the usual Lie derivative).

Proof. First we start with a 1-form ξ, when

$$
\begin{aligned}
v\lrcorner\mathrm{d}(\xi_i\,\mathrm{d}x^i)+\mathrm{d}(v\lrcorner\xi_i\,\mathrm{d}x^i) &= v^j\,\xi_{i,j}\,\mathrm{d}x^i - v^i\,\xi_{i,j}\,\mathrm{d}x^j + v^i\,\xi_{i,j}\,\mathrm{d}x^j + v^i{}_{,j}\,\xi_i\,\mathrm{d}x^j\\
&= v^j\,(\xi_{i,j}-\Gamma^k_{ji}\,\xi_k)\,\mathrm{d}x^i + (v^i{}_{,j}+\Gamma^i_{jk}\,v^k)\,\xi_i\,\mathrm{d}x^j + v^j\,T^k_{ji}\,\xi_k\,\mathrm{d}x^i\ .
\end{aligned}
$$

Now we extend this by induction, for $\xi\in\Omega^1(M)$,

$$
\begin{aligned}
v\lrcorner\mathrm{d}(\xi\wedge\eta) &= v\lrcorner(\mathrm{d}\xi\wedge\eta-\xi\wedge\mathrm{d}\eta)\\
&= (v\lrcorner\mathrm{d}\xi)\wedge\eta+\mathrm{d}\xi\wedge(v\lrcorner\eta)-(v\lrcorner\xi)\,\mathrm{d}\eta+\xi\wedge(v\lrcorner\mathrm{d}\eta)\ ,\\
\mathrm{d}(v\lrcorner(\xi\wedge\eta)) &= \mathrm{d}((v\lrcorner\xi)\wedge\eta)-\mathrm{d}(\xi\wedge(v\lrcorner\eta))\\
&= \mathrm{d}(v\lrcorner\xi)\wedge\eta+(v\lrcorner\xi)\,\mathrm{d}(\eta)-\mathrm{d}\xi\wedge(v\lrcorner\eta)+\xi\wedge\mathrm{d}(v\lrcorner\eta)\ .
\end{aligned}
$$

Then, assuming the $\eta\in\Omega^n(M)$ and that the result works for n,

$$
\begin{aligned}
&v\lrcorner\mathrm{d}(\xi\wedge\eta)+\mathrm{d}(v\lrcorner(\xi\wedge\eta))\\
&\quad = \big(\mathrm{d}(v\lrcorner\xi)+v\lrcorner\mathrm{d}\xi\big)\wedge\eta+\xi\wedge\big(v\lrcorner\mathrm{d}\eta+\mathrm{d}(v\lrcorner\eta)\big)\\
&\quad = \big(v\lrcorner\nabla\xi+\wedge(\nabla(v)\lrcorner\xi)+v^j\,T^k_{ji}\,\mathrm{d}x^i\wedge(\partial_k\lrcorner\xi)\big)\wedge\eta\\
&\qquad + \xi\wedge\big(v\lrcorner\nabla\eta+\wedge(\nabla(v)\lrcorner\eta)+v^j\,T^k_{ji}\,\mathrm{d}x^i\wedge(\partial_k\lrcorner\eta)\big)\ . \quad □
\end{aligned}
$$

Proposition 4.3. *Let H^{ij} be as in Lemma 4.1. Then*

$$
\xi\wedge_1\eta=\xi\wedge_Q\eta+\lambda\,(-1)^{|\xi|+1}\,H^{ij}\wedge(\partial_i\lrcorner\xi)\wedge(\partial_j\lrcorner\eta)
$$

is associative to $O(\lambda^2)$ and the Leibniz rule holds to this order if and only if

$$H^{ij} = H^{ji}, \quad \mathrm{d}H^{ij} + \Gamma^i_{rp}\,\mathrm{d}x^p \wedge H^{rj} + \Gamma^j_{rp}\,\mathrm{d}x^p \wedge H^{ir} = 0 \quad \forall i, j.$$

Proof. We write

$$\xi \wedge_1 \eta = \xi \wedge_Q \eta + \lambda\, \xi \widehat{\wedge} \eta$$

where

$$\xi \widehat{\wedge} \eta = (-1)^{|\xi|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta)\,,$$

and for the moment H^{ij} is an arbitrary collection of 2-forms (the first part holds in general). For the first part, we compute

$$\begin{aligned}(\xi \widehat{\wedge} \eta) \wedge \zeta &= (-1)^{|\xi|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta) \wedge \zeta\,,\\ (\xi \wedge \eta) \widehat{\wedge} \zeta &= (-1)^{|\xi|+|\eta|+1} H^{ij} \wedge (\partial_i \lrcorner\, (\xi \wedge \eta)) \wedge (\partial_j \lrcorner\, \zeta)\\ &= (-1)^{|\xi|+|\eta|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \eta \wedge (\partial_j \lrcorner\, \zeta)\\ &\quad + (-1)^{|\eta|+1} H^{ij} \wedge \xi \wedge (\partial_i \lrcorner\, \eta) \wedge (\partial_j \lrcorner\, \zeta)\,,\end{aligned}$$

and

$$\begin{aligned}\xi \wedge (\eta \widehat{\wedge} \zeta) &= (-1)^{|\eta|+1} \xi \wedge H^{ij} \wedge (\partial_i \lrcorner\, \eta) \wedge (\partial_j \lrcorner\, \zeta)\,,\\ \xi \widehat{\wedge} (\eta \wedge \zeta) &= (-1)^{|\xi|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, (\eta \wedge \zeta))\\ &= (-1)^{|\xi|+|\eta|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \eta \wedge (\partial_j \lrcorner\, \zeta)\\ &\quad + (-1)^{|\xi|+1} H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta) \wedge \zeta\,.\end{aligned}$$

Hence

$$(\xi \widehat{\wedge} \eta) \wedge \zeta + (\xi \wedge \eta) \widehat{\wedge} \zeta = \xi \wedge (\eta \widehat{\wedge} \zeta) + \xi \widehat{\wedge} (\eta \wedge \zeta)\,,$$

which given that $\wedge_Q$ is necessarily associative to $O(\lambda^2)$ by functoriality gives the result stated.

Next, using again the given definition of $\xi \widehat{\wedge} \eta$,

$$\begin{aligned}\mathrm{d}(\xi \widehat{\wedge} \eta) &= (-1)^{|\xi|+1} \mathrm{d}H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta) + (-1)^{|\xi|+1} H^{ij} \wedge \mathrm{d}(\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta)\\ &\quad + H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \mathrm{d}(\partial_j \lrcorner\, \eta)\\ &= (-1)^{|\xi|} (\Gamma^i_{rt}\,\mathrm{d}x^t \wedge H^{rj} + \Gamma^j_{rt}\,\mathrm{d}x^t \wedge H^{ir} - G^{ij}) \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta)\\ &\quad + (-1)^{|\xi|+1} H^{ij} \wedge \mathrm{d}(\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta) + H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \mathrm{d}(\partial_j \lrcorner\, \eta)\,.\end{aligned}$$

From Lemma 4.2 we use

$$\begin{aligned}\partial_j \lrcorner\, \mathrm{d}\xi + \mathrm{d}(\partial_j \lrcorner\, \xi) &= \partial_j \lrcorner\, \nabla\xi + \wedge(\nabla(\partial_j) \lrcorner\, \xi) + T^k_{jt}\,\mathrm{d}x^t \wedge (\partial_k \lrcorner\, \xi)\\ &= \nabla_j \xi + \mathrm{d}x^t \wedge \Gamma^s_{tj}\,(\partial_s \lrcorner\, \xi) + T^s_{jt}\,\mathrm{d}x^t \wedge (\partial_s \lrcorner\, \xi)\\ &= \nabla_j \xi + \mathrm{d}x^t \wedge \Gamma^s_{jt}\,(\partial_s \lrcorner\, \xi)\end{aligned}$$

to give

$$\begin{aligned}\mathrm{d}(\xi \widehat{\wedge} \eta) &= (-1)^{|\xi|} (\Gamma^i_{rt}\,\mathrm{d}x^t \wedge H^{rj} + \Gamma^j_{rt}\,\mathrm{d}x^t \wedge H^{ir} - G^{ij}) \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta)\\ &\quad + (-1)^{|\xi|+1} H^{ij} \wedge \big(\nabla_i \xi + \mathrm{d}x^t \wedge \Gamma^s_{it}\,(\partial_s \lrcorner\, \xi) - \partial_i \lrcorner\, \mathrm{d}\xi\big) \wedge (\partial_j \lrcorner\, \eta)\\ &\quad + H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \big(\nabla_j \eta + \mathrm{d}x^t \wedge \Gamma^s_{jt}\,(\partial_s \lrcorner\, \eta) - \partial_j \lrcorner\, \mathrm{d}\eta\big)\,.\end{aligned}$$

Comparing these fragments, we find

$$\begin{aligned}&\mathrm{d}(\xi \widehat{\wedge} \eta) - \mathrm{d}(\xi) \widehat{\wedge} \eta - (-1)^{|\xi|} \xi \widehat{\wedge} \mathrm{d}(\eta)\\ &= H^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge \nabla_j \eta - (-1)^{|\xi|} H^{ij} \wedge \nabla_i \xi \wedge (\partial_j \lrcorner\, \eta) - (-1)^{|\xi|} G^{ij} \wedge (\partial_i \lrcorner\, \xi) \wedge (\partial_j \lrcorner\, \eta)\,,\end{aligned}$$

where $G^{ij} := \mathrm{d}H^{ij} + \Gamma^i_{rp}\,\mathrm{d}x^p \wedge H^{rj} + \Gamma^j_{rp}\,\mathrm{d}x^p \wedge H^{ir}$. Again, this expression holds for any collection H^{ij}.

Now comparing with Lemma 4.1 and taking H^{ij} as defined there, we see that the Leibniz rule holds with respect to $\wedge_1$ if and only if H^{ij} is symmetric and $G^{ij} = 0$. To see that these have to hold separately, one may take η in degree 0 so that the interior product $\partial_j \lrcorner\, \eta = 0$. □

This gives conditions on the curvature and torsion contained in H^{ij} to obtain a differential graded algebra to $O(\lambda^2)$.

4.2. **Results on curvature, torsion and the tensor N.** Here we do some calculations in Riemannian geometry with torsion in order to simplify our two conditions in Proposition 4.3 on the tensor H^{ij}. We use [17] and [42] for the Bianchi identities with torsion;

$$(B1) \qquad \sum_{\text{cyclic permutations}(abc)} \Big(T^k_{bc;a} - R^k{}_{abc} - T^k_{ai}\, T^i_{bc}\Big) \;=\; 0 \;,$$

$$(B2) \qquad \sum_{\text{cyclic permutations}(abc)} \Big(R^k{}_{jbc;a} - R^k{}_{jai}\, T^i_{bc}\Big) \;=\; 0 \;.$$

When working with covariant derivatives with torsion we also have be aware of a technical point on the use of the semicolon notation for the covariant derivative, which only occurs if we use it more than once. If K is a tensor with various indices then by definition $K_{;i} = \nabla_i K$ while $K_{;ij} \neq \nabla_j \nabla_i K$ because in $K_{;ij}$ we take the jth covariant derivative of $K_{;i}$ including i with the existing tensor indices of K, with the result that we have an extra term $-\Gamma^p_{ji} K_{;p}$ which does not appear in $\nabla_j \nabla_i K$. Doing the same for $K_{;ji}$ and comparing implies that

$$K_{;ij} - K_{;ji} = [\nabla_j, \nabla_i] K - T^p_{ji}\, K_{;p}$$

where the commutator $[\nabla_j, \nabla_i]$ is given by the curvature as usual; we see that there is an extra term involving the torsion.

Theorem 4.4. *Suppose that ∇ on $\Omega^1(M)$ is Poisson-compatible. Then the conditions in Proposition 4.3 on H^{ij} hold and we have a differential graded algebra $\Omega(A)$ to $O(\lambda^2)$ and defined by $(\wedge_1, \mathrm{d})$.*

Proof. (1) We first claim that given the compatibility condition (3.1), the 2-forms H^{ij} in Lemma 4.1 obey $H^{ij} = H^{ji}$. To prove this, we differentiate the compatibility condition to obtain

$$\begin{aligned} 0 &= \omega^{ij}{}_{;mn} + \omega^{ik}{}_{;n}\, T^j_{km} + \omega^{kj}{}_{;n}\, T^i_{km} + \omega^{ik}\, T^j_{km;n} + \omega^{kj}\, T^i_{km;n} \\ &= \omega^{ij}{}_{;mn} - (\omega^{is}\, T^k_{sn} + \omega^{sk}\, T^i_{sn})\, T^j_{km} - (\omega^{ks}\, T^j_{sn} + \omega^{sj}\, T^k_{sn})\, T^i_{km} + \omega^{ik}\, T^j_{km;n} + \omega^{kj}\, T^i_{km;n} \end{aligned}$$

which we rearrange as

$$\omega^{ij}{}_{;mn} = \omega^{is}\, T^k_{sn}\, T^j_{km} + \omega^{sj}\, T^k_{sn}\, T^i_{km} + \omega^{sk}(T^i_{sn}\, T^j_{km} + T^i_{sm}\, T^j_{kn}) - \omega^{ik}\, T^j_{km;n} - \omega^{kj}\, T^i_{km;n}$$

Now use

$$\begin{aligned} \omega^{ij}{}_{;mn} - \omega^{ij}{}_{;nm} &= \omega^{sj}\, R^i{}_{snm} + \omega^{is}\, R^j{}_{snm} - T^p_{nm}\, \omega^{ij}{}_{;p} \\ &= \omega^{sj}\, R^i{}_{snm} + \omega^{is}\, R^j{}_{snm} + T^p_{nm}\, (\omega^{ik} T^j_{kp} + \omega^{kj} T^i_{kp}) \;, \end{aligned}$$

where we have used the compatibility condition again, to get

$$\begin{aligned} \omega^{sj}\, R^i{}_{snm} + \omega^{is}\, R^j{}_{snm} &= \omega^{is}(T^k_{sn}\, T^j_{km} - T^k_{sm}\, T^j_{kn}) + \omega^{sj}(T^k_{sn}\, T^i_{km} - T^k_{sm}\, T^i_{kn}) \\ &\quad - \omega^{is}(T^j_{sm;n} - T^j_{sn;m}) - \omega^{sj}(T^i_{sm;n} - T^i_{sn;m}) \\ &\quad - T^k_{nm}\, (\omega^{is} T^j_{sk} + \omega^{sj} T^i_{sk}) \\ &= \omega^{is}(T^k_{sn}\, T^j_{km} - T^k_{sm}\, T^j_{kn} - T^k_{nm}\, T^j_{sk}) - \omega^{is}(T^j_{sm;n} - T^j_{sn;m}) \\ &\quad + \omega^{sj}(T^k_{sn}\, T^i_{km} - T^k_{sm}\, T^i_{kn} - T^k_{nm}\, T^i_{sk}) - \omega^{sj}(T^i_{sm;n} - T^i_{sn;m}) \;, \end{aligned}$$

which we rearrange to obtain

$$\begin{aligned} 0 &= \omega^{is}\big((T^k_{ns}\, T^j_{mk} + T^k_{sm}\, T^j_{nk} + T^k_{mn}\, T^j_{sk}) - (T^j_{sm;n} + T^j_{ns;m}) + R^j{}_{smn}\big) \\ &\quad + \omega^{sj}\big((T^k_{ns}\, T^i_{mk} + T^k_{sm}\, T^i_{nk} + T^k_{mn}\, T^i_{sk}) - (T^i_{sm;n} + T^i_{ns;m}) + R^i{}_{smn}\big) \;. \end{aligned}$$

Using (B1) gives the symmetry of H^{ij}.

$$0 = \omega^{is}(T^j_{mn;s} - R^j{}_{mns} - R^j{}_{nsm}) + \omega^{sj}(T^i_{mn;s} - R^i{}_{mns} - R^i{}_{nsm}) \;.$$

(2) We next claim that if the compatibility condition (3.1) holds then the 2-forms H^{ij} in Lemma 4.1 obey $\mathrm{d}H^{ij}+\Gamma^i_{rp}\,\mathrm{d}x^p\wedge H^{rj}+\Gamma^j_{rp}\,\mathrm{d}x^p\wedge H^{ir}=0$. To prove this we calculate $\mathrm{d}H^{ij}$, noting that the i,j are fixed indices and are not summed with the vector or covector basis. This is the reason for the extra Christoffel symbols entering the following expression:

$$\begin{aligned}\nabla_p(H^{ij}) &= \tfrac14\,\nabla_p\big(\omega^{is}\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^m\wedge\mathrm{d}x^n\big)\\ &= \tfrac14\omega^{is}{}_{;p}\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^m\wedge\mathrm{d}x^n+\tfrac14\omega^{is}\big(T^j_{nm;sp}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-\Gamma^i_{pr}\,H^{rj}-\Gamma^j_{pr}\,H^{ir}\ .\end{aligned}$$

Thus we have, using the compatibility condition,

$$\begin{aligned}&\mathrm{d}x^p\wedge\nabla_p(H^{ij})+\Gamma^i_{rp}\,\mathrm{d}x^p\wedge H^{rj}+\Gamma^j_{rp}\,\mathrm{d}x^p\wedge H^{ir}\\ &=\tfrac14\omega^{is}{}_{;p}\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n+\tfrac14\omega^{is}\big(T^j_{nm;sp}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-T^i_{pr}\,\mathrm{d}x^p\wedge H^{rj}-T^j_{pr}\,\mathrm{d}x^p\wedge H^{ir}\\ &=-\tfrac14\,(\omega^{it}T^s_{tp}+\omega^{ts}T^i_{tp})\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad+\tfrac14\omega^{is}\big(T^j_{nm;sp}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n-T^i_{pr}\,\mathrm{d}x^p\wedge H^{rj}-T^j_{pr}\,\mathrm{d}x^p\wedge H^{ir}\ .\end{aligned}$$

Using (B1) and then differentiating, we see that

$$\sum_{\mathrm{cyclic}(pmn)}\big(T^j_{nm;p}-R^j{}_{pnm}-T^j_{pr}\,T^r_{nm}\big)=0\ ,$$
$$\sum_{\mathrm{cyclic}(pmn)}\big(T^j_{nm;ps}-R^j{}_{pnm;s}-T^j_{pr;s}\,T^r_{nm}-T^j_{pr}\,T^r_{nm;s}\big)=0\ .$$

Since the 3-form has cyclic symmetry in (pmn),

$$\begin{aligned}&\mathrm{d}x^p\wedge\nabla_p(H^{ij})+\Gamma^i_{rp}\,\mathrm{d}x^p\wedge H^{rj}+\Gamma^j_{rp}\,\mathrm{d}x^p\wedge H^{ir}\\ &=-\tfrac14\,(\omega^{it}T^s_{tp}+\omega^{ts}T^i_{tp})\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad+\tfrac14\omega^{is}\big(T^j_{nm;sp}-T^j_{nm;ps}+R^j{}_{pnm;s}+T^j_{pr;s}\,T^r_{nm}+T^j_{pr}\,T^r_{nm;s}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-T^i_{pr}\,\mathrm{d}x^p\wedge H^{rj}-T^j_{pr}\,\mathrm{d}x^p\wedge H^{ir}\\ &=-\tfrac14\,(\omega^{it}T^s_{tp}+\omega^{ts}T^i_{tp})\big(T^j_{nm;s}-2\,R^j{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad+\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}-T^j_{rm}\,R^r{}_{nps}-T^j_{nr}\,R^r{}_{mps}-T^j_{nm;r}\,T^r_{ps}+R^j{}_{pnm;s}\\ &\quad+T^j_{pr;s}\,T^r_{nm}+T^j_{pr}\,T^r_{nm;s}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-T^i_{pr}\,\mathrm{d}x^p\wedge H^{rj}-T^j_{pr}\,\mathrm{d}x^p\wedge H^{ir}\\ &=-\tfrac14\,\big(\omega^{is}T^r_{sp}\,(T^j_{nm;r}-2\,R^j{}_{nmr})+\omega^{rs}T^i_{rp}\,(T^j_{nm;s}-2\,R^j{}_{nms})\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad+\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}-T^j_{rm}\,R^r{}_{nps}-T^j_{nr}\,R^r{}_{mps}-T^j_{nm;r}\,T^r_{ps}+R^j{}_{pnm;s}\\ &\quad+T^j_{pr;s}\,T^r_{nm}+T^j_{pr}\,T^r_{nm;s}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-\tfrac14\omega^{rs}(T^j_{nm;s}-2R^j{}_{nms})\,T^i_{pr}\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n-\tfrac14\omega^{is}\,T^j_{pr}\,\big(T^r_{nm;s}-2R^r{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &=-\tfrac14\,\big(\omega^{is}T^r_{sp}\,(-2\,R^j{}_{nmr})\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad+\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}-T^j_{rm}\,R^r{}_{nps}-T^j_{nr}\,R^r{}_{mps}+R^j{}_{pnm;s}+T^j_{pr;s}\,T^r_{nm}-2\,R^j{}_{nms;p}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &\quad-\tfrac14\omega^{is}\,T^j_{pr}\,\big(-2R^r{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &=\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}-T^j_{rm}\,R^r{}_{nps}-T^j_{nr}\,R^r{}_{mps}+R^j{}_{pnm;s}+T^j_{pr;s}\,T^r_{nm}-2\,R^j{}_{nms;p}\\ &\quad+2\,T^r_{sp}\,R^j{}_{nmr}+2\,T^j_{pr}\,R^r{}_{nms}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\ .\end{aligned}$$

Given the overall $\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n$ factor, we can make the following substitutions:
$-R^j{}_{nms;p}\mapsto-R^j{}_{pns;m}\mapsto R^j{}_{psn;m}\mapsto-R^j{}_{psm;n}\mapsto R^j{}_{pms;n}$
$T^r_{sp}\,R^j{}_{nmr}\mapsto T^r_{sm}\,R^j{}_{pnr}\mapsto-T^r_{ms}\,R^j{}_{pnr}\mapsto T^r_{ns}\,R^j{}_{pmr}\mapsto T^r_{ns}\,R^j{}_{pmr}\mapsto-T^r_{sn}\,R^j{}_{pmr}$
$T^j_{pr}\,R^r{}_{nms}\mapsto T^j_{nr}\,R^r{}_{mps}\mapsto-T^j_{mr}\,R^r{}_{nps}\mapsto T^j_{rm}\,R^r{}_{nps}$
Using these we can rewrite the previous equations, and then use (B2) to obtain

$$\begin{aligned}&\mathrm{d}x^p\wedge\nabla_p(H^{ij})+\Gamma^i_{rp}\,\mathrm{d}x^p\wedge H^{rj}+\Gamma^j_{rp}\,\mathrm{d}x^p\wedge H^{ir}\\ &=\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}-T^j_{rm}\,R^r{}_{nps}-T^j_{nr}\,R^r{}_{mps}+T^j_{pr;s}\,T^r_{nm}\\ &\quad+R^j{}_{pnm;s}+R^j{}_{psn;m}+R^j{}_{pms;n}-T^r_{ms}\,R^j{}_{pnr}-T^r_{sn}\,R^j{}_{pmr}\\ &\quad+T^j_{nr}\,R^r{}_{mps}+T^j_{rm}\,R^r{}_{nps}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\\ &=\tfrac14\omega^{is}\big(T^r_{nm}\,R^j{}_{rps}+T^j_{pr;s}\,T^r_{nm}+T^r_{nm}\,R^j{}_{psr}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\end{aligned}$$

$$= \tfrac{1}{4}\,\omega^{is}\,T^r_{nm}\left(R^j{}_{rps} + T^j_{pr;s} + R^j{}_{psr}\right)\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\ .$$

We also need

$$\begin{aligned}
&\tfrac{1}{2}\,T^t_{vu}\,\mathrm{d}x^v\wedge\mathrm{d}x^u\wedge(\partial_t \lrcorner\, H^{ij})\\
&\quad= \tfrac{1}{2}\,T^t_{vu}\,\mathrm{d}x^v\wedge\mathrm{d}x^u\wedge(\partial_t \lrcorner\,(\tfrac{1}{4}\omega^{is}\big(T^j_{nm;s} - 2\,R^j{}_{nms}\big)\,\mathrm{d}x^m\wedge\mathrm{d}x^n))\\
&\quad= \tfrac{1}{8}\,T^p_{vu}\,\omega^{is}\big(T^j_{np;s} - 2\,R^j{}_{nps}\big)\,\mathrm{d}x^v\wedge\mathrm{d}x^u\wedge\mathrm{d}x^n\\
&\qquad- \tfrac{1}{8}\,T^p_{vu}\,\omega^{is}\big(T^j_{pn;s} - 2\,R^j{}_{pns}\big)\,\mathrm{d}x^v\wedge\mathrm{d}x^u\wedge\mathrm{d}x^n\\
&\quad= \tfrac{1}{4}\,T^p_{vu}\,\omega^{is}\big(T^j_{np;s} - R^j{}_{nps} + R^j{}_{pns}\big)\,\mathrm{d}x^v\wedge\mathrm{d}x^u\wedge\mathrm{d}x^n\\
&\quad= \tfrac{1}{4}\,T^r_{mn}\,\omega^{is}\big(T^j_{pr;s} + R^j{}_{psr} + R^j{}_{rps}\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n\ .
\end{aligned}$$

Our claim then follows by using

$$\mathrm{d}H^{ij} = \mathrm{d}x^k\wedge\nabla_k H^{ij} + \tfrac{1}{2}\,T^t_{kn}\,\mathrm{d}x^k\wedge\mathrm{d}x^n\wedge(\partial_t \lrcorner\, H^{ij})\ .$$

Now parts (1) and (2) prove precisely the conditions required in Proposition 4.3 and we conclude the result. □

Proposition 4.5. *For $\lambda^* = -\lambda$, the above $\Omega(A)$ is a $*$-DGA to $O(\lambda^2)$.*

Proof. As both d and $\star$ are undeformed, it is automatic that $\mathrm{d}(\xi^*) = (\mathrm{d}\xi)^*$. Next

$$\begin{aligned}
\eta^*\wedge_1\xi^* &= \eta^*\wedge\xi^* + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\eta^*\wedge\nabla_j\xi^* + \lambda\,(-1)^{|\eta|+1}\,H^{ij}\wedge(\partial_i\lrcorner\,\eta^*)\wedge(\partial_j\lrcorner\,\xi^*)\\
&= \eta^*\wedge\xi^* + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\eta^*\wedge\nabla_j\xi^* + \lambda\,(-1)^{|\eta|+1}\,H^{ij}\wedge(\partial_i\lrcorner\,\eta)^*\wedge(\partial_j\lrcorner\,\xi)^*\\
&= (-1)^{|\xi|\,|\eta|}(\xi\wedge\eta)^* + (-1)^{|\xi|\,|\eta|}\tfrac{\lambda}{2}\,\omega^{ij}\,(\nabla_j\xi\wedge\nabla_i\eta)^*\\
&\quad+ \lambda\,(-1)^{|\eta|+1+(|\eta|-1)(|\xi|-1)}\,H^{ij}\wedge((\partial_j\lrcorner\,\xi)\wedge(\partial_i\lrcorner\,\eta))^*\\
&= (-1)^{|\xi|\,|\eta|}(\xi\wedge\eta)^* + (-1)^{|\xi|\,|\eta|}(\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\xi\wedge\nabla_j\eta)^*\\
&\quad+ \lambda\,(-1)^{|\xi|\,|\eta|+|\xi|+2}\,(H^{ij}\wedge(\partial_j\lrcorner\,\xi)\wedge(\partial_i\lrcorner\,\eta))^*\\
&= (-1)^{|\xi|\,|\eta|}(\xi\wedge\eta)^* + (-1)^{|\xi|\,|\eta|}(\tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i\xi\wedge\nabla_j\eta)^*\\
&\quad+ (-1)^{|\xi|\,|\eta|}\,(\lambda\,(-1)^{|\xi|+1}\,H^{ij}\wedge(\partial_j\lrcorner\,\xi)\wedge(\partial_i\lrcorner\,\eta))^* \quad \square
\end{aligned}$$

4.3. **Quantum torsion of the quantisation ∇_Q of the background connection.** Here we compute the quantum torsion of the quantum connection ∇_Q given by applying Theorem 3.5 to the background connection ∇ on $\Omega^1(M)$ itself. We have already touched upon ∇_Q and its generalised braiding σ_Q at the end of Section 3.3. The torsion of a left connection in noncommutative geometry was covered in Section 2.2 and is automatically a left-module map. It is 'torsion compatible'[8] if it is also a right (hence bi-)module map. In our case $T_{\nabla_Q} = \wedge_1\nabla_Q - \mathrm{d} : \Omega^1(A)\to\Omega^2(A)$ while the torsion of ∇ is the map $T = \wedge\nabla - \mathrm{d} : \Omega^1(M)\to\Omega^2(M)$ written above as a tensor.

Proposition 4.6. *The quantum torsion of ∇_Q to $O(\lambda^2)$ on a 1-form ξ is*

$$T_{\nabla_Q}(\xi) = T(\xi) + \tfrac{\lambda}{4}\,\omega^{is}\,T^j_{nm;s}\,\mathrm{d}x^m\wedge\mathrm{d}x^n\partial_j\lrcorner\,\nabla_i\xi$$

and is a right module map to $O(\lambda^2)$ if and only if $\omega^{ij}\,\nabla_j T = 0$.

Proof. Here all covariant derivatives are the background connection on $\Omega^1(M)$ and we use ∇_Q from Theorem 3.5 and $\wedge_1$ from the preceding sections to find

$$\begin{aligned}
\wedge_1\nabla_Q\xi &= \wedge_1\,q^{-1}\nabla\xi - \tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\wedge[\nabla_k,\nabla_j]\nabla_i\xi\\
&= \wedge_1\,q^{-1}(\mathrm{d}x^k\otimes_0\nabla_k\xi) - \tfrac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^k\wedge[\nabla_k,\nabla_j]\nabla_i\xi\\
&= \mathrm{d}x^k\wedge\nabla_k\xi + \lambda\,H^{ij}\,(\partial_j\lrcorner\,\nabla_i\xi) - \tfrac{\lambda}{2}\,\omega^{is}\,\mathrm{d}x^m\wedge[\nabla_m,\nabla_s]\nabla_i\xi\\
&= \mathrm{d}x^k\wedge\nabla_k\xi + \tfrac{\lambda}{4}\,(\partial_j\lrcorner\,\nabla_i\xi)\,\omega^{is}\,\big(T^j_{nm;s} - 2R^j{}_{nms}\big)\,\mathrm{d}x^m\wedge\mathrm{d}x^n\\
&\quad+ \tfrac{\lambda}{2}\,\omega^{is}\,R^j{}_{nms}\,\mathrm{d}x^m\wedge\mathrm{d}x^n(\partial_j\lrcorner\,\nabla_i\xi)
\end{aligned}$$

using the definition of H^{ij} in Lemma 4.1 for the last equality. This simplifies further to give the result stated for T_{∇_Q}. On the right hand side the leading term is the classical torsion T now viewed as a map $\Omega^1(A) \to \Omega^2(A)$ given that these are essentially identified as vector spaces with their classical counterparts. For the second part

$$\begin{aligned} T_{\nabla_Q}(\xi \bullet a) &= T(\xi a + \tfrac{\lambda}{2}\omega^{ij}(\nabla_i\xi)a_{,j}) + \tfrac{\lambda}{2}\omega^{is}((\partial_j \,\lrcorner\, \nabla_i\xi)a + \xi_j a_{,i})\nabla_s T^j \\ &= T_{\nabla_Q}(\xi)a + \tfrac{\lambda}{2}\omega^{ij}T(\nabla_i\xi)a_{,j} + \tfrac{\lambda}{2}\omega^{is}\xi_j a_{,i}\nabla_s T^j \\ &= T_{\nabla_Q}(\xi)\bullet a + \tfrac{\lambda}{2}\omega^{ij}a_{,j}\big(T(\nabla_i\xi) - \xi_k(\nabla_j T^k) - \nabla_i(T(\xi))\big) \\ &= -\lambda\omega^{ij}a_{,j}\xi_k\nabla_i T^k \end{aligned}$$

where we used the formula for T_{∇_Q} and the definitions of $\bullet$. □

We also note that the functor $Q : \mathcal{D}_0 \to \mathcal{D}_1$ in Theorem 3.5 also gives us that the various ∇_Q on different degrees of $\Omega(A)$ (quantising the ∇ on the corresponding degree of $\Omega(M)$) are compatible with the $\wedge_Q$ to $O(\lambda^2)$ as this was true classically and we apply the functor. This is no longer true of $\wedge_1$ due to its order λ correction.

Proposition 4.7.

$$\begin{aligned} &(\mathrm{id}\otimes_1 \wedge_1)\nabla_{Q\otimes_1 Q}(\xi\otimes_1\eta) - \nabla_Q(\xi\wedge_1\eta) \\ &\quad = \lambda\,(-1)^{|\xi|}\,\mathrm{d}x^k\otimes_1(\nabla_k H^{ij} + \Gamma^i_{kp}H^{pj} + \Gamma^j_{kp}H^{ip})\wedge(\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta) \end{aligned}$$

to $O(\lambda^2)$.

Proof: From modifying (3.11) we get

$$(4.2)\qquad \begin{array}{ccc} Q(\Omega(M))\otimes_1 Q(\Omega(M)) & \xrightarrow{\quad q \quad} & Q(\Omega(M)\otimes_0\Omega(M)) \\ \downarrow{\scriptstyle \nabla_{Q(\Omega(M))\otimes_1 Q(\Omega(M))}} & & \downarrow{\scriptstyle \nabla_{Q(\Omega(M)\otimes_0\Omega(M))}} \\ Q(\Omega(M))\otimes_1 Q(\Omega(M))\otimes_1 Q(\Omega(M)) & \xrightarrow[\mathrm{id}\otimes q]{} & Q(\Omega(M))\otimes_1 Q(\Omega(M)\otimes_0\Omega(M)) \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes_1(\wedge)} \\ & & Q(\Omega(M))\otimes_1 Q(\Omega(M)) \end{array}$$

As classically $\wedge$ intertwines the covariant derivatives,

$$\begin{aligned} &(\mathrm{id}\otimes_1(\wedge q))\nabla_{Q(\Omega(M))\otimes_1 Q(\Omega(M))} = \nabla_{Q(\Omega(M))}(\wedge q) : \\ &\qquad\qquad Q(\Omega(M))\otimes_1 Q(\Omega(M)) \to Q(\Omega(M))\otimes_1 Q(\Omega(M)) \ . \end{aligned}$$

In the notation of Proposition 4.3 we now look at $\xi\wedge_1\eta = \xi\wedge_Q\eta + \lambda\,\xi\,\widehat{\wedge}\,\eta$ where

$$\xi\,\widehat{\wedge}\,\eta = (-1)^{|\xi|+1}H^{ij}\wedge(\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta) \ .$$

Then we obtain

$$\begin{aligned} &\lambda\,(\mathrm{id}\otimes_1\widehat{\wedge})\nabla_{Q(\Omega(M))\otimes_1 Q(\Omega(M))}(\xi\otimes_1\eta) \\ &\quad = \lambda\,(\mathrm{id}\otimes_1\widehat{\wedge})(\mathrm{d}x^k\otimes_1(\nabla_k\xi\otimes_1\eta + \xi\otimes_1\nabla_k\eta)) = \lambda\,\mathrm{d}x^k\otimes_1(\nabla_k\xi\,\widehat{\wedge}\,\eta + \xi\,\widehat{\wedge}\,\nabla_k\eta) \\ &\quad = \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1 H^{ij}\wedge((\partial_i \,\lrcorner\, \nabla_k\xi)\wedge(\partial_j \,\lrcorner\, \eta) + (\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \nabla_k\eta)) \ . \end{aligned}$$

Also, using $\nabla_i(v \,\lrcorner\, \xi) = \nabla_i(v) \,\lrcorner\, \xi + v \,\lrcorner\, \nabla_i\xi$

$$\begin{aligned} &\lambda\,\nabla_{Q(\Omega(M))}(\xi\,\widehat{\wedge}\,\eta) \\ &\quad = \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1\nabla_k(H^{ij}\wedge(\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta)) \\ &\quad = \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1 H^{ij}\wedge((\partial_i \,\lrcorner\, \nabla_k\xi)\wedge(\partial_j \,\lrcorner\, \eta) + (\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \nabla_k\eta)) \\ &\qquad + \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1\nabla_k H^{ij}\wedge(\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta) \\ &\qquad + \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1 H^{ij}\wedge((\Gamma^p_{ki}\partial_p \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta) + (\partial_i \,\lrcorner\, \xi)\wedge(\Gamma^p_{kj}\partial_p \,\lrcorner\, \eta)) \\ &\quad = \lambda\,(\mathrm{id}\otimes_1\widehat{\wedge})\nabla_{Q(\Omega(M))\otimes_1 Q(\Omega(M))}(\xi\otimes_1\eta) \\ &\qquad + \lambda\,(-1)^{|\xi|+1}\,\mathrm{d}x^k\otimes_1(\nabla_k H^{ij} + \Gamma^i_{kp}H^{pj} + \Gamma^j_{kp}H^{ip})\wedge(\partial_i \,\lrcorner\, \xi)\wedge(\partial_j \,\lrcorner\, \eta) \ . \quad \square \end{aligned}$$

4.4. **Quantizing other linear connections relative to the background one.** Here we extend the above to other connections $\nabla_S = \nabla + S$ on $\Omega^1(M)$ different from the background one ∇, where $S(\xi) = \xi_p\, S^p_{nm}\, \mathrm{d}x^n \otimes \mathrm{d}x^m$ for $\xi \in \Omega^1(M)$. Our quantisation to $O(\lambda^2)$ is achieved on the same $\Omega^1(A)$ as already obtained from ∇ by virtue of the construction in Corollary 3.10.

Proposition 4.8. *The torsion of ∇_{QS} to $O(\lambda^2)$ on a 1-form ξ is*

$$\begin{aligned} T_{\nabla_{QS}}(\xi) &= T_{\nabla_S}(\xi) + \tfrac{\lambda}{4}\, \xi_{p;i}\, \omega^{ij}\, (T^p_{nm;j} - 2\, S^p_{nm;j})\, \mathrm{d}x^m \wedge \mathrm{d}x^n \\ &\quad + \lambda\, \xi_p\, (S^p_{nm}\, H^{nm} + \tfrac{1}{2}\, \omega^{ij}\, S^p_{nm;\hat{j}i}\, \mathrm{d}x^n \wedge \mathrm{d}x^m)\ . \end{aligned}$$

Note that the hat on $\hat{j}$ denotes that the j index does not take part in the covariant differentiation in the i direction.

Proof. The definition of the quantum torsion of a connection and of ∇_{QS} gives

$$T_{\nabla_{QS}}(\xi) = T_{\nabla_Q}(\xi) + \wedge_1\, q^{-1} S(\xi) + \wedge\, \tfrac{\lambda}{2}\, \omega^{ij}\, \nabla_i \circ \mathbb{W}_j(S)(\xi)$$

in which

$$\begin{aligned} \wedge_1\, q^{-1} S(\xi) &= \xi_p\, S^p_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m + \lambda\, \xi_p\, S^p_{nm}\, H^{nm}\ , \\ \wedge\, \tfrac{\lambda}{2}\, \omega^{ij}\, \nabla_i \circ \mathbb{W}_j(S)(\xi) &= \wedge\, \tfrac{\lambda}{2}\, \omega^{ij}\, \nabla_i(\xi_p\, S^p_{nm;j}\, \mathrm{d}x^n \otimes \mathrm{d}x^m) \\ &= \tfrac{\lambda}{2}\, \omega^{ij}\, \nabla_i(\xi_p\, S^p_{nm;j}\, \mathrm{d}x^n \wedge \mathrm{d}x^m) \\ &= \tfrac{\lambda}{2}\, \omega^{ij}\, (\xi_{p;i}\, S^p_{nm;j} + \xi_p\, S^p_{nm;\hat{j}i})\, \mathrm{d}x^n \wedge \mathrm{d}x^m\ . \end{aligned}$$

By Proposition 4.6 we have

$$\begin{aligned} T_{\nabla_{QS}}(\xi) &= T_{\nabla}(\xi) + \tfrac{\lambda}{4}\, \xi_{j;i}\, \omega^{is}\, T^j_{nm;s}\, \mathrm{d}x^m \wedge \mathrm{d}x^n + \wedge S(\xi) + \lambda\, \xi_p\, S^p_{nm}\, H^{nm} \\ &\quad + \tfrac{\lambda}{2}\, \omega^{ij}\, (\xi_{p;i}\, S^p_{nm;j} + \xi_p\, S^p_{nm;\hat{j}i})\, \mathrm{d}x^n \wedge \mathrm{d}x^m \\ &= T_{\nabla_S}(\xi) + \tfrac{\lambda}{4}\, \xi_{p;i}\, \omega^{ij}\, T^p_{nm;j}\, \mathrm{d}x^m \wedge \mathrm{d}x^n + \lambda\, \xi_p\, S^p_{nm}\, H^{nm} \\ &\quad + \tfrac{\lambda}{2}\, \omega^{ij}\, (\xi_{p;i}\, S^p_{nm;j} + \xi_p\, S^p_{nm;\hat{j}i})\, \mathrm{d}x^n \wedge \mathrm{d}x^m \\ &= T_{\nabla_S}(\xi) + (\tfrac{\lambda}{4}\, \xi_{p;i}\, \omega^{ij}\, T^p_{nm;j} - \tfrac{\lambda}{2}\, \omega^{ij}\, \xi_{p;i}\, S^p_{nm;j})\, \mathrm{d}x^m \wedge \mathrm{d}x^n \\ &\quad + \lambda\, \xi_p\, S^p_{nm}\, H^{nm} + \tfrac{\lambda}{2}\, \omega^{ij}\, \xi_p\, S^p_{nm;\hat{j}i}\, \mathrm{d}x^n \wedge \mathrm{d}x^m \end{aligned}$$

which we recognise as the expression stated. □

Corollary 4.9. *If ∇_S is torsion free then $T_{\nabla_{QS}}(\xi) = \frac{\lambda}{2}\, \xi_p\, A^p_{nm}\, \mathrm{d}x^m \wedge \mathrm{d}x^n$ to $O(\lambda^2)$ with*

$$A^p_{nm} = \tfrac{1}{4}\omega^{is}\, (S^p_{ij} + S^p_{ji})\, (T^j_{nm;s} - R^j{}_{nms} + R^j{}_{mns}) - \tfrac{1}{4}\, \omega^{ij}\, (T^s_{nm}\, R^p{}_{sij} - T^p_{sm}\, R^s{}_{nij} + T^p_{sn}\, R^s{}_{mij})\ .$$

Proof. We begin with

$$\wedge\, \nabla_S(\mathrm{d}x^p) = \wedge\, \nabla(\mathrm{d}x^p) + S^p_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m$$

so $0 = T_{\nabla}(\mathrm{d}x^p) + S^p_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m$ and from (2.1) we deduce

$$S^p_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m = \tfrac{1}{2}\, T^p_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m\ . \tag{4.3}$$

Then Proposition 4.8 gives

$$T_{\nabla_{QS}}(\xi) = T_{\nabla_S}(\xi) + \tfrac{\lambda}{2}\, \xi_p\, (2\, S^p_{nm}\, H^{nm} + \tfrac{1}{2}\, \omega^{ij}\, T^p_{nm;\hat{j}i}\, \mathrm{d}x^n \wedge \mathrm{d}x^m)\ ,$$

in which we use the formula for the curvature of a tensor and the symmetry of H^{nm}. □

We see that quantisation of a torsion free covariant derivative introduces quantum torsion λA at order λ A^i_{mn} looks like a classical torsion tensor and is given by the expression stated. We similarly look at Lemma 3.12 to measure the deviation of ∇_{QS} from being star preserving and again find an error of order λ. We will then modify ∇_{QS} to kill both quantum corrections.

Lemma 4.10. *If $\lambda^* = -\lambda$ and S is real then the difference $D^a_{ijnm}\overline{\frac{\lambda}{2}\,\omega^{ij}\,\mathrm{d}x^n\otimes\mathrm{d}x^m}$ to $O(\lambda^2)$ in going clockwise minus anticlockwise round the diagram in Lemma 3.12 starting from $Q(\mathrm{d}x^a)$ is given by*

$$D^a_{ijnm} = 2\,S^a_{ip}\,S^p_{nm;j} - (S^b_{nm}\,R^a{}_{bij} - S^a_{rm}\,R^r{}_{nij} - S^a_{nr}\,R^r{}_{mij}) - 2\,S^a_{jr}\,R^r{}_{mni}$$

Proof. Putting $e = e^* = \mathrm{d}x^a$ in Lemma 3.12, and using $\mathbb{V}_j(S)(\xi) = \xi_p\,S^p_{nm;j}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$ we get

$$\mathbb{V}_j(S)(S_i(\mathrm{d}x^a)) = \mathbb{V}_j(S)(S^a_{ir}\,\mathrm{d}x^r) = S^a_{ip}\,S^p_{km;j}\,\mathrm{d}x^k\otimes\mathrm{d}x^m\ ,$$
$$\mathrm{d}x^k\otimes[\nabla_k,\nabla_i]S_j(\mathrm{d}x^a) = \mathrm{d}x^k\otimes[\nabla_k,\nabla_i](S^a_{jr}\,\mathrm{d}x^r) = -\,S^a_{jr}\,R^r{}_{mki}\,\mathrm{d}x^k\otimes\mathrm{d}x^m\ .$$

Now we use the antisymmetry of ω^{ij} to get

$$\omega^{ij}\,\nabla_{\Omega^1\otimes\Omega^1 i}(\mathbb{V}_j(S))(\mathrm{d}x^a) = \tfrac{1}{2}\,\omega^{ij}\,(S^b_{km}\,R^a{}_{bij} - S^a_{rm}\,R^r{}_{kij} - S^a_{kr}\,R^r{}_{mij})\mathrm{d}x^k\otimes\mathrm{d}x^m\ .$$

We note that this is equivalent to the derivative of $\star : Q(\Omega^1(M)) \to Q(\overline{\Omega^1(M)})$ being

$$\nabla_{QS}(\star)(Q(\mathrm{d}x^a)) = \tfrac{\lambda}{2}\,\omega^{ij}\,D^a_{ijkm}\,\mathrm{d}x^k\otimes\overline{\mathrm{d}x^m}$$

and we see that this is not necessarily zero. □

Next we consider adding a correction, so

$$\nabla_1 = \nabla_{QS} + \lambda K$$

where $K : \Omega^1(M) \to \Omega^1(M)\otimes_0\Omega^1(M)$ is given by $K(\xi) = \xi_p\,K^p_{nm}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$.

Theorem 4.11. *If $\lambda^* = -\lambda$ and S is real then there is a unique real K such that $\nabla_1 = \nabla_{QS} + \lambda K$ is star preserving to $O(\lambda^2)$ (namely $K^a_{nm} = \frac{1}{4}\,\omega^{ij}\,D^a_{ijnm}$). Moreover, if ∇_S is torsion free then ∇_1 is quantum torsion free to this order.*

Proof. We look at the following diagram:

$$\begin{array}{ccccc}
\overline{Q(\Omega^1(M))} = Q(\overline{\Omega^1(M)}) & \xleftarrow{\ \star\ } & Q(\Omega^1(M)) & \xrightarrow{\ \lambda K\ } & Q(\Omega^1(M))\otimes_1 Q(\Omega^1(M)) \\
\Big\downarrow{\scriptstyle \overline{\lambda K}} & & & & \Big\downarrow{\scriptstyle \star\otimes_1\star} \\
\overline{Q(\Omega^1(M))\otimes_1 Q(\Omega^1(M))} & \xleftarrow{\ \overline{\sigma_{QS}}\ } & \overline{Q(\Omega^1(M))\otimes_1 Q(\Omega^1(M))} & \xleftarrow{\ \Upsilon^{-1}\ } & \overline{Q(\Omega^1(M))}\otimes_1\overline{Q(\Omega^1(M))}
\end{array}$$

where at this order σ_{QS} is simply transposition. Hence for $\nabla_{QS} + \lambda K$ the effect of adding K is to add

$$-\,\overline{\lambda\,(K^a_{nm} + (K^a_{nm})^*)\,\mathrm{d}x^n\otimes\mathrm{d}x^m}\ .$$

to the difference in Lemma 4.10. This gives the unique value if we assume K is real for the connection to be $\star$-preserving. Adding K also adds $\lambda\,\xi_a\,K^a_{nm}\,\mathrm{d}x^n\wedge\mathrm{d}x^m$ to the formula for the torsion in Proposition 4.8 so if K has the unique real value stated and if ∇_S is torsion free, and using (4.3),

$$\begin{aligned}
K^a_{nm}\,\mathrm{d}x^n\wedge\mathrm{d}x^m &= \tfrac{1}{4}\,\omega^{ij}\,D^a_{ijnm}\,\mathrm{d}x^n\wedge\mathrm{d}x^m \\
&= \tfrac{1}{4}\,\omega^{ij}\big(S^a_{ip}\,T^p_{nm;j} - (\tfrac{1}{2}\,T^b_{nm}\,R^a{}_{bij} - S^a_{rm}\,R^r{}_{nij} + S^a_{mr}\,R^r{}_{nij}) - 2\,S^a_{jr}\,R^r{}_{mni}\big)\,\mathrm{d}x^n\wedge\mathrm{d}x^m \\
&= \tfrac{1}{4}\,\omega^{ij}\big(S^a_{ip}\,T^p_{nm;j} - (\tfrac{1}{2}\,T^b_{nm}\,R^a{}_{bij} + T^a_{mr}\,R^r{}_{nij}) + 2\,S^a_{ir}\,R^r{}_{mnj}\big)\,\mathrm{d}x^n\wedge\mathrm{d}x^m \\
&= \tfrac{1}{4}\,S^a_{ip}\,\omega^{ij}\big(T^p_{nm;j} + 2\,R^p{}_{mnj}\big)\,\mathrm{d}x^n\wedge\mathrm{d}x^m - \tfrac{1}{4}\,\omega^{ij}\big(\tfrac{1}{2}\,T^b_{nm}\,R^a{}_{bij} + T^a_{mr}\,R^r{}_{nij}\big)\,\mathrm{d}x^n\wedge\mathrm{d}x^m \\
&= -\,S^a_{ip}\,H^{ip} - \tfrac{1}{8}\,\omega^{ij}(T^b_{nm}\,R^a{}_{bij} + 2\,T^a_{mr}\,R^r{}_{nij})\,\mathrm{d}x^n\wedge\mathrm{d}x^m
\end{aligned}$$

where the 2nd equality used antisymmetry in m, n given the wedge product. Now Corollary 4.9 tells us that $\nabla_{QS} + \lambda K$ is torsion free. □

We have achieved a unique star-preserving quantisation to $O(\lambda^2)$ of any real connection on $\Omega^1(M)$, which is quantum torsion-free to our order if the classical connection is torsion-free.

5. Semiquantisation of Riemannian geometry

We are now in position to semiquantize Riemannian geometry on our above datum (ω, ∇). We need to proceed carefully, as there are various places where modifications arise, and there are typically two connections involved. Throughout this section suppose $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \in \Omega^{\otimes 2}(M)$ is a Riemannian metric on M. We start with the quantum metric and the quantisation ∇_Q of the background connection ∇.

5.1. Quantized metric.

We obtain to first order a quantum metric $g_1 \in \Omega^{\otimes_1 2}A$ characterised by quantum symmetry and centrality. The former is the statement that g_1 is in the kernel of $\wedge_1 : \Omega^{\otimes_1 2}A \to \Omega^2 A$ and the latter, that $a.g_1 = g_1.a$ for all $a \in A$, is needed to be able to apply the metric to situations involving tensor products over the algebra (i.e. the fibrewise tensor product of bundles), without which using the metric would become much more complicated.

As with the wedge product, we start with a functorial part of the quantum metric

$$g_Q := q^{-1}_{\Omega^1,\Omega^1}(g) = g_{ij}\mathrm{d}x^i \otimes_1 \mathrm{d}x^j + \tfrac{\lambda}{2}\omega^{ij}(g_{ms,i} - g_{ks}\Gamma^k_{im})\mathrm{d}x^m \otimes_1 \Gamma^s_{jn}\mathrm{d}x^n \; . \tag{5.1}$$

and of the quantum connection

$$\nabla_Q \mathrm{d}x^i = -\Big(\Gamma^i_{mn} + \frac{\lambda}{2}\omega^{sj}(\Gamma^i_{mk,s}\Gamma^k_{jn} - \Gamma^i_{kt}\Gamma^k_{sm}\Gamma^t_{jn} - \Gamma^i_{jk}R^k{}_{nms})\Big)\mathrm{d}x^m \otimes_1 \mathrm{d}x^n \tag{5.2}$$

by application of our functor in Section 3.

Lemma 5.1. *If we have $\nabla g = 0$ then we also have $\nabla_Q g_Q = 0$ to $O(\lambda^2)$ as an application of Theorem 3.5. Moreover, if $\lambda^* = -\lambda$ then g_Q is 'real' and ∇_Q is $\star$-preserving to this order.*

Proof. We consider the metric as a morphism $\tilde g : C^\infty(M) \to \Omega^1(M) \otimes_0 \Omega^1(M)$ in $\mathcal{D}_0$, where $g = \tilde g(1)$ and $\Omega^1(M)$, is equipped with the background connection (assumed now to be metric compatible). Then $q^{-1}_{\Omega^1,\Omega^1}Q(\tilde g) : Q(C^\infty(M)) \to Q(\Omega^1(M)) \otimes_1 Q(\Omega^1(M))$ and we evaluate this on 1 to give the element $g_Q \in \Omega^1(A) \otimes_1 \Omega^1(A)$. In this case the morphism property of $q_{\Omega^1(M),\Omega^1(M)}$ implies (suppressing M for clarity)

$$\nabla_{Q(\Omega^1)\otimes_1 Q(\Omega^1)} q^{-1}_{\Omega^1,\Omega^1} \circ Q(\tilde g)(1) = (\mathrm{id} \otimes q^{-1}_{\Omega^1,\Omega^1})\nabla_{Q(\Omega^1\otimes_0\Omega^1)}Q(\tilde g)(1)$$

and the right hand side is zero since $\nabla_{\Omega^1\otimes_0\Omega^1} g = 0$. One can also see this another way, which some readers may prefer: By Lemma 3.6 (which is best summarised by the commuting diagram (3.17)), as long as the corresponding qs are inserted, the tensor product of the quantized connections is the same as the quantisation of the tensor product connection. We take a special case of (3.17), remembering that $\Omega^1(A) = Q(\Omega^1(M))$.

$$\begin{array}{ccc} Q(\Omega^1(M)) \otimes_1 Q(\Omega^1(M)) & \xrightarrow{\;q\;} & Q(\Omega^{\otimes 2}(M)) \\ \downarrow{\scriptstyle \nabla_{Q\otimes_1 Q}} & & \downarrow{\scriptstyle \nabla_{Q(\Omega^{\otimes 2}(M))}} \\ \Omega^1(A) \otimes_1 Q(\Omega^1(M)) \otimes_1 Q(\Omega^1(M)) & \xrightarrow[\mathrm{id}\otimes q]{} & \Omega^1(A) \otimes_1 Q(\Omega^{\otimes 2}(M)) \end{array} \tag{5.3}$$

Now we suppose that classically the background connection preserves the classical Riemannian metric $g \in \Omega^{\otimes 2}(M)$, i.e. that $\nabla_{\Omega^{\otimes 2}(M)} g = 0$. By Lemma 3.13 we have $\nabla_{Q(\Omega^{\otimes 2}(M))} g = 0$, which gives g central in the quantized system. Also by (5.3) we see that $g_Q = q^{-1} g \in \Omega^1 A \otimes_1 \Omega^1 A$ is indeed preserved by the tensor product of the quantized connections $\nabla_{Q\otimes_1 Q}$. Moreover, we know from Lemma 3.8 that ∇_Q preserves the star operation hence in this case we also have hermitian-metric compatibility with g_Q in the sense

$$(\bar\nabla_Q \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_Q)(\star \otimes \mathrm{id}) g_Q = 0 \; .$$

Over $\mathbb{C}$, reality of g_Q in the sense $\Upsilon^{-1}(\star\otimes_1\star)\,g_Q=\overline{g_Q}$ reduces by (3.15) to the classical statement for g and $\star\otimes_0\star$, which is trivial certainly if the classical coefficients g_{ij} are real and symmetric. □

However, g_Q is not necessarily 'quantum symmetric'. We can correct for this by an adjustment at order λ.

Proposition 5.2. *Let (ω,∇) be a Poisson tensor with Poisson-compatible connection and define the associated 'generalised Ricci 2-form' and adjusted metric*

$$\mathcal{R}=g_{ij}H^{ij},\quad g_1=g_Q-\lambda\, {q_{\Omega^1,\Omega^1}}^{-1}\mathcal{R}$$

where on the right the 2-form is lifted to an antisymmetric tensor. Suppose that $\nabla g=0$.

(1) *If the lowered T_{ijk} is totally antisymmetric then $\mathrm{d}\mathcal{R}=0$.*
(2) *$\wedge_1(g_1)=0$, and $q^2\nabla_Q g_1=-\lambda\nabla\mathcal{R}$ to $O(\lambda^2)$. Here $\nabla_Q g_1=0$ to this order if and only if $\nabla\mathcal{R}=0$.*
(3) *If $\lambda^*=-\lambda$ then g_1 is 'real' (and ∇_Q is star-preserving) to $O(\lambda^2)$.*

Proof. (1) We use the formula for $\mathrm{d}H^{ij}$ proven in part (2) of Theorem 4.4 in the following,

$$\begin{aligned}\mathrm{d}(g_{ij}\,H^{ij})&=g_{ij,p}\,\mathrm{d}x^p\wedge H^{ij}+g_{ij}\,\mathrm{d}H^{ij}\\&=g_{ij,p}\,\mathrm{d}x^p\wedge H^{ij}-g_{ij}\,(\Gamma^i_{rp}\,\mathrm{d}x^p\wedge H^{rj}+\Gamma^j_{rp}\,\mathrm{d}x^p\wedge H^{ir})\\&=(g_{ij,p}-g_{rj}\,\Gamma^r_{ip}-g_{ir}\,\Gamma^r_{jp})\,\mathrm{d}x^p\wedge H^{ij}\ .\end{aligned}$$

If ∇ preserves the metric we also have

$$0=\nabla_p(g_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j)=(g_{ij,p}-g_{rj}\,\Gamma^r_{pi}-g_{ir}\,\Gamma^r_{pj})\,\mathrm{d}x^i\otimes\mathrm{d}x^j\ ,$$

and using this, if the lowered T_{ijk} is totally antisymmetric

$$\begin{aligned}\mathrm{d}(g_{ij}\,H^{ij})&=(g_{rj}\,\Gamma^r_{pi}+g_{ir}\,\Gamma^r_{pj}-g_{rj}\,\Gamma^r_{ip}-g_{ir}\,\Gamma^r_{jp})\,\mathrm{d}x^p\wedge H^{ij}\\&=(g_{rj}\,T^r_{pi}+g_{ir}\,T^r_{pj})\,\mathrm{d}x^p\wedge H^{ij}=(T_{jpi}+T_{ipj})\,\mathrm{d}x^p\wedge H^{ij}=0\ .\end{aligned}\tag{5.4}$$

(2) Clearly $\wedge_1(g_Q)=\lambda\mathcal{R}$ so $\wedge_1(g_1)=0$. Likewise $q^2\nabla_Q g_1=q^2\nabla_Q g_Q-\lambda\nabla\mathcal{R}=-\lambda\nabla\mathcal{R}$ by Lemma 5.1, where the last term here is viewed as an element of $\Omega^1(M)^{\otimes_0 3}$ by an antisymmetric lift. The antisymmetric lift commutes with ∇ so $\nabla_1 g_1=0$ if and only if $\nabla\mathcal{R}=0$ on $\mathcal{R}$ as a 2-form. To give the formulae here more explicitly, we remember our 2-form conventions so that

$$\mathcal{R}=\frac{1}{2}\mathcal{R}_{nm}\mathrm{d}x^m\wedge\mathrm{d}x^n,\quad \mathcal{R}_{nm}=\tfrac{1}{2}\,g_{ij}\omega^{is}(T^j_{nm;s}-{R^j}_{nms}+{R^j}_{mns}).\tag{5.5}$$

in which case,

$$g_1=g_Q+\frac{\lambda}{2}\mathcal{R}_{mn}\mathrm{d}x^m\otimes_1\mathrm{d}x^n.$$

(3) Over $\mathbb{C}$, we also have the condition $\Upsilon^{-1}(\star\otimes_1\star)g_1=\overline{g_1}$, as the correction is both imaginary and antisymmetric. ∇_Q is still star-preserving because that statement is not dependent on the metric (which means that it is also hermitian-metric compatible with the corresponding hermitian metric $(\star\otimes\mathrm{id})g_1$). □

In general we may not have either of these properties of $\mathcal{R}$ but we do have $\nabla_Q g_1$ being order λ and that is enough to make g_1 commute with elements of A to order λ which is what we wanted to retain at this point. The terminology for $\mathcal{R}$ comes from the Kähler case which is a subcase of the following special case.

Corollary 5.3. *If the background connection ∇ is taken to be the Levi-Civita one,*
(1) Poisson-compatibility reduces to ω covariantly constant.
(2) ∇_Q is quantum torsion free to $O(\lambda^2)$ and $\mathcal{R}=\frac{1}{2}\omega^{ji}R_{inmj}\mathrm{d}x^m\wedge\mathrm{d}x^n$ is closed.
(3) $\nabla_Q g_1=O(\lambda^2)$, i.e. ∇_Q is a quantum-Levi-Civita connection for g_1, to this order if and only if $\nabla\mathcal{R}=0$.

Proof. This is a special case of Proposition 5.2. For the quantum torsion we use Proposition 4.6 where the torsion T of ∇ is currently being assumed to be zero. In this case $\mathrm{d}\mathcal{R}=0$ as $T=0$ is antisymmetric. Note that if $\nabla\mathcal{R}\neq 0$ we still have $\nabla_Q g_1$ is order λ by Lemma 5.1. □

5.2. **Relating general ∇ and the Levi-Civita $\widehat{\nabla}$.** In general there is no reason to take the background connection ∇ to be the same as the classical Levi-Civita connection $\widehat{\nabla}$ for our chosen metric on M. The role of the former in controlling the quantisation of the differential structure is very different one from the role of the latter in controlling the geometry. We still need $\nabla g=0$ as a quantisation condition on the metric since the quantum metric has to be central to $O(\lambda^2)$ in the noncommutative geometry (it being shown in [10] that this is necessary for the existence of a bimodule map (,) inverse to the metric.) There is no reason here to think that ∇ should have zero torsion T and indeed Lemma 3.1 tells us that it cannot be torsion free unless ω is covariantly constant. In this case we can write $\widehat{\nabla}$ in the general form $\nabla_S=\nabla+S$ for some *contorsion tensor* $S:\Omega^1(M)\to\Omega^1(M)\otimes_0\Omega^1(M)$ which will then allow us to quantise it via $Q(S)$. Moreover, it is well-known (see [28]) that given an arbitrary tensor T of the correct type there is a unique metric compatible covariant derivative ∇ with that torsion, given by Christoffel symbols

$$\Gamma^a_{bc}=\widehat{\Gamma}^a_{bc}+\tfrac12 g^{ad}(T_{dbc}-T_{bcd}-T_{cbd}) \tag{5.6}$$

where $T_{abc}=g_{ad}T^d_{bc}$ and $\widehat{\Gamma}^a_{bc}$ are the Christoffel symbols for the Levi-Civita connection so that $\nabla_S(\mathrm{d}x^a)=-\widehat{\Gamma}^a_{bc}\,\mathrm{d}x^b\otimes\mathrm{d}x^c$. Hence

$$S^a_{bc}=\tfrac12 g^{ad}(T_{dbc}-T_{bcd}-T_{cbd})\ . \tag{5.7}$$

As a quick check of conventions, note that this formula is consistent with (4.3). Throughout this section T is arbitrary which fixes ∇ such that this is metric compatible, and S is the above function of T so that $\nabla_S=\widehat{\nabla}$, the Levi-Civita connection.

Lemma 5.4. *The curvatures are related by*

$$\widehat{R}^l{}_{ijk}=R^l{}_{ijk}-S^l_{ki;j}+S^l_{ji;k}+S^m_{kj}\,S^l_{mi}-S^m_{jk}\,S^l_{mi}+S^m_{ki}\,S^l_{jm}-S^m_{ji}\,S^l_{km}\ ,$$

where semicolon is derivative with respect to ∇.

Proof. This is elementary: $\widehat{\Gamma}^m_{ji}=\Gamma^m_{ji}-S^m_{ji}$ so that

$$\begin{aligned}\widehat{R}^l{}_{ijk}&=\widehat{\Gamma}^l_{ki,j}-\widehat{\Gamma}^l_{ji,k}+\widehat{\Gamma}^m_{ki}\,\widehat{\Gamma}^l_{jm}-\widehat{\Gamma}^m_{ji}\,\widehat{\Gamma}^l_{km}\\&=R^l{}_{ijk}-S^l_{ki,j}+S^l_{ji,k}-\Gamma^m_{ki}\,S^l_{jm}+\Gamma^m_{ji}\,S^l_{km}-S^m_{ki}\,\Gamma^l{}_{jm}+S^m_{ji}\,\Gamma^l{}_{km}+S^m_{ki}\,S^l_{jm}-S^m_{ji}\,S^l_{km}\\&=R^l{}_{ijk}-S^l_{ki;j}+S^l_{ji;k}-T^m_{jk}\,S^l_{mi}+S^m_{ki}\,S^l_{jm}-S^m_{ji}\,S^l_{km}\end{aligned}$$

and we then write T in terms of S to obtain the answer stated. □

This gives a different point of view on some of the formulae below, if we wish to rewrite expressions in terms of the Levi-Civita connection. In the same vein:

Proposition 5.5. *Suppose that a connection ∇ is metric-compatible. Then (∇,ω) are Poisson-compatible if and only if*

$$(\widehat{\nabla}_k\omega)^{ij}+\omega^{ir}\,S^j_{rk}-\omega^{jr}\,S^i_{rk}=0$$

or equivalently

$$\omega^{jm}S^i_{mk}=\tfrac12\Big((\widehat{\nabla}_k\omega)^{ij}-(\widehat{\nabla}_r\omega)^{mj}\,g^{ri}\,g_{mk}+(\widehat{\nabla}_r\omega)^{im}\,g^{rj}\,g_{mk}\Big).$$

Proof. The compatibility condition gives

$$\begin{aligned}0=(\widehat{\nabla}_m\omega)^{ij}&+\omega^{ik}\,(T^j_{km}+\tfrac12 g^{jd}(T_{dmk}-T_{mkd}-T_{kmd}))\\&+\omega^{kj}\,(T^i_{km}+\tfrac12 g^{id}(T_{dmk}-T_{mkd}-T_{kmd}))\end{aligned}$$

$$= (\widehat{\nabla}_m\omega)^{ij} + \omega^{ik}\,\tfrac{1}{2}g^{jd}(T_{dkm} + T_{mdk} - T_{kmd}) + \omega^{kj}\,\tfrac{1}{2}g^{id}(T_{dkm} + T_{mdk} - T_{kmd})$$
$$= (\widehat{\nabla}_m\omega)^{ij} + \tfrac{1}{2}\,(\omega^{ik}\,g^{jd} - \omega^{jk}\,g^{id})(T_{dkm} + T_{mdk} - T_{kmd})$$

which is the first condition stated in terms of S. From this,

$$(\widehat{\nabla}_m\omega)^{ij}\,g_{ir}\,g_{js} = -\omega^{ik}\,g_{ir}\,S_{skm} + \omega^{jk}\,g_{js}S_{rkm}\ .$$

Now define

$$-\Theta_{mrs} := (\widehat{\nabla}_m\omega)^{ij}\,g_{ir}\,g_{js} - \omega^{jk}\,g_{js}2S_{rkm} = -\,\omega^{ik}\,g_{ir}\,S_{skm} - \omega^{jk}\,g_{js}S_{rkm}$$

and note that Θ_{mrs} is symmetric on swapping r, s. Hence

$$\Theta_{mrs} = -\,(\widehat{\nabla}_m\omega)^{ij}\,g_{ir}\,g_{js} + 2\,\omega^{jk}\,g_{js}S_{rkm}\ , \tag{5.8}$$

$$2\,\omega^{jk}S_{rkm} = (\widehat{\nabla}_m\omega)^{ij}\,g_{ir} + \Theta_{mrs}\,g^{sj}\ . \tag{5.9}$$

From (5.8) we obtain the following condition, which we repeat with permuted indices

$$\Theta_{mrs} + \Theta_{rms} = -\,(\widehat{\nabla}_m\omega)^{ij}\,g_{ir}\,g_{js} - (\widehat{\nabla}_r\omega)^{ij}\,g_{im}\,g_{js}\ ,$$
$$\Theta_{rsm} + \Theta_{srm} = -\,(\widehat{\nabla}_r\omega)^{ij}\,g_{is}\,g_{jm} - (\widehat{\nabla}_s\omega)^{ij}\,g_{ir}\,g_{jm}\ ,$$
$$\Theta_{smr} + \Theta_{msr} = -\,(\widehat{\nabla}_s\omega)^{ij}\,g_{im}\,g_{jr} - (\widehat{\nabla}_m\omega)^{ij}\,g_{is}\,g_{jr}\ .$$

Taking the first of these equations, subtracting the second and adding the third gives

$$\Theta_{mrs} = (\widehat{\nabla}_r\omega)^{ij}\,g_{is}\,g_{jm} + (\widehat{\nabla}_s\omega)^{ij}\,g_{ir}\,g_{jm}\ .$$

Now we rewrite (5.9) as

$$2\,\omega^{jk}S_{rkm} = (\widehat{\nabla}_m\omega)^{ij}\,g_{ir} + ((\widehat{\nabla}_r\omega)^{it}\,g_{is}\,g_{tm} + (\widehat{\nabla}_s\omega)^{it}\,g_{ir}\,g_{tm})\,g^{sj}$$
$$= (\widehat{\nabla}_m\omega)^{ij}\,g_{ir} - (\widehat{\nabla}_r\omega)^{ij}\,g_{im} + (\widehat{\nabla}_s\omega)^{it}\,g_{ir}\,g_{tm}\,g^{sj}$$

which we write as stated. □

5.3. **Quantising the Levi-Civita connection and metric compatibility.** Now we look for a quantum Levi-Civita connection in the general case where the background connection ∇ may not be the Levi-Civita connection $\widehat{\nabla}$. As in Section 5.1 we assume a metric $g \in \Omega^{1^{\otimes 2}}(M)$ and $\nabla g = 0$ and as in Section 5.2 we let S be a function of the torsion T of ∇ such that $\nabla_S = \nabla + S = \widehat{\nabla}$. We consider straight quantum metric compatibility in this section (which makes sense over any field) and the hermitian version in the next section (recall that the two versions of the metric-compatibility coincide if the quantum connection is star-preserving).

Lemma 5.6. *For ∇_S the Levi-Civita connection, the quantum metric compatibility tensor and quantum torsion $T_{\nabla_{QS}}(\xi) = \frac{\lambda}{2}\,\xi_p A^p_{nm}\mathrm{d}x^m \wedge \mathrm{d}x^n$ to $O(\lambda^2)$ are given respectively by*

$$q^2\,\nabla_{QS\otimes_1 QS}(g_Q) = -\,\lambda\,\omega^{ij}\,g_{rs}\,S^s_{jn}(R^r{}_{mki} + S^r_{km;i})(\mathrm{d}x^k \otimes \mathrm{d}x^m \otimes \mathrm{d}x^n)$$

$$A^p_{nm} = -\tfrac{1}{4}\omega^{ij}\left(g^{pd}\,(T_{isd} + T_{sid})\,(T^s_{nm;j} - R^s{}_{nmj} + R^s{}_{mnj}) + T^s_{nm}\,R^p{}_{sij} - T^p_{sm}\,R^s{}_{nij} + T^p_{sn}\,R^s{}_{mij}\right)$$

where $q^2 := q_{\Omega^1,\Omega^1\otimes_0\Omega^1}(\mathrm{id} \otimes q_{\Omega^1,\Omega^1}) = q_{\Omega^1\otimes_0\Omega^1,\Omega^1}(q_{\Omega^1,\Omega^1} \otimes \mathrm{id})$.

Proof. Following the general theory in Proposition 3.11 we set

$$H = S \otimes \mathrm{id}_F + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes S) : \Omega^{1^{\otimes 2}}(M) \to \Omega^1(M) \otimes_0 \Omega^{1^{\otimes 2}}(M)$$

and note that $H(g) = 0$ since both ∇ and ∇_S preserve g. Hence $Q(H)(g) = 0$ and $\nabla_{QH}(g) = 0$ by application of Lemma 3.13. Now applying Proposition 3.11 gives

$$q^2\,\nabla_{QS\otimes_1 QS}(q^{-1}g) = (q\,\nabla_{QH}\,q + \lambda\,\mathrm{rem})(q^{-1}g) = \lambda\,\mathrm{rem}(q^{-1}g)\ ,$$
$$\mathrm{rem}(e \otimes_1 f) = \omega^{ij}\,(\mathrm{d}x^k \otimes [\nabla_k, \nabla_i]e - \mathbb{W}_i(S)(e)) \otimes S_j(f) \tag{5.10}$$

where q^2 is as stated in the lemma, $S(f) = \mathrm{d}x^k \otimes S_k(f)$ and $q^{-1}g := q_{\Omega^1,\Omega^1}{}^{-1}g$. Then by (5.10),

$$q^2\,\nabla_{QS\otimes_1 QS}(q^{-1}g) = \lambda\,\mathrm{rem}(q^{-1}g) = \lambda\,\mathrm{rem}(g_{rs}\,\mathrm{d}x^r \otimes_1 \mathrm{d}x^s)$$
$$= \lambda\,\omega^{ij}\,g_{rs}\,(\mathrm{d}x^k \otimes [\nabla_k, \nabla_i](\mathrm{d}x^r) - \mathbb{W}_i(S)(\mathrm{d}x^r)) \otimes S_j(\mathrm{d}x^s)$$

$$= \lambda\,\omega^{ij}\,g_{rs}\,S^s_{jn}\,(\mathrm{d}x^k\otimes[\nabla_k,\nabla_i](\mathrm{d}x^r) - \mathbb{V}_i(S)(\mathrm{d}x^r))\otimes\mathrm{d}x^n$$

which we write as stated. For the torsion we used $S^p_{ij} + S^p_{ji} = -g^{pd}(T_{ijd}+T_{jid})$ in Corollary 4.9 and relabelled. □

We see that the quantisation ∇_{QS} given by the procedure outlined in Section 3.4 is only quantum metric compatible to an error of order λ. However, the quantum metric g_1 in Section 5.1 has an order λ correction to g_Q as above we similarly make an order λ correction to ∇_{QS}.

Theorem 5.7. *Let ∇_S be the Levi-Civita connection. There is a unique quantum connection of the form $\nabla_1 = \nabla_{QS} + \lambda K$ such that the quantum torsion and merely the symmetric part of $\nabla_1 g_1$ vanish to $O(\lambda^2)$. The antisymmetric part,*

$$(\mathrm{id}\otimes\wedge)q^2\,\nabla_1 g_1 = -\lambda\left(\widehat{\nabla}\mathcal{R} + \omega^{ij}\,g_{rs}\,S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k\otimes\mathrm{d}x^m\wedge\mathrm{d}x^n\right)\ ,$$

to $O(\lambda^2)$ is independent of K. A fully metric compatible torsion free ∇_1 exists if and only if the above expression vanishes, in which case it is given by the unique ∇_1 discussed.

Proof. Here q^2 is the same as in Lemma 5.6. If we write $K(\xi) = \xi_p\,K^p_{nm}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$, then the results in Lemma 5.6 (with semicolon given by the background connection) are clearly adjusted to

$$\begin{aligned} q^2\,\nabla_1(g_1) &= -\lambda\,\omega^{ij}\,g_{rs}\,S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k\otimes\mathrm{d}x^m\otimes\mathrm{d}x^n \\ &\quad - \tfrac{\lambda}{4}\,\nabla_{S\otimes S}(g_{ij}\,\omega^{is}(T^j_{nm;s} - R^j{}_{nms} + R^j{}_{mns})\,\mathrm{d}x^m\otimes\mathrm{d}x^n) \\ &\quad + \lambda\,(g_{pn}\,K^p_{km} + g_{mp}\,K^p_{kn})\,\mathrm{d}x^k\otimes\mathrm{d}x^m\otimes\mathrm{d}x^n \\ T_{\nabla_1}(\xi) &= \tfrac{\lambda}{2}\,\xi_p\,(K^p_{nm} - K^p_{mn} - A^p_{nm})\,\mathrm{d}x^n\wedge\mathrm{d}x^m\ . \end{aligned}$$

Looking at the first expression reveals that the second term is purely antisymmetric in nm, whereas the third term (the only one to contain the order λ correction K^a_{bc}) is purely symmetric in nm. Hence there is nothing we can do by adding K^a_{bc} to make the part of the metric compatibility tensor which is antisymmetric in nm vanish, it will have the value stated, but we show that we can choose K^a_{bc} to make the part which is symmetric in nm vanish, namely by setting

$$g_{np}\,K^p_{km} + g_{mp}\,K^p_{kn} = B_{knm}$$

where

$$B_{knm} = \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jn}(R^r{}_{mki} + S^r_{km;i}) + S^s_{jm}(R^r{}_{nki} + S^r_{kn;i}))$$

while for vanishing torsion, clearly we need $K^p_{nm} - K^p_{mn} = A^p_{nm}$. If we set $K_{nkm} = g_{np}\,K^p_{km}$ then these conditions become

$$K_{nkm} + K_{mkn} = B_{knm}\ ,\quad K_{knm} - K_{kmn} = g_{kp}\,A^p_{nm}\ .$$

Now

$$K_{nkm} = B_{knm} - K_{mkn} = B_{knm} + g_{mp}\,A^p_{nk} - K_{mnk}\ ,$$

and continuing in this manner six times gives a unique value of K,

$$K_{nkm} = \frac{1}{2}\left(B_{knm} - B_{nkm} + B_{mnk} + g_{mp}\,A^p_{nk} + g_{kp}\,A^p_{nm} + g_{np}\,A^p_{km}\right)\ . \tag{5.11}$$

where A describes the quantum torsion of ∇_{QS} as in Lemma 5.6 and

$$\begin{aligned} &B_{knm} - B_{nkm} + B_{mnk} \\ &= \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jn}(R^r{}_{mki} + S^r_{km;i}) + S^s_{jm}(R^r{}_{nki} + S^r_{kn;i})) \\ &\quad - \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jk}(R^r{}_{mni} + S^r_{nm;i}) + S^s_{jm}(R^r{}_{kni} + S^r_{nk;i})) \\ &\quad + \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jn}(R^r{}_{kmi} + S^r_{mk;i}) + S^s_{jk}(R^r{}_{nmi} + S^r_{mn;i})) \\ &= \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jn}(R^r{}_{mki} + R^r{}_{kmi} + S^r_{mk;i} + S^r_{km;i}) + S^s_{jm}(R^r{}_{nki} - R^r{}_{kni} - S^r_{nk;i} + S^r_{kn;i}) \\ &\quad + S^s_{jk}(R^r{}_{nmi} - R^r{}_{mni} - S^r_{nm;i} + S^r_{mn;i})) \\ &= \tfrac{1}{2}\,\omega^{ij}\,g_{rs}\,(S^s_{jn}(R^r{}_{mki} + R^r{}_{kmi} - g^{rd}\,(T_{mkd;i} + T_{kmd;i})) + S^s_{jm}(R^r{}_{nki} - R^r{}_{kni} + T^r_{kn;i}) \end{aligned}$$

$$+ S^s_{jk}(R^r{}_{nmi} - R^r{}_{mni} + T^r_{mn;i}))$$

using $S^r_{mk} + S^r_{km} = -g^{rd}(T_{mkd} + T_{kmd})$. □

This clearly reduces to Corollary 5.3 in the case where $T = 0$ but more generally we have a free parameter, the value of T for the background connection provided only that (∇, ω) are Poisson-compatible. We might hope to use this freedom to set $R = 0$ so that our differential calculus remains associative at the next order in λ, and/or we might hope to choose T so that the antisymmetric part of the quantum metric compatibility tensor also vanishes. Otherwise we still have a 'best possible' choice of ∇_1 given by (5.11).

5.4. **Hermitian-metric compatibility.** We again assume that our Poisson-compatible connection ∇ obeys $\nabla g = 0$ and that $\nabla_S = \nabla + S = \widehat{\nabla}$ the Levi-Civita connection for g. We set $\nabla_1 = \nabla_{QS} + \lambda K$, for some real K, and ask this time that ∇_1 is hermitian-metric compatible with the hermitian metric $(\star\otimes\mathrm{id})g_1$ corresponding to g_1. This is a potentially different condition from straight metric compatibility unless ∇_1 is star-preserving, in which case it is equivalent.

Proposition 5.8. *If $\lambda^* = -\lambda$ and ∇_S is the Levi-Civita connection then the condition for $\nabla_{QS} + \lambda K$ to be hermitian-metric compatible with g_1 to $O(\lambda^2)$ is*

$$K_{npm} - K_{mpn} = \mathcal{R}_{nm\hat{;}p} + \tfrac{1}{2}\,\omega^{ij}\,(g_{rm}\,\mathbb{V}_i(\mathbb{V}_j(S))^r_{pn} - g_{nr}\,\mathbb{V}_i(\mathbb{V}_j(S))^r_{pm})$$

where $\hat{;}$ denotes the Levi-Civita derivative. This condition on K can always be solved simultaneously with vanishing of the quantum torsion.

Proof. (1) If we write the quantum correction to the metric in Proposition 5.2 as $g_1 = g_Q - \lambda\, g_c$, then hermitian-metric compatibility tensor for ∇_{QS} becomes

$$\begin{aligned}&((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{q_{\Omega^1\otimes\Omega^1}{}^{-1}Q(S)}\otimes\mathrm{id} + \mathrm{id}\otimes q_{\Omega^1\otimes\Omega^1}{}^{-1}Q(S))(\star\otimes\mathrm{id})g_Q\\ &-\lambda(\bar{\nabla}_S\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_S)(\star\otimes\mathrm{id})g_c\end{aligned}\tag{5.12}$$

From Proposition 3.4 we can write this as

$$\begin{aligned}&(q_{\overline{\Omega^1}\otimes\Omega^1}{}^{-1}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{Q(S)}\otimes_1\mathrm{id} + \mathrm{id}\otimes_1 q_{\Omega^1\otimes\Omega^1}{}^{-1}Q(S))(\star\otimes\mathrm{id})g_Q\\ &-\lambda(\bar{\nabla}_S\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_S)(\star\otimes\mathrm{id})g_c\ .\end{aligned}$$

The definition of $Q(S)$ gives

$$\begin{aligned}&(q_{\overline{\Omega^1}\otimes\Omega^1}{}^{-1}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_1\mathrm{id} + \mathrm{id}\otimes_1 q_{\Omega^1\otimes\Omega^1}{}^{-1}S)(\star\otimes\mathrm{id})g_Q\\ &+\tfrac{\lambda}{2}\,\omega^{ij}\,(-(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_i\circ\mathbb{V}_j(S)}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_i\circ\mathbb{V}_j(S))(\star\otimes\mathrm{id})g\\ &-\lambda((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_S}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_S)(\star\otimes\mathrm{id})g_c\ .\end{aligned}$$

Now apply $q^2 := q_{\overline{\Omega^1},\Omega^1\otimes_0\Omega^1}(\mathrm{id}\otimes q_{\Omega^1,\Omega^1}) = q_{\overline{\Omega^1}\otimes_0\Omega^1,\Omega^1}(q_{\overline{\Omega^1},\Omega^1}\otimes\mathrm{id})$ and use Proposition 3.3,

$$\begin{aligned}&q^2(q_{\overline{\Omega^1}\otimes\Omega^1}{}^{-1}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_1\mathrm{id} + \mathrm{id}\otimes_1 q_{\Omega^1\otimes\Omega^1}{}^{-1}S)(\star\otimes\mathrm{id})g_Q\\ &= q^2(q_{\overline{\Omega^1}\otimes\Omega^1}{}^{-1}(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_1\mathrm{id} + \mathrm{id}\otimes_1 q_{\Omega^1\otimes\Omega^1}{}^{-1}S)(\star\otimes\mathrm{id})\,q^{-1}g\\ &= ((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0 S)(\star\otimes\mathrm{id})\,g\\ &\quad+\tfrac{\lambda}{2}\,\omega^{ij}\,(\nabla_i((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S})\otimes_0\nabla_j+\nabla_i\otimes_0\mathbb{V}_j(S))(\star\otimes\mathrm{id})\,g\\ &= ((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0 S)(\star\otimes\mathrm{id})\,g\\ &\quad+\tfrac{\lambda}{2}\,\omega^{ij}\,((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\mathbb{V}_i(S)}\otimes_0\nabla_j+\nabla_i\otimes_0\mathbb{V}_j(S))(\star\otimes\mathrm{id})\,g\\ &= ((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0 S)(\star\otimes\mathrm{id})\,g\\ &\quad-\tfrac{\lambda}{2}\,\omega^{ij}\,((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\mathbb{V}_i(S)\circ\nabla_j}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0\mathbb{V}_j(S)\circ\nabla_i)(\star\otimes\mathrm{id})\,g\end{aligned}$$

as g is preserved by ∇ so that $(\nabla_i\otimes\mathrm{id})g = -(\mathrm{id}\otimes\nabla_i)g$.

Then q^2 applied to (5.12) gives

$$\begin{aligned}&((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0 S)(\star\otimes\mathrm{id})\,g\\ &-\tfrac{\lambda}{2}\,\omega^{ij}\,(-(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\mathbb{V}_j(S)\circ\nabla_i}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0\mathbb{V}_j(S)\circ\nabla_i)(\star\otimes\mathrm{id})\,g\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{\lambda}{2}\,\omega^{ij}\,(-(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_i\circ\mathbb{W}_j(S)}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_i\circ\mathbb{W}_j(S))(\star\otimes\mathrm{id})g\\
&-\lambda((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_S}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_S)(\star\otimes\mathrm{id})g_c\\
=&((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{S}\otimes_0\mathrm{id}+\mathrm{id}\otimes_0 S)(\star\otimes\mathrm{id})\,g\\
&+\tfrac{\lambda}{2}\,\omega^{ij}\,(-(\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\mathbb{W}_i(\mathbb{W}_j(S))}\otimes\mathrm{id}+\mathrm{id}\otimes\mathbb{W}_i(\mathbb{W}_j(S)))(\star\otimes\mathrm{id})g\\
&-\lambda((\mathrm{id}\otimes\star^{-1})\Upsilon\,\overline{\nabla_S}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_S)(\star\otimes\mathrm{id})g_c\ . \qquad (5.13)
\end{aligned}
$$

Now set $g=g_{nm}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$ and $\mathbb{W}_i(\mathbb{W}_j(S))(\mathrm{d}x^a)=\mathbb{W}_i(\mathbb{W}_j(S))^a_{nm}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$, and using the reality of S the first two lines of the result of (5.13) become

$$
\begin{aligned}
&(g_{rm}\,S^r_{pn}+g_{nr}\,S^r_{pm})\,\overline{\mathrm{d}x^n}\otimes\mathrm{d}x^p\otimes\mathrm{d}x^m\\
&+\tfrac{\lambda}{2}\,\omega^{ij}\,(-g_{rm}\,\mathbb{W}_i(\mathbb{W}_j(S))^r_{pn}+g_{nr}\,\mathbb{W}_i(\mathbb{W}_j(S))^r_{pm})\,\overline{\mathrm{d}x^n}\otimes\mathrm{d}x^p\otimes\mathrm{d}x^m\ ,
\end{aligned}
$$

and the first line of this vanishes as ∇_S preserves g. Now we write $g_c=-\frac{1}{2}\mathcal{R}_{nm}\,\mathrm{d}x^n\otimes\mathrm{d}x^m$ where $\mathcal{R}_{nm}$ is antisymmetric giving

$$
q^2(\bar\nabla_{QS}\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_{QS})(\star\otimes\mathrm{id})g_1=-\,\lambda\,C_{npm}\,\overline{\mathrm{d}x^n}\otimes\mathrm{d}x^p\otimes\mathrm{d}x^m;
$$

$$
C_{npm}=-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}+\tfrac{1}{2}\,\omega^{ij}\,(g_{rm}\,\mathbb{W}_i(\mathbb{W}_j(S))^r_{pn}-g_{nr}\,\mathbb{W}_i(\mathbb{W}_j(S))^r_{pm}).
$$

(2) Now we look at $\nabla_1=\nabla_{QS}+\lambda K$, then clearly

$$
(\bar\nabla_1\otimes\mathrm{id}+\mathrm{id}\otimes\nabla_1)(\star\otimes\mathrm{id})g_1=\lambda\,(g_{na}\,K^a_{pm}-g_{ma}\,K^a_{pn}-C_{npm})\,\overline{\mathrm{d}x^n}\otimes\mathrm{d}x^p\otimes\mathrm{d}x^m
$$

so we need to solve $K_{npm}-K_{mpn}=C_{npm}$ to preserve the hermitian metric, and also $K_{knm}-K_{kmn}=g_{ks}\,A^s_{nm}$ if we want to have zero torsion as in the previous section. These equations have a required compatibility condition

$$
C_{npm}+C_{mnp}+C_{pmn}+g_{ms}\,A^s_{pn}+g_{ps}\,A^s_{nm}+g_{ns}\,A^s_{mp}=0\ .
$$

We use the formula (5.7) for S^a_{bc} in terms of the torsion to write

$$
\begin{aligned}
C_{npm}&=-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}+\tfrac{1}{2}\,\omega^{ij}\,(g_{rm}\,S^r_{pn;\hat{j}i}-g_{nr}\,S^r_{pm;\hat{j}i})\\
&=-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}+\tfrac{1}{4}\,\omega^{ij}\,((T_{mpn}-T_{pnm}-T_{npm})-(T_{npm}-T_{pmn}-T_{mpn}))_{;\hat{j}i}\\
&=-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}+\tfrac{1}{2}\,\omega^{ij}\,(T_{mpn}-T_{pnm}-T_{npm})_{;\hat{j}i}\ ,
\end{aligned}
$$

and taking the cyclic sum gives

$$
C_{npm}+C_{mnp}+C_{pmn}=-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}-\tfrac{1}{2}\,\mathcal{R}_{pn\hat{;}m}-\tfrac{1}{2}\,\mathcal{R}_{mp\hat{;}n}+\tfrac{1}{2}\,\omega^{ij}\,(T_{mpn}+T_{pnm}-T_{npm})_{;\hat{j}i}\ .
$$

We have

$$
\begin{aligned}
-\,g_{pa}\,A^a_{nm}&=\tfrac{1}{4}\omega^{ij}\,(T_{isp}+T_{sip})\,(T^s_{nm;j}-R^s{}_{nmj}+R^s{}_{mnj})\\
&\quad+\tfrac{1}{4}\,\omega^{ij}\,g_{pa}\,(T^s_{nm}\,R^a{}_{sij}-T^a_{sm}\,R^s{}_{nij}+T^a_{sn}\,R^s{}_{mij})\\
&=\tfrac{1}{4}\omega^{ij}\,(T_{isp}+T_{sip})\,(T^s_{nm;j}-R^s{}_{nmj}+R^s{}_{mnj})+\tfrac{1}{2}\,\omega^{ij}\,T_{pnm;\hat{j}i}\ ,
\end{aligned}
$$

so now the cyclic sum becomes

$$
\begin{aligned}
C_{npm}+C_{mnp}+C_{pmn}=&-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}-\tfrac{1}{4}\omega^{ij}\,(T_{isp}+T_{sip})\,(T^s_{nm;j}-R^s{}_{nmj}+R^s{}_{mnj})\\
&-\tfrac{1}{2}\,\mathcal{R}_{pn\hat{;}m}-\tfrac{1}{4}\omega^{ij}\,(T_{ism}+T_{sim})\,(T^s_{pn;j}-R^s{}_{pnj}+R^s{}_{npj})\\
&-\tfrac{1}{2}\,\mathcal{R}_{mp\hat{;}n}-\tfrac{1}{4}\omega^{ij}\,(T_{isn}+T_{sin})\,(T^s_{mp;j}-R^s{}_{mpj}+R^s{}_{pmj})\ . \qquad (5.14)
\end{aligned}
$$

This is totally antisymmetric in npm, so we may equivalently consider the 3-form

$$
\begin{aligned}
\alpha&=\big(-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}-\tfrac{1}{4}\omega^{ij}\,(T_{isp}+T_{sip})\,(T^s_{nm;j}-R^s{}_{nmj}+R^s{}_{mnj})\big)\,\mathrm{d}x^p\wedge\mathrm{d}x^n\wedge\mathrm{d}x^m\\
&=\mathrm{d}x^p\wedge(-\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p}\,\mathrm{d}x^n\wedge\mathrm{d}x^m)+(T_{isp}+T_{sip})\,\mathrm{d}x^p\wedge H^{is}\\
&=\mathrm{d}x^p\wedge\widehat\nabla_p(g_{ij}\,H^{ij})+2\,T_{isp}\,\mathrm{d}x^p\wedge H^{is}
\end{aligned}
$$

where we use $H^{ij}=\frac{1}{4}\omega^{is}(T^j_{nm;s}-2R^j{}_{nms})\mathrm{d}x^m\wedge\mathrm{d}x^n$ and the symmetry of H^{ij}. Now we have as in (5.4) (but not requiring this to be zero)

$$
\mathrm{d}\mathcal{R}=(T_{jpi}+T_{ipj})\,\mathrm{d}x^p\wedge H^{ij}=-\,2\,T_{ijp}\,\mathrm{d}x^p\wedge H^{ij}\ ,
$$

so vanishing of $\alpha=\mathrm{d}x^p\wedge\widehat\nabla_p(\mathcal{R})-\mathrm{d}\mathcal{R}$ is the condition for a joint solution. But this is zero as the Levi-Civita connection is torsion free. □

Note that Proposition 5.8 does not say that such a torsion free quantum connection preserving the hermitian metric is unique. If we take the collection of K_{ijk} for all permutations of the ijk, then the equations fix relative values such $K_{ijk} - K_{kij}$ but we can add a number to K_{ijk} as long as we add the same amount to each $K_{\pi(i,j,k)}$ where π is a permutation of the indices.

Corollary 5.9. *Let $\lambda^* = -\lambda$ and ∇_S be the Levi-Civita connection. If a torsion free metric compatible quantum connection of the form $\nabla_1 = \nabla_{QS} + \lambda K$ exists to $O(\lambda^2)$ then it is star-preserving to this order and coincides with the unique star-preserving quantum connection in Theorem 4.11.*

Proof. From Lemma 4.10 and Theorem 4.11 the star preserving connection is given by $K^a_{nm} = \frac{1}{4}\,\omega^{ij} D^a_{ijnm}$, or

$$\begin{aligned} K^a_{nm} &= \tfrac{1}{4}\,\omega^{ij}\,(2\,S^a_{ip}\,S^p_{nm;j} - (S^b_{nm}\,R^a{}_{bij} - S^a_{rm}\,R^r{}_{nij} - S^a_{nr}\,R^r{}_{mij}) - 2\,S^a_{jr}\,R^r{}_{mni}) \\ &= \tfrac{1}{4}\,\omega^{ij}\,(2\,S^a_{ip}\,S^p_{nm;j} + 2\,S^a_{ip}\,R^p{}_{mnj} - (S^b_{nm}\,R^a{}_{bij} - S^a_{rm}\,R^r{}_{nij} - S^a_{nr}\,R^r{}_{mij})) \\ &= \tfrac{1}{2}\,\omega^{ij}\,S^a_{ip}\,(S^p_{nm;j} + R^p{}_{mnj}) - \tfrac{1}{4}\,\omega^{ij}\,([\mathbb{V}_i,\mathbb{V}_j](S))^a_{nm} \\ &= \tfrac{1}{2}\,\omega^{ij}\,S^a_{ip}\,(S^p_{nm;j} + R^p{}_{mnj}) - \tfrac{1}{2}\,\omega^{ij}\,\mathbb{V}_i(\mathbb{V}_j(S))^a_{nm}\ . \end{aligned}$$

From this we get

$$K_{npm} = \tfrac{1}{2}\,g_{an}\,\omega^{ij}\,S^a_{is}\,(S^s_{pm;j} + R^s{}_{mpj}) - \tfrac{1}{2}\,g_{nr}\,\omega^{ij}\,\mathbb{V}_i(\mathbb{V}_j(S))^r_{pm}\ .$$

From Proposition 5.8 the condition for $\nabla_1 = \nabla_{QS} + \lambda K$ to be hermitian-metric compatible is the following, where $\hat{;}$ denotes Levi-Civita derivative

$$K_{npm} - K_{mpn} = -\tfrac{1}{2}\,\mathcal{R}_{nm\hat{;}p} + \tfrac{1}{2}\,\omega^{ij}\,(g_{rm}\,\nabla_{\Omega^1\otimes E i}(\mathbb{V}_j(S))^r_{pn} - g_{nr}\,\nabla_{\Omega^1\otimes E i}(\mathbb{V}_j(S))^r_{pm})\ ,$$

so on substituting for K_{npm} we find the single condition

$$\widehat{\nabla}\mathcal{R} = -\,\omega^{ij}\,g_{rs}\,S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k \otimes \mathrm{d}x^m \wedge \mathrm{d}x^n\ .$$

This is the same as the condition for existence of a fully metric compatible torsion free connection of our assumed form in Theorem 5.7. So, if such a connection exists, our star-preserving one gives it. The converse direction is also proved, but obvious (if our star-preserving connection is hermitian-metric compatible then it is also straight metric compatible and hence the stated condition must hold by Theorem 5.7.) □

6. Quantized surfaces and Kähler-Einstein manifolds

We have seen that our theory applies in particular to any Riemannian manifold equipped with a covariantly constant Poisson-bivector, with the choice $\nabla = \widehat{\nabla}$. We then always have a quantum differential algebra by Theorem 4.4 and Corollary 5.3 says that the nicest case is when the ω-contracted Ricci tensor is covariantly constant. In this case we have a quantum symmetric g_1 and a quantum-Levi-Civita connection for it.

Proposition 6.1. *In the case of a Kähler manifold, $\mathcal{R}$ in Corollary 5.3 is the Ricci 2-form. A sufficient condition for this to be covariantly constant is for the metric to be Kähler-Einstein.*

Proof. Here $\omega^{ij} = -g^{ik}J_k{}^j = J_k{}^i g^{kj}$ where $J^2 = -\mathrm{id}$ and $\mathcal{R} = \frac{1}{2}\mathcal{R}_{nm}\mathrm{d}x^m \wedge \mathrm{d}x^n$ in our conventions so in Corollary 5.3 we have $\mathcal{R}_{nm} = \omega^{ji}R_{inmj} = g_{kj}\omega^{ji}R_{inm}{}^k = -J_j{}^i R_{inm}{}^j$. Now we use standard complexified local coordinates $z^a, \bar{z}^a$ in which $J_a{}^b = \imath\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -\imath\delta_{\bar{a}}{}^{\bar{b}}$. The only nonzero elements of Riemann are then of the form

$$R_{\bar{a}bc}{}^d = -R_{b\bar{a}c}{}^d, \quad R_{a\bar{b}\bar{c}}{}^{\bar{d}} = -R_{\bar{b}a\bar{c}}{}^{\bar{d}}.$$

Hence $\mathcal{R}_{\bar{n}m} = -\imath R_{a\bar{n}m}{}^a = \imath R_{\bar{n}am}{}^a = \imath\mathrm{Ricci}_{\bar{n}m}$ and similarly $\mathcal{R}_{n\bar{m}} = -\imath\mathrm{Ricci}_{n\bar{m}} = -\mathcal{R}_{\bar{m}n}$ by symmetry of Ricci. Then $\mathcal{R}_{ij} = -J_i{}^k\mathrm{Ricci}_{kj}$ in our conventions for 2-form components. Equivalently, $\mathcal{R} = \frac{1}{2}\mathcal{R}_{a\bar{b}}\mathrm{d}z^{\bar{b}} \wedge \mathrm{d}z^a + \frac{1}{2}\mathcal{R}_{\bar{b}a}\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b = \imath\mathrm{Ricci}_{a\bar{b}}\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b$ as usual. Clearly in the Kähler-Einstein case we have also that Ricci = αg for some constant α. Then $\mathcal{R}_{ij} = -J_i{}^k \alpha g_{kj} = -\alpha\omega_{ij}$ in terms of the inverse ω_{ij} of the Poisson tensor, or $\mathcal{R} = \alpha\omega/2$ in terms of the symplectic 2-form $\omega = \omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$. This is covariantly constant by our assumption of Poisson-compatibility by Lemma 3.1. □

Note that the Ricci 2-form here is closed and represents the 1st Chern class. It is known that every Kähler manifold with $c_1 \le 0$ admits a Kähler-Einstein metric and that this is also true under certain stability conditions for positive values. This includes Calabi-Yau manifolds (admitting a Ricci flat metric) and $\mathbb{C}P^n$ with its Fubini-study metric. Also note that on a Kähler manifold the J is also covariantly constant and we may hope to have a noncommutative complex structure in the sense of [12] to $O(\lambda^2)$. This will be considered elsewhere.

Any orientable surface can be given the structure of a Kähler manifold so that the above applies. In fact we do not make use above of the full Kähler structure and in the case of an orientable surface we can consider any metric and Poisson tensor $\omega = -\mathrm{Vol}^{-1}$ as obtained from the volume form, which will be covariantly constant. The generalised Ricci 2-form is then a constant multiple of $S\mathrm{Vol}$ where S is the Ricci scalar (this follows from the Ricci tensor being $gS/2$ for any surface). So $\mathcal{R}$ will be covariantly constant if and only if S is constant, i.e. the case of constant curvature.

Some general formulae for any surface are as follows, in local coordinates (x, y). Here $\mathrm{Vol} = \sqrt{\det(g)}\mathrm{d}x\mathrm{d}y$ where $g = (g_{ij})$ is the metric. The Poisson tensor $\omega = -\mathrm{Vol}^{-1}$ is then

$$\omega = w\left(\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\otimes\frac{\partial}{\partial x}\right); \quad \omega^{12} = w(x,y) := \frac{1}{\sqrt{\det(g)}}$$

which of course gives our product as $x \bullet x = x^2$, $y \bullet y = y^2$, $x \bullet y = xy + \frac{\lambda}{2}w$, $y \bullet x = xy - \frac{\lambda}{2}w$, or commutation relations $[x, y]_\bullet = \lambda w$ on the generators. Similarly, the bimodule commutation relations from the form of ω are

$$[f, \xi]_\bullet = \lambda w\left(\frac{\partial f}{\partial x}\nabla_y - \frac{\partial f}{\partial y}\nabla_x\right)\xi$$

where ∇_x, ∇_y are the covariant derivatives along $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively. In terms of Γ,

$$[f, \mathrm{d}x^j]_\bullet = \lambda w(f_{,2}\Gamma^j{}_{1m}\mathrm{d}x^m - f_{,1}\Gamma^j{}_{2m}\mathrm{d}x^m)$$

or on generators and with $\epsilon^{12} = 1$ antisymmetric,

$$[x^i, \mathrm{d}x^j]_\bullet = -\lambda w\epsilon^{in}\Gamma^j_{nm}\mathrm{d}x^m.$$

There are similar expressions for $\bullet$ itself in terms of the classical product plus half of the relevant commutator.

Next, Ricci = $\frac{S}{2}g$ implies by symmetries of the Riemann tensor that

$$R_{1212} = \tfrac{1}{2}S\det(g) =: \rho(x,y)$$

say, with other components determined by its symmetries. In this case

$$\mathcal{R}_{12} = -\mathcal{R}_{21} = -\omega^{is}R_{i12s} = w\rho, \quad \mathcal{R} = -w\rho\mathrm{d}x\mathrm{d}y = -\frac{S}{2}\mathrm{Vol}$$

and

$$H^{ij} = -\frac{1}{2}\omega^{is}R^i{}_{nms}\mathrm{d}x^m\mathrm{d}x^n$$

which we compute first with j lowered by the metric as

$$H^1{}_1 = H^2{}_2 = -\frac{w\rho}{2}\mathrm{d}x\mathrm{d}y = -\frac{S}{4}\mathrm{Vol}, \quad H^1{}_2 = H^2{}_1 = 0$$

so we conclude in terms of the inverse metric that

$$H^{ij} = -\tfrac{1}{4}Sg^{ij}\mathrm{Vol}.$$

By Theorem 4.4 we necessarily have a differential graded algebra to $O(\lambda^2)$. Here Proposition 4.3 in our case becomes

$$\mathrm{d}x^i \bullet \mathrm{d}x^j = \mathrm{d}x^i \wedge \mathrm{d}x^j + \tfrac{\lambda}{2} w \left(\Gamma^i_{11}\Gamma^j_{22} - 2\Gamma^i_{12}\Gamma^j_{12} + \Gamma^i_{22}\Gamma^j_{11}\right) \mathrm{d}x \wedge \mathrm{d}y - \tfrac{\lambda S}{4} g^{ij}\mathrm{Vol}$$

so that the anticommutation relations for the quantum wedge product have the form

$$\{\mathrm{d}x^i, \mathrm{d}x^j\}_\bullet = \lambda \left(w^2 \left(\Gamma^i_{11}\Gamma^j_{22} - 2\Gamma^i_{12}\Gamma^j_{12} + \Gamma^i_{22}\Gamma^j_{11}\right) - \tfrac{S}{2} g^{ij}\right) \mathrm{Vol}.$$

Finally the quantized metric, from (5.1) and since $\nabla g = 0$,

$$g_1 = g_Q + \lambda \mathcal{R}_{12}\widetilde{\mathrm{Vol}} = \tilde{g} + \frac{\lambda w}{2}\epsilon^{ij} g_{ma}\Gamma^a_{ib}\Gamma^b_{jn}\mathrm{d}x^m \otimes_1 \mathrm{d}x^n + \frac{\lambda S}{2}\widetilde{\mathrm{Vol}} \tag{6.1}$$

where the first two terms are g_Q and

$$\tilde{g} := g_{ij}\mathrm{d}x^i \otimes_1 \mathrm{d}x^j, \quad \widetilde{\mathrm{Vol}} := \frac{1}{2w}(\mathrm{d}x \otimes_1 \mathrm{d}y - \mathrm{d}y \otimes_1 \mathrm{d}x)$$

are shorthand notations. Similarly, the connection ∇_Q is computed from the local formula (5.2). As explained, we will have $\nabla_Q g_1 = 0$ at order λ if and only if S is constant. We compute further details for the two basic examples.

6.1. **Quantized hyperbolic space.** As the basic example we look at the Poincaré upper half plane with its hyperbolic metric

$$M = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}, \quad g = \frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$$

which is readily found to have nonzero Christoffel symbols

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -y^{-1}\ , \quad \Gamma^2_{11} = y^{-1}$$

or $\Gamma^i_{1j} = -\epsilon^{ij}y^{-1}$ and $\Gamma^i_{2j} = -\delta^i_j y^{-1}$. The bivector $\omega^{12} = -\omega^{21} = y^2$ is easily seen to be the unique solution to (3.1) up to normalisation. This is the inverse of the volume form $\mathrm{Vol} = y^{-2}\mathrm{d}x\mathrm{d}y$.

Clearly from the Poisson tensor

$$\omega = y^2\Big(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}\Big)$$

we have $[x,y]_\bullet = \lambda y^2$, which relations also occur for the standard bicrossproduct model spacetime in 2-dimensions in terms of inverted coordinates in [10]. Also note that $[x, y^{-1}]_\bullet = \lambda$. Note that although the relations do extend to an obvious associative algebra A, this is not unique and not immediately relevant.

Next, from Γ we see that

$$[f, \mathrm{d}x]_\bullet = \lambda y\Big(\frac{\partial f}{\partial x}\mathrm{d}x - \frac{\partial f}{\partial y}\mathrm{d}y\Big), \quad [f, \mathrm{d}y]_\bullet = \lambda y\Big(\frac{\partial f}{\partial y}\mathrm{d}x + \frac{\partial f}{\partial x}\mathrm{d}y\Big)$$

or on generators we have

$$[x, \mathrm{d}x]_\bullet = [y, \mathrm{d}y]_\bullet = \lambda y \mathrm{d}x, \quad [x, \mathrm{d}y]_\bullet = -[y, \mathrm{d}x]_\bullet = \lambda y \mathrm{d}y.$$

There are similar expressions for $\bullet$ itself in terms of the classical product.

The Ricci scalar here is $S = -2$ so

$$\mathcal{R} = \mathrm{Vol}, \quad H^{ij} = \frac{1}{2}g^{ij}\mathrm{Vol}$$

and from the latter we obtain

$$\mathrm{d}x^i \bullet \mathrm{d}x^j = \mathrm{d}x^i \wedge \mathrm{d}x^j + \tfrac{\lambda}{2} y^2 \left(\Gamma^i_{11}\Gamma^j_{22} - 2\Gamma^i_{12}\Gamma^j_{12} + \Gamma^i_{22}\Gamma^j_{11}\right) \mathrm{d}x \wedge \mathrm{d}y + \tfrac{\lambda}{2}\delta_{ij}\mathrm{d}x \wedge \mathrm{d}y$$

which from the form of Γ simplifies further to

$$\mathrm{d}x^i \bullet \mathrm{d}x^j = \mathrm{d}x^i \wedge \mathrm{d}x^j - \tfrac{\lambda}{2}\delta_{ij}\mathrm{d}x \wedge \mathrm{d}y, \quad \{\mathrm{d}x^i, \mathrm{d}x^j\}_\bullet = -\lambda\delta_{ij}\mathrm{d}x \wedge \mathrm{d}y$$

which has a 'Clifford algebra-like' form. The result here is the same as obtained by applying d to the bimodule relations, i.e. is consistent with the maximal prolongation of the first order calculus.

Finally, we have our constructions of noncommutative Riemannian geometry. In our case

$$\epsilon^{ij} g_{ma} \Gamma^a_{ib} \Gamma^b_{jn} = 0$$

so that g_Q has the same form as classically but with $\otimes_1$ and

$$g_1 = \frac{\mathrm{d}x^i}{y^2} \otimes_1 \mathrm{d}x^i - \lambda \widetilde{\mathrm{Vol}}$$

(sum over i). Similarly, one may compute using the form of Γ in (5.2) that

$$\nabla_Q \mathrm{d}x^i = \frac{\mathrm{d}y}{y} \otimes_1 \mathrm{d}x^i + \frac{\mathrm{d}x}{y} \otimes_1 \epsilon_{ij} \mathrm{d}x^j$$

which again has the same form as classically. There is an associated generalised brading σ_Q making this a bimodule connection. As per our general theory, ∇_Q is quantum torsion free and metric compatible with g_1.

All constructions above are invariant under $SL_2(\mathbb{R})$ and hence under the modular group and other discrete subgroups. Indeed, the metric is well known to be invariant. The volume form can also easily be seen to be and correspondingly ω is invariant. As these are the only inputs into the theory it follows that the deformed structures are likewise compatible with this action. The quotient of the constructions corresponds to replacing the Poincaré upper half plane by a Riemann surface of constant negative curvature, constructed as quotient. Development of the fuller noncommutative and nonassociative geometry to the point of contact with modular forms and with physics such as the fractional quantum Hall effect[31] are interesting directions for further work.

6.2. **Quantized sphere.** The case of a surface of constant positive curvature, the sphere, is the $n = 1$ case of $\mathbb{C}P^n$ which will be covered elsewhere in holomorphic coordinates. Here we give it is as an example of the analysis for surfaces above.

Our construction is global but we focus on the upper hemisphere in standard cartesian coordinates, with similar formulae for the lower hemisphere. Thus for now,

$$M = \{(x,y) \mid x^2 + y^2 < 1\}, \quad z = \sqrt{1 - x^2 - y^2},$$

$$g = \frac{1}{z^2}\big((1-y^2)\mathrm{d}x \otimes \mathrm{d}x + xy(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + (1-x^2)\mathrm{d}y \otimes \mathrm{d}y\big)$$

which is readily found to have symmetric Christoffel symbols

$$\Gamma^1_{11} = \frac{x}{z^2}(1-y^2), \quad \Gamma^1_{22} = \frac{x}{z^2}(1-x^2), \quad \Gamma^1_{12} = \frac{x^2 y}{z^2}$$

$$\Gamma^2_{11} = \frac{y}{z^2}(1-y^2), \quad \Gamma^2_{22} = \frac{y}{z^2}(1-x^2), \quad \Gamma^2_{12} = \frac{xy^2}{z^2}.$$

or compactly $\Gamma^i_{jk} = x^i g_{jk}$. The inverse of the volume form $\mathrm{Vol} = z^{-1}\mathrm{d}x\mathrm{d}y$ gives the Poisson bivector

$$\omega = z\Big(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}\Big)$$

so we have relations

$$[x,y]_\bullet = \lambda z, \quad [z,x]_\bullet = \lambda y, \quad [y,z]_\bullet = \lambda x,$$

the standard relations of the fuzzy sphere. In this case there is an obvious associative quantisation to all orders as the enveloping algebra $U(su_2)$ modulo a constant value of the quadratic Casimir. It is known that this algebra A does not admit an associative 3D rotationally invariant calculus[6] so there won't be a zero-curvature Poisson-compatible connection ∇. We use the Levi-Civita connection according to Corollary 5.3. Then from Γ we have

$$[f, \mathrm{d}x^j]_\bullet = -\lambda z\, x^j f_{,i}\epsilon^{ik} g_{km}\mathrm{d}x^m$$

for the bimodule relations of the quantum differential calculus, where $\epsilon^{12} = 1$ is antisymmetric. Explicitly,

$$[x, \mathrm{d}x^i]_\bullet = -\lambda \frac{x^i}{z}(xy\mathrm{d}x + (1-x^2)\mathrm{d}y), \quad [y, \mathrm{d}x^i]_\bullet = \lambda\frac{x^i}{z}((1-y^2)\mathrm{d}x + xy\mathrm{d}y).$$

Next, the Ricci scalar of the unit sphere is $S = 2$ so

$$\mathcal{R} = -\mathrm{Vol}, \quad H^{ij} = -\frac{1}{2}g^{ij}\mathrm{Vol}\ , \quad \text{where } g^{ij} = \begin{pmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \end{pmatrix}.$$

From this and Γ we obtain

$$\mathrm{d}x^i \wedge_\bullet \mathrm{d}x^j = \mathrm{d}x^i \wedge \mathrm{d}x^j + \lambda\big(x^i x^j - \tfrac{1}{2}\, g^{ij}\big)\mathrm{Vol}, \quad \{\mathrm{d}x^i, \mathrm{d}x^j\}_\bullet = \lambda\left(2x^i x^j - g^{ij}\right)\mathrm{Vol}$$

for the exterior algebra relations. One can verify that this is the maximal prolongation of the bimodule relations.

Finally, we note that from the form of g that $x^a g_{ai} = x^i z^{-2}$ and $\omega^{ab}g_{ai}g_{bj} = \epsilon_{ij}z^{-1}$. The first of these and the form of Γ gives us

$$g_{ij,k} = \Gamma^a_{ki}g_{aj} + \Gamma^a_{kj}g_{ia} = x^a\big(g_{ki}g_{aj} + g_{kj}g_{ia}\big) = z^{-2}\big(x^j g_{ki} + x^i g_{kj}\big)$$

using metric compatibility. One may then compute the quantum metric and connection from (6.1) and (5.2) respectively as

$$g_1 = \tilde{g} + \frac{\lambda}{2z^3}x^m \mathrm{d}x^m \otimes_1 x^a\epsilon_{an}\mathrm{d}x^n + \lambda\widetilde{\mathrm{Vol}}$$

$$\nabla_Q \mathrm{d}x^i = -x^i\tilde{g} - \lambda x^i\widetilde{\mathrm{Vol}} - \frac{\lambda}{2z}x^m\mathrm{d}x^m \otimes_1 (\epsilon^{ib}g_{bn} + \frac{x^i x^b}{z^2}\epsilon_{bn})\mathrm{d}x^n$$

$$= -x^i g_1 - \frac{\lambda}{2z}x^m \mathrm{d}x^m \otimes_1 \epsilon^{ib}g_{bn}\mathrm{d}x^n = -x^i \bullet g_1$$

(sum over m). Here on the left $x^i\tilde{g}$ is a shorthand notation for the previously defined element of $\Omega^1 \otimes_1 \Omega^1$ but now with an extra classical x^i in the definition. One can think of it as made with the classical product when the classical and quantum vector spaces are identified, and ditto for $x^i g_1$. The expression $x^i \bullet g_1$ is computed with the quantized product but only on the first tensor factor of $g_1 \in \Omega^1 \otimes_1 \Omega^1$ (since this is the relevant bimodule structure).

The formulae are analogous in any other cartesian coordinate patch of $M = S^2$ and together can be expressed in a spherically symmetric way in terms of variables x, y, z, which we denote collectively as x^i, $i = 1, 2, 3$, with relations $\sum_i (x^i)^2 = 1$ and

$$[x^i, x^j] = \lambda\epsilon^{ij}{}_k x^k, \quad [x^i, \mathrm{d}x^j] = \lambda x^j \epsilon^i{}_{km}x^k\mathrm{d}x^m, \quad \{\mathrm{d}x^i, \mathrm{d}x^j\}_1 = \lambda(3x^i x^j - \delta^{ij})$$

to $O(\lambda^2)$. Here $\epsilon_{123} = 1$ is totally antisymmetric with the same values for any raised indices. In fact, this example recovers the first order differentials previously obtained in [6] as a cochain twist of $\Omega(S^2)$ by a certain action of the Lorentz group on the sphere (the so-called 'sphere at infinity'). This can in principle be used to construct the full noncommutative nonassociative Riemannian geometry by cochain twist, including a Dirac operator, in the formalism of [9].

7. Bicrossproduct and black-hole models

In this section we give two examples where we cannot take $\nabla = \widehat{\nabla}$. The black-hole is our main goal but before doing that we warm-up with the easier bicrossproduct model example to illustrate all our semiclassical theory but where the algebraic version is already exactly solved by computer algebra[10]. In both cases there is a natural g_1 and a unique

∇_1 that is star-preserving, quantum torsion free and quantum metric compatible. In the bicrossproduct model we use an existing differential calculus, derived from the theory of quantum groups, giving ω and a flat ∇ while $\nabla g = 0$ then forces the metric, while for the black hole model we fix g and solve for reasonable ω and ∇, which has curvature. The bicrossproduct model can be viewed as having a strong gravitational source, so strong that even light can't escape (so something like the inside of a black hole, but with decaying rather than zero Ricci tensor) [10], as well as a different, cosmological, interpretation. Both models will be developed further in a sequel [24].

7.1. **2D bicrossproduct model.** Setting $x^0 = t$ and $x^1 = r$, we have $\omega^{10} = -\omega^{01} = r$ as the semiclassical data behind the bicrossproduct model commutation relations $[t, r]_\bullet = \lambda r$. It is known that this model has a standard 2D differential calculus with nonzero relations

$$[r, \mathrm{d}t]_\bullet = \lambda \mathrm{d}r, \quad [t, \mathrm{d}t]_\bullet = \lambda \mathrm{d}t,$$

which has as its underlying semiclassical data a connection with Christoffel symbols $\Gamma^0_{01} = -r^{-1}$ and $\Gamma^0_{10} = r^{-1}$ and all other Christoffel symbols zero. This has torsion $T^0_{10} = -T^0_{01} = 2\,r^{-1}$ and $T^1_{ij} = 0$ and one can check that it Poisson-compatible. One can calculate

$$\begin{aligned} T^i_{01;p} &= T^i_{01,p} + \Gamma^i_{pn}\, T^n_{01} - \Gamma^n_{p0}\, T^i_{n1} - \Gamma^n_{p1}\, T^i_{0n} \\ &= \delta_{0i}\,(T^0_{01,p} + \Gamma^0_{pn}\, T^n_{01} - \Gamma^n_{p0}\, T^0_{n1} - \Gamma^n_{p1}\, T^0_{0n}) \\ &= \delta_{0i}\,(T^0_{01,p} + \Gamma^0_{p0}\, T^0_{01} - \Gamma^0_{p0}\, T^0_{01}) = \delta_{0i}\,\delta_{1p}\, T^0_{01,1} = 2\,r^{-2}\,\delta_{0i}\,\delta_{1p} \end{aligned}$$

and that the curvature is zero, as it should since the standard calculus is associative to all orders. To see this, without loss of generality, we look at $j = 0$, $k = 1$:

$$\begin{aligned} R^l{}_{i01} &= \frac{\partial \Gamma^l_{1i}}{\partial x^0} - \frac{\partial \Gamma^l_{0i}}{\partial x^1} + \Gamma^m_{1i}\,\Gamma^l_{0m} - \Gamma^m_{0i}\,\Gamma^l_{1m} \\ &= \delta_{0l}\,(\frac{\partial \Gamma^0_{1i}}{\partial t} - \frac{\partial \Gamma^0_{0i}}{\partial r} + \Gamma^0_{1i}\,\Gamma^0_{00} - \Gamma^0_{0i}\,\Gamma^0_{10}) \\ &= \delta_{0l}\,(-\frac{\partial \Gamma^0_{0i}}{\partial t} - \Gamma^0_{0i}\,\Gamma^0_{10}) = \delta_{0l}\,\delta_{1i}\,(-\frac{\partial \Gamma^0_{01}}{\partial r} - \Gamma^0_{01}\,\Gamma^0_{10}) = 0\ . \end{aligned}$$

Next we compute,

$$\begin{aligned} H^{ij} &:= \tfrac{1}{4}\omega^{is}\big(T^j_{nm;s} - 2R^j_{nms}\big)\,\mathrm{d}x^m \wedge \mathrm{d}x^n \\ &= \tfrac{1}{4}\,\delta_{0j}\,\omega^{is} T^0_{nm;1}\,\delta_{1s}\,\mathrm{d}x^m \wedge \mathrm{d}x^n = \tfrac{1}{4}\,\delta_{0j}\,\omega^{i1} T^0_{nm;1}\,\mathrm{d}x^m \wedge \mathrm{d}x^n \\ &= \tfrac{1}{4}\,\delta_{0j}\,\delta_{0i}\,\omega^{01} T^0_{nm;1}\,\mathrm{d}x^m \wedge \mathrm{d}x^n \\ &= \tfrac{1}{4}\,\delta_{0j}\,\delta_{0i}\,\omega^{01}(T^0_{01;1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^0 + T^0_{10;1}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1) \\ &= \tfrac{1}{4}\,\delta_{0j}\,\delta_{0i}\,\omega^{01} 2\, T^0_{01;1}\,\mathrm{d}r \wedge \mathrm{d}t \\ &= \tfrac{1}{2}\,\delta_{0j}\,\delta_{0i}\,(-r) 2\,r^{-2}\,\mathrm{d}r \wedge \mathrm{d}t = \delta_{0j}\,\delta_{0i}\,r^{-1}\,\mathrm{d}t \wedge \mathrm{d}r\ . \end{aligned}$$

The wedge product obeying the Leibniz rule in Theorem 4.4 is then;

$$\begin{aligned} \xi \wedge_1 \eta &= \xi \wedge \eta + \tfrac{\lambda}{2}\omega^{ij}\,\nabla_i \xi \wedge \nabla_j \eta \\ &\quad + (-1)^{|\xi|+1}\,\lambda\, r^{-1}\,\mathrm{d}t \wedge \mathrm{d}r \wedge (\partial_0 \lrcorner\, \xi) \wedge (\partial_0 \lrcorner\, \eta)\ . \end{aligned} \tag{7.1}$$

For ξ and η being either $\mathrm{d}r$ or $\mathrm{d}t$, the only potentially deformed case is

$$\begin{aligned} \mathrm{d}t \wedge_1 \mathrm{d}t &= \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_i(\mathrm{d}t) \wedge \nabla_j(\mathrm{d}t) + \lambda\, r^{-1}\,\mathrm{d}t \wedge \mathrm{d}r \wedge (\partial_0 \lrcorner\, \mathrm{d}t) \wedge (\partial_0 \lrcorner\, \mathrm{d}t) \\ &= \tfrac{\lambda}{2}\,\big(\omega^{01}\,\nabla_0(\mathrm{d}t) \wedge \nabla_1(\mathrm{d}t) + \omega^{10}\,\nabla_1(\mathrm{d}t) \wedge \nabla_0(\mathrm{d}t)\big) + \lambda\, r^{-1}\,\mathrm{d}t \wedge \mathrm{d}r \\ &= \tfrac{\lambda}{2}\,\omega^{01}\,\big(\nabla_0(\mathrm{d}t) \wedge \nabla_1(\mathrm{d}t) - \nabla_1(\mathrm{d}t) \wedge \nabla_0(\mathrm{d}t)\big) + \lambda\, r^{-1}\,\mathrm{d}t \wedge \mathrm{d}r = 0\ . \end{aligned}$$

The exterior algebra among these basis elements is therefore undeformed, in agreement with the noncommutative algebraic picture where this is known (and holds to all orders).

Our goal is to study the semiclassical geometry of this model using our functorial methods. First of all, the above connection is not compatible with the flat metric, but *is* compatible with the metric

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = b\,r^2\,\mathrm{d}t \otimes \mathrm{d}t - b\,r\,t\,(\mathrm{d}t \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}t) + (1 + b\,t^2)\,\mathrm{d}r \otimes \mathrm{d}r\ .$$

where b is a non-zero real parameter. This is our semiclassical analogue of the obstruction discovered in [10]. For our purposes it is better to write the metric as the following, where $v = r\,\mathrm{d}t - t\,\mathrm{d}r$

$$g = \mathrm{d}r \otimes \mathrm{d}r + b\,v \otimes v\ .$$

Note that ∇ applied to both $\mathrm{d}r$ gives zero. We quantize the classical bicrossproduct spacetime with this metric. First

$$q^{-1}(g) = \mathrm{d}r \otimes_1 \mathrm{d}r + b\,v \otimes_1 v\ .$$

From the expression for H^{ij}, we have $\mathcal{R} = g_{ij}H^{ij} = b\,r\,\mathrm{d}t \wedge \mathrm{d}r = b\,v \wedge \mathrm{d}r = \pm\sqrt{|b|}\mathrm{Vol}$ and according to our general scheme, we take

$$g_1 = \mathrm{d}r \otimes_1 \mathrm{d}r + b\,v \otimes_1 v + \tfrac{b\lambda}{2}(\mathrm{d}r \otimes_1 v - v \otimes_1 \mathrm{d}r).$$

To compare with [10], if we let

$$\nu := r \bullet \mathrm{d}t - t \bullet \mathrm{d}r = v + \tfrac{\lambda}{2}\mathrm{d}r, \quad \nu^* := (\mathrm{d}t) \bullet r - (\mathrm{d}r) \bullet t = v - \tfrac{\lambda}{2}\mathrm{d}r \tag{7.2}$$

and identify these with v, v^* in [10] (apologies for the clash of notation) then the quantum metric there gives the same answer as g_1 above, i.e. this is the leading order part of the noncommutative geometry. From Theorem 3.5 we get ∇_Q vanishing on both $\mathrm{d}r$ and v, and for all 1-forms ξ, $\sigma_Q(\mathrm{d}r \otimes_1 \xi) = \xi \otimes_1 \mathrm{d}r$ and $\sigma_Q(v \otimes_1 \xi) = \xi \otimes_1 v$.

Next we express the classical Levi-Civita connection for the above metric in the form ∇_S. We use (5.7) together with the only nonvanishing downstairs torsions being $T_{010} = -T_{001} = 2\,b\,r$ and $T_{110} = -T_{101} = -2\,b\,t$ and

$$S^a_{bc} = \tfrac{1}{2} g^{a0}(T_{0bc} - T_{bc0} - T_{cb0}) + \tfrac{1}{2} g^{a1}(T_{1bc} - T_{bc1} - T_{cb1})\ ,$$

to give

$$S^a_{11} = -\,g^{a0}\,T_{110}\ ,\ S^a_{00} = -\,g^{a1}\,T_{001}\ ,\ S^a_{10} = g^{a1}\,T_{110}\ ,\ S^a_{01} = g^{a0}\,T_{001}\ .$$

The upstairs metric is given by

$$g^{00} = (1 + b\,t^2)/(b\,r^2)\ ,\ g^{01} = g^{10} = t\,r^{-1}\ ,\ g^{11} = 1\ .$$

$$S^a_{11} = 2\,b\,t\,g^{a0}\ ,\ S^a_{00} = 2\,b\,r\,g^{a1}\ ,\ S^a_{10} = -\,2\,b\,t\,g^{a1}\ ,\ S^a_{01} = -\,2\,b\,r\,g^{a0},$$

which we write compactly as $S^a_{ij} = 2b\epsilon_{im}x^m\epsilon_{jn}g^{an}$, where $\epsilon_{01} = 1$ is antisymmetric. Then its covariant derivative is zero in the t direction, and in the r direction we have

$$S^0_{\mu\nu;1} = \begin{pmatrix} -\frac{2bt}{r} & \frac{2(1+bt^2)}{r^2} \\ \frac{2bt^2}{r^2} & -\frac{2t(1+bt^2)}{r^3} \end{pmatrix}, \quad S^1_{\mu\nu;1} = \begin{pmatrix} -2b & \frac{2bt}{r} \\ \frac{2bt}{r} & -\frac{2bt^2}{r^2} \end{pmatrix}.$$

We also have $\widehat{\nabla}\mathcal{R} = 0$ since $\mathcal{R}$ was a multiple of the volume form, and $R = 0$ for the curvature of ∇, so the obstruction in Theorem 5.7 for a torsion free metric compatible quantum connection is

$$\widehat{\nabla}\mathcal{R} + \omega^{ij}\,g_{rs}\,S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k \otimes \mathrm{d}x^m \wedge \mathrm{d}x^n = \omega^{ij}\,g_{rs}\,S^s_{jn}S^r_{km;i}\,\mathrm{d}x^k \otimes \mathrm{d}x^m \wedge \mathrm{d}x^n = 0$$

when we put in the compact form of S and its covariant derivative. Hence Theorem 5.7 tells us that there is a unique such quantum connection of the form $\nabla_1 = \nabla_{QS} + \lambda K$. Corollary 5.9 tells us that this is also the unique star-preserving connection of this form. In short, all obstructions vanish and we have a unique quantum Levi-Civita connection with all our desired properties.

It only remains to compute ∇_1. We take the liberty of changing the basis to write, for K real,

$$K(v) = K^v_{vv}\,v \otimes v + K^v_{rv}\,\mathrm{d}r \otimes v + K^v_{vr}\,v \otimes \mathrm{d}r + K^v_{rr}\,\mathrm{d}r \otimes \mathrm{d}r\ ,$$
$$K(\mathrm{d}r) = K^r_{vv}\,v \otimes v + K^r_{rv}\,\mathrm{d}r \otimes v + K^r_{vr}\,v \otimes \mathrm{d}r + K^r_{rr}\,\mathrm{d}r \otimes \mathrm{d}r\ .$$

Proposition 7.1. *The unique star-preserving quantum connection of the form* $\nabla_1 = \nabla_{QS} + \lambda K$ *is also torsion free and metric compatible ('quantum Levi-Civita') and given by non-zero components*

$$K^r_{vr} = K^v_{vv} = -2\,b\,r^{-1}$$

in our basis, leading to

$$\nabla_1 \mathrm{d}r = 2bv\,r^{-1} \otimes_1 v - 2b\lambda\,r^{-1} v \otimes_1 \mathrm{d}r, \quad \nabla_1 v = -2v\,r^{-1} \otimes_1 \mathrm{d}r - 2b\lambda\,r^{-1} v \otimes_1 v\ .$$

Proof. Note that $v^* = v$ and $\mathrm{d}r^* = \mathrm{d}r$ and also that Theorem 4.11 tells us the value of K which can be computed out as the value stated. But we still need to compute ∇_1 and, moreover, since this is an illustrative example we will also verify its properties directly as a nontrivial check of all our main theorems.

First we compute S as an operator from the components stated above (or one can readily compute the classical Levi-Civita connection and find S as the difference between this and ∇). Either way,

$$S(\mathrm{d}r) = 2\,b\,r^{-1}\,v \otimes v, \quad S(\mathrm{d}t) = 2\,b\,t\,r^{-2}\,v \otimes v - 2\,r^{-2}\,v \otimes \mathrm{d}r, \quad S(v) = -2\,r^{-1}\,v \otimes \mathrm{d}r$$

Next we compute ∇_{QS} and its associated generalised braiding. In the following calculation, ∇_0, ∇_1 denote the components ∇_i of the classical connection ∇ (apologies for the clash of notation). We have $\mathbb{W}_0(S) = 0$ and

$$\mathbb{W}_1(S)(v) = \nabla_1(S(v)) = \nabla_1(-2\,r^{-1}\,v \otimes \mathrm{d}r) = 2\,r^{-2}\,v \otimes \mathrm{d}r\ ,$$
$$\mathbb{W}_1(S)(\mathrm{d}r) = \nabla_1(S(\mathrm{d}r)) = \nabla_1(2\,b\,r^{-1}\,v \otimes v) = -\,2\,b\,r^{-2}\,v \otimes v$$

since $\nabla_i(\mathrm{d}r) = \nabla_i(v) = 0$. From Corollary 3.10,

$$\begin{aligned}\sigma_{QS}(v \otimes_1 \xi) &= \sigma_Q(v \otimes_1 \xi) + \lambda\,\omega^{01}\,\xi_0\,\mathbb{W}_1(S)(v)\\ &= \xi \otimes_1 v - \lambda\,r\,\xi_0\,\mathbb{W}_1(S)(v) = \xi \otimes_1 v - 2\,\lambda\,\xi_0\,r^{-1}\,v \otimes \mathrm{d}r\ ,\\ \sigma_{QS}(\mathrm{d}r \otimes_1 \xi) &= \xi \otimes_1 \mathrm{d}r - \lambda\,r\,\xi_0\,\mathbb{W}_1(S)(\mathrm{d}r) = \xi \otimes_1 \mathrm{d}r + 2\,\lambda\,\xi_0\,b\,r^{-1}\,v \otimes v\ .\end{aligned}$$

and

$$\begin{aligned}Q(S)(v) &= S(v) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1 \otimes E i}(\mathbb{W}_j(S)(v)) = S(v) + \tfrac{\lambda}{2}\,\omega^{01}\,\nabla_0(\mathbb{W}_1(S)(v))\\ &= S(v) = -2\,r^{-1}\,v \otimes \mathrm{d}r\ ,\\ Q(S)(\mathrm{d}r) &= S(\mathrm{d}r) + \tfrac{\lambda}{2}\,\omega^{ij}\,\nabla_{\Omega^1 \otimes E i}(\mathbb{W}_j(S)(\mathrm{d}r)) = S(\mathrm{d}r) + \tfrac{\lambda}{2}\,\omega^{01}\,\nabla_0(\mathbb{W}_1(S)(\mathrm{d}r))\\ &= S(\mathrm{d}r) = 2\,b\,r^{-1}\,v \otimes v\ .\end{aligned}$$

Then

$$\nabla_{QS}(v) = \nabla_Q(v) + q^{-1} Q(S)(v) = -2\,q^{-1}(r^{-1}\,v \otimes \mathrm{d}r) = -2\,r^{-1}\,v \otimes_1 \mathrm{d}r\ ,$$
$$\nabla_{QS}(\mathrm{d}r) = \nabla_Q(\mathrm{d}r) + q^{-1} Q(S)(\mathrm{d}r) = 2\,b\,q^{-1}(r^{-1}\,v \otimes v) = 2\,b\,r^{-1}\,v \otimes_1 v\ .$$

We can add this to the K obtained from Theorem 4.11 to obtain the result stated for the quantum Levi-Civita connection. □

One can also check that this quantum connection is indeed the part to $O(\lambda^2)$ of the full connection found in [10] by algebraic methods, provided we make the identification (7.2). In summary, all steps can be made to work in the 2D bicrossproduct model quantum spacetime including a quantum metric g_1 and quantisation of the Levi-Civita connection and our Poisson-level analysis agrees with the previous algebraic approach for this model. In [24] we further compute the quantum Laplacian to $O(\lambda^2)$ and fully diagonalise it in terms of Kummer M and U functions. Meanwhile in [37] we solve for g for a different choice of calculus on the same algebra (with $n-1$ commuting spatial variables in place of r) and this time are forced to the Bertotti-Robinson metric of $S^{n-2} \times dS_2$.

7.2. **Semiquantisation of the Schwarzschild black hole.** We take polar coordinates plus t for 4-dimensional space, where ϕ is the angle of rotation about the z-axis and θ is the angle to the z-axis. We take any static isotropic form of metric (including the Schwarzschild case)

$$g = -e^{N(r)}\mathrm{d}t \otimes \mathrm{d}t + e^{P(r)}\mathrm{d}r \otimes \mathrm{d}r + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi) \tag{7.3}$$

The Levi-Civita Christoffel symbols are zero except for

$$\begin{aligned}&\widehat{\Gamma}^0_{01} = \widehat{\Gamma}^0_{10} = \tfrac{1}{2}\,N', \quad \widehat{\Gamma}^1_{11} = \tfrac{1}{2}\,P', \quad \widehat{\Gamma}^1_{00} = \tfrac{1}{2}\,N'\,e^{N-P}\\ &\widehat{\Gamma}^1_{22} = -r\,e^{-P}, \quad \widehat{\Gamma}^1_{33} = -r\,e^{-P}\sin^2(\theta), \quad \widehat{\Gamma}^2_{12} = \widehat{\Gamma}^2_{21} = \widehat{\Gamma}^3_{13} = \widehat{\Gamma}^3_{31} = r^{-1}\\ &\widehat{\Gamma}^2_{33} = -\sin(\theta)\cos(\theta), \quad \widehat{\Gamma}^3_{23} = \widehat{\Gamma}^3_{32} = \cot(\theta)\ .\end{aligned} \tag{7.4}$$

We shall only consider rotationally invariant Poisson tensors ω. Consider a bivector and rotation invariance in the spherical polar coordinate system. To generate the Lie algebra of the rotation group, we only need two infinitesimal rotations, about the z axis and about the y axis. For the first, denoting change under the infinitesimal rotation by δ, we get $\delta(\theta) = 0$, $\delta(\phi) = 1$, and $\delta(\mathrm{d}\theta) = \delta_A(\mathrm{d}\phi) = 0$. The infinitesimal rotation about the y axis is rather more complicated in polar coordinates:

$$\begin{aligned}&\delta(\theta) = \cos\phi\ ,\ \delta(\phi) = -\cot\theta\sin\phi\ ,\ \delta(\mathrm{d}\theta) = -\sin\phi\,\mathrm{d}\phi\ ,\\ &\delta(\mathrm{d}\phi) = -\cot\theta\cos\phi\,\mathrm{d}\phi + \csc^2\theta\sin\phi\,\mathrm{d}\theta\ .\end{aligned}$$

It is now easily checked that a rotation invariant 2-form on the sphere is, up to a multiple, $\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$. It follows that a rotation invariant bivector on the sphere is, up to a multiple, given in polars by $\omega^{23} = \csc\theta$.

Proposition 7.2. *If ω is rotationally invariant and independent of x^0, then only $\omega^{01} = -\omega^{10} = k(r)$ and $\omega^{23} = -\omega^{32} = f(r)/\sin\theta$ are non-zero. The condition to be a Poisson tensor is that $\omega^{01}\,\omega^{23}{}_{,1} = 0$, i.e. $k(r)\,f'(r) = 0$.*

Proof. We now suppose that ω is rotationally invariant as a bivector field. To analyse this, we use our Minkowski-polar coordinates to view $E^i = \omega^{0i}$ as a spatial vector in polar coordinates and to view ω^{ij} where $i, j \neq 0$ as a spatial 2-form which we view as another vector, B. Now consider their values at the north pole of a sphere of radius r. Under rotation about the z-axis the north pole does not move so there is no orbital angular momentum. There is, however, rotation of the vector indices unless both E, B point along the z-axis. This applies equally at any point of the sphere, i.e. E, B must point radially. Equation (3.3) gives the Poisson result. □

We now write the Christoffel symbols Γ^a_{bc} for the background connection ∇ in terms of its torsion T and use Mathematica. Then we obtain the following result:

Proposition 7.3. *Assume time independence and axial symmetry (i.e. that the torsions T_{ijk} are independent of the coordinates t and ϕ). Then the general solution for the Poisson-compatibility and metric-compatibility conditions for (∇, ω) is given by $\omega^{23} = 1/\sin\theta$ (up to a constant multiple set to one), $\omega^{01} = 0$, and the following restrictions on T_{ijk}, apart from the obvious $T_{ijk} = -T_{ikj}$:*

$$\begin{array}{ccc} T_{012} = T_{201} + T_{102} & T_{013} = T_{301} + T_{103} & T_{023} = 0\\ T_{123} = 0 & T_{202} = 0 & T_{203} = -T_{302}\\ T_{212} = r & T_{213} = -T_{312} & T_{223} = 0\\ T_{303} = 0 & T_{313} = r\,\sin^2(\theta) & T_{323} = 0\end{array}$$

As T_{313} and T_{212} are non-zero, we cannot take for ∇ the Levi-Civita connection. To reduce the moduli space to a manageable size we further assume that T is rotationally invariant. This then gives the following as the only non-zero torsions, apart from the obvious $T_{ijk} = -T_{ikj}$:

$$\begin{array}{ccc} T_{001} = f_1(r) & T_{101} = f_2(r) & T_{203} = -T_{302} = -f_3(r)\sin\theta \\ T_{212} = r & T_{313} = r\sin^2(\theta) & T_{213} = -T_{312} = -f_4(r)\sin\theta \end{array}$$

where $f_1(r), f_2(r), f_3(r), f_4(r)$ are arbitrary functions of r only. At least under this simplifying assumption we then get the following value of H^{ij}, independently of any choice in the torsions:

$$H^{ij} = \begin{cases} -\frac{1}{2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & i=j=2 \\ -\frac{1}{2}\csc\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & i=j=3 \\ 0 & \text{otherwise} \end{cases}.$$

From this $\mathcal{R} = g_{ij}\,H^{ij} = -r^2\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi$. Moreover, we find in Theorem 5.7 (remembering that semicolon refers to the background connection) that there is no obstruction to a full quantum Levi-Civita connection and $\nabla_1 g_1 = 0$ exactly at our first order level.

Finally, we specialise further to the Schwarzschild case, where $e^N = \mathrm{c}^2(1 - r_s/r)$ and $e^P = (1 - r_s/r)^{-1}$, where r_s is the Schwarzschild radius. A short calculation then gives

Lemma 7.4. *For the Schwarzschild metric the non-zero* ${R^i}_{jkl}$*, up to the obvious* ${R^i}_{jkl} = -{R^i}_{jlk}$ *are*

$${R^1}_{010} = {R^0}_{110} = -\frac{f_1'(r) + \mathrm{c}^2\,r_s\,r^{-3}}{\mathrm{c}^2\,(1 - r_s/r)} \qquad {R^2}_{310} = \sin\theta\,(2\,f_3(r) - r\,f_3'(r))\,r^{-3}$$

$${R^3}_{210} = -\csc\theta\,(2\,f_3(r) - r\,f_3'(r))\,r^{-3} \qquad {R^3}_{223} = -1 \qquad {R^2}_{323} = \sin^2\theta.$$

In particular, the curvature cannot vanish entirely.

We also have (using row i column j notation)

$$S^0_{ij} = \begin{pmatrix} 0 & -e^{-N}f_1(r) & 0 & 0 \\ 0 & -e^{-N}f_2(r) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^1_{ij} = \begin{pmatrix} -e^{-P}f_1(r) & 0 & 0 & 0 \\ -e^{-P}f_2(r) & 0 & 0 & 0 \\ 0 & 0 & e^{-P}r & 0 \\ 0 & 0 & 0 & e^{-P}r\sin^2(\theta) \end{pmatrix},$$

$$S^2_{ij} = \begin{pmatrix} 0 & 0 & 0 & -\frac{f_3(r)\sin(\theta)}{r^2} \\ 0 & 0 & 0 & -\frac{f_4(r)\sin(\theta)}{r^2} \\ 0 & -\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^3_{ij} = \begin{pmatrix} 0 & 0 & \frac{f_3(r)\csc(\theta)}{r^2} & 0 \\ 0 & 0 & \frac{\csc(\theta)f_4(r)}{r^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 & 0 \end{pmatrix},$$

and the Christoffel symbols for the background connection are are

$$\Gamma^0_{ij} = \begin{pmatrix} 0 & \frac{N'(r)}{2} - e^{-N}f_1(r) & 0 & 0 \\ \frac{N'(r)}{2} & -e^{-N}f_2(r) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^2_{ij} = \begin{pmatrix} 0 & 0 & 0 & -\frac{f_3(r)\sin(\theta)}{r^2} \\ 0 & 0 & \frac{1}{r} & -\frac{f_4(r)\sin(\theta)}{r^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\cos(\theta)\sin(\theta) \end{pmatrix},$$

$$\Gamma^1_{ij} = \begin{pmatrix} \frac{1}{2}e^{-P}\left(e^N N'(r) - 2f_1(r)\right) & 0 & 0 & 0 \\ -e^{-P}f_2(r) & -\frac{1}{2}N'(r) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^3_{ij} = \begin{pmatrix} 0 & 0 & \frac{f_3(r)\csc(\theta)}{r^2} & 0 \\ 0 & 0 & \frac{\csc(\theta)f_4(r)}{r^2} & \frac{1}{r} \\ 0 & 0 & 0 & \cot(\theta) \\ 0 & 0 & \cot(\theta) & 0 \end{pmatrix}.$$

We can chose the f_i to minimise but not eliminate either the torsion T or the curvature R of ∇, i.e. we can't set $\nabla = \widehat{\nabla}$ and the quantum differentials will be nonassociative at order λ^2. It is also clear from the form of the Poisson tensor that the first two rows of the Γ matrices do contribute to the relations of the calculus, so we have a unique black hole differential calculus at our order and this has $\mathrm{d}t, \mathrm{d}r$ central. One can also observe that the remaining angular sector for each r, t, as a unit sphere, has the same quantum calculus as for the sphere in Section 6.2.

According to our Poisson-Riemannian theory, once we know that the quantum Levi-Civita connection exists to $O(\lambda^2)$, which we have established above, it can be computed in the form $\nabla_1 = \nabla_{QS} + \lambda K$ either from Theorem 5.7 or, in view of Corollary 5.9, as the unique $*$-preserving quantisation of the classical Levi-Civitia connection provided by Theorem4.11. One finds the non-zero components

$$K^1_{23} = -K^1_{32} = \frac{e^{-P}}{2r}\sin\theta \tag{7.5}$$

where the quantum torsion free connection with vanishing symmetric part of the metric compatibility tensor gives the only nonzero $A_{ijk} = g_{is}A^s_{jk}$ as

$$A_{123} = -A_{132} = r\sin\theta$$

in Lemma 5.6 and the only nonzero B_{knm} in the proof of Theorem 5.7 are

$$B_{213} = B_{231} = \tfrac{1}{2}\, r\,\sin\theta, \quad B_{312} = B_{321} = -\tfrac{1}{2}\, r\,\sin\theta\ .$$

In summary, we find that we inevitably have curvature of ∇ and hence a nonassociative calculus at order λ^2 if we try to quantize the black-hole and keep rotational invariance and classical dimension, an anomaly in line with experience in quantum group models[5]. As with those models, the alternative is to quantize associatively but have an extra cotangent dimension as in the wave-operator quantisation of the black hole achieved to all orders in [33]. The above nonassociative model is explored further at order λ in the sequel[24], including the quantum Ricci tensor. This sequel also shows that the above uniqueness and central $r, t, \mathrm{d}r, \mathrm{d}t$ phenomena apply generically for spherically symmetric spacetimes in the absence of certain degeneracies.

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