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LEAST ACTION NODAL SOLUTIONS
FOR THE QUADRATIC CHOQUARD EQUATION

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ABSTRACT. We prove the existence of a minimal action nodal solution for the quadratic Choquard equation

$$-\Delta u + u = \left(I_\alpha * |u|^2\right)u \quad \text{in } \mathbb{R}^N,$$

where I_α is the Riesz potential of order $\alpha \in (0, N)$. The solution is constructed as the limit of minimal action nodal solutions for the nonlinear Choquard equations

$$-\Delta u + u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

when $p \searrow 2$. The existence of minimal action nodal solutions for $p > 2$ can be proved using a variational minimax procedure over Nehari nodal set. No minimal action nodal solutions exist when $p < 2$.

1. INTRODUCTION

We study least action nodal solutions of the quadratic Choquard equation

$$-\Delta u + u = (I_\alpha * |u|^2)u \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{C}_2$}$$

for $N \in \mathbb{N}$ and $\alpha \in (0, N)$. Here $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined for each $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \qquad \text{where} \qquad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}.$$

For $N = 3$ and $\alpha = 2$ equation ($\mathcal{C}_2$) is the *Choquard–Pekar equation* which goes back to the 1954's work by S. I. Pekar on quantum theory of a polaron at rest [12, Section 2.1; 27] and to 1976's model of P. Choquard of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [17]. In the 1990's the same equation reemerged as a model of self-gravitating matter [15, 25] and is known in that context as the *Schrödinger–Newton equation*. Equation ($\mathcal{C}_2$) is also studied as a nonrelativistic model of boson stars [13, 16].

Mathematically, the existence and some qualitative properties of solutions of Choquard equation ($\mathcal{C}_2$) have been studied by variational methods in the early 1980s, see [17; 19; 20, Chapter III; 24] for earlier work on the problem. Recently, nonlinear Choquard equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{C}_p$}$$

with a parameter $p > 1$ attracted interest of mathematicians, see $[3, 7, 8, 10, 23, 26]$ and further references therein; see also $[9, 11]$ for modifications of $(\mathcal{C}_p)$ involving fractional Laplacian. Most of the recent works on Choquard type equations $(\mathcal{C}_p)$ so far were dedicated to the study of positive solutions. Nodal solutions were studied in $[7; 8; 10; 14; 30, \text{theorem } 9.5]$.

Equation $(\mathcal{C}_p)$ is the Euler equation of the Choquard action functional $\mathcal{A}_p$ which is defined for each function u in the Sobolev space $H^1(\mathbb{R}^N)$ by

$$\mathcal{A}_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p.$$

By the Hardy–Littlewood–Sobolev inequality, if $s \in (1, \frac{N}{\alpha})$ then for every $v \in L^s(\mathbb{R}^N)$, $I_\alpha * v \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} \le C\Big(\int_{\mathbb{R}^N} |v|^s\Big)^{\frac{N}{N-\alpha s}}, \tag{1.1}$$

(see for example $[18, \text{theorem } 4.3]$). In view of the classical Sobolev embedding, the action functional $\mathcal{A}_p$ is well-defined and continuously differentiable if and only if

$$\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}.$$

By testing equation $(\mathcal{C}_p)$ against u, the natural *Nehari constraint* $\langle \mathcal{A}'(u), u \rangle = 0$ appears. Then positive solutions of $(\mathcal{C}_p)$ can be obtained by studying the infimum

$$c_{0,p} = \inf\{\mathcal{A}_p(u) \ : \ u \in \mathcal{N}_{0,p}\}$$

over the *Nehari manifold*

$$\mathcal{N}_{0,p} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \ : \ \langle \mathcal{A}'_p(u), u \rangle = 0\}.$$

It turns out that the infimum $c_{0,p}$ is achieved when

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha},$$

and these assumptions are optimal $[17, 19, 26]$.

Besides positive solutions minimising $c_{0,p}$, which are known as *ground-states* or *least action solutions*, additional solutions can be constructed by several variational construction. In particular, one can consider *least action nodal solutions*, the sign-changing counterpart of least actions solutions. One way to search for such solutions is to consider the infimum

$$c_{\text{nod},p} = \inf\{\mathcal{A}_p(u) \ : \ u \in \mathcal{N}_{0,p}\}$$

over the *Nehari nodal set*

$$\mathcal{N}_{\text{nod},p} = \{u \in H^1(\mathbb{R}^N) \ : \ u^+ \ne 0 \ne u^-,$$
$$\langle \mathcal{A}'_p(u), u^+ \rangle = 0 \text{ and } \langle \mathcal{A}'_p(u), u^- \rangle = 0\},$$

where $u = u^+ - u^-$. Such construction has been performed for local elliptic problems on bounded domains of $\mathbb{R}^N$, see $[4\text{–}6]$, whereas the approach fails for the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{2p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which has no least action nodal solutions [1, lemma 2.4; 14; 31]. Moreover, the least action energy on the Nehari nodal set is not approximated by nodal solutions of (1.2), see [31, theorem 1.5].

Surprisingly, it has been proved that unlike its local counterpart (1.2), the nonlocal Choquard equation $(\mathcal{C}_p)$ admits least action nodal solutions when

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{1}{2}$$

[14, theorem 2]; while the infimum $c_{\mathrm{nod},p}$ is not achieved when

$$\frac{1}{2} < \frac{1}{p} < \frac{N}{N+\alpha},$$

because $c_{\mathrm{nod},p} = c_{0,p}$ [14, theorem 3]. The borderline quadratic case $p = 2$ was not covered by either existence or non-existence proofs in [14], because of the possible degeneracy of the minimax reformulation of the problem that was introduced (see [14, eq. (3.3)]) and of difficulties in controlling the norms of the positive and negative parts of Palais–Smale sequences.

The goal of the present work is to study the existence of least action nodal solutions for Choquard equation $(\mathcal{C}_p)$ in the physically most relevant quadratic case $p = 2$. Because the minimax procedure for capturing $c_{\mathrm{nod},p}$ introduced in [14] apparently fails for $p = 2$, a different approach is needed. Instead of directly minimizing $c_{\mathrm{nod},2}$, our strategy will be to employ Choquard equations $(\mathcal{C}_p)$ with $p > 2$ as a regularisation family for the quadratic Choquard equation $(\mathcal{C}_2)$ and to pass to the limit when $p \searrow 2$. Our main result is the following.

Theorem 1. *If $N \in \mathbb{N}$ and $\alpha \in ((N-4)^+, N)$, then there exists a weak solution $u \in H^1(\mathbb{R}^N)$ of Choquard equation $(\mathcal{C}_2)$ such that $u^+ \neq 0 \neq u^-$ and $\mathcal{A}_2(u) = c_{\mathrm{nod},2}$.*

The constructed nodal solution u is regular, that is, $u \in L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, see [26, proposition 4.1]. The conditions of the theorem are optimal, as for $\alpha \notin ((N-4)^+, N)$ no sufficiently regular solutions to $(\mathcal{C}_2)$ exist in $H^1(\mathbb{R}^N)$ by the Pohožaev identity [26, theorem 2].

In order to prove theorem (1), we will approximate a least action nodal solution of quadratic Choquard equation $(\mathcal{C}_2)$ by renormalised least action nodal solution of $(\mathcal{C}_p)$ with $p \searrow 2$. To do this, in section 2 we first establish continuity of the energy level $c_{0,p}$ with respect to p. Then in section 3 we prove theorem 1 by showing that as $p \searrow 2$, positive and negative parts of the renormalised least action nodal solutions of $(\mathcal{C}_p)$ do not vanish and do not diverge apart from each other.

2. Continuity of the critical levels

In the course of the proof of theorem 1, we will need the following strict inequality on critical levels.

Proposition 2.1. *If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$, then*

$$c_{\mathrm{nod},p} < 2c_{0,p}.$$

This follows directly from [14, propositions 2.4 and 3.7]. The construction in the latter reference is done by taking translated pa ositive and a negative copy of the groundstate of $(\mathcal{C}_p)$ and by estimating carefully the balance between the truncation and the effect of the Riesz potential interaction. When $p < 2$, it is known that $c_{\text{nod},p} = c_{0,p}$ [14, theorem 3] and proposition 2.1 loses its interest.

Because we shall approximate the quadratic case $p = 2$ by $p > 2$, it will be useful to have some information about the continuity of $c_{0,p}$.

Proposition 2.2. *The function $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+}) \mapsto c_{0,p} \in \mathbb{R}$ is continuous.*

Proof. It can be observed that

$$c_{0,p} = \inf\left\{ \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\right)^{\frac{p}{p-1}}}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\right)^{\frac{1}{p-1}}} \right) \; : \; u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

Since for every $u \in H^1(\mathbb{R}^N)$, the function

$$p \in (\tfrac{N+\alpha}{N}, \tfrac{N+\alpha}{(N-2)_+}) \mapsto \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

is continuous, the function $p \mapsto c_{0,p}$ is then upper semicontinuous as an infimum of continuous functions.

We now consider the more delicate question of the lower semicontinuity. There exists a family of functions $u_p \in H^1(\mathbb{R}^N)$ such that $(\mathcal{C}_p)$ holds and $\mathcal{A}_p(u_p) = c_{0,p}$. By the upper semicontinuity, it follows that the function $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+}) \mapsto u_p \in H^1(\mathbb{R}^N)$ is locally bounded.

In view of the equation $(\mathcal{C}_p)$ and of the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_p|^p)|u_p|^p \le C_1 \left(\int_{\mathbb{R}^N} |u_p|^{\frac{2Np}{N+\alpha}} \right)^{\frac{N+\alpha}{N}},$$

where constant C_1 could be chosen uniformly bounded when p remains in a compact subset of $(\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+})$. This implies that [21, lemma I.1; 26, lemma 2.3; 28, (2.4); 32, lemma 1.21]

$$\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2$$
$$\le C_2 \left(\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \right)^{\frac{N+\alpha}{N}} \left(\sup_{a \in \mathbb{R}^N} \int_{B_1(a)} |u_p|^{\frac{2Np}{N+\alpha}} \right)^{(N+\alpha)\left(\frac{1}{N} - \frac{N+\alpha}{p}\right)},$$

where C_2 can be also chosen uniformly bounded when p is in a compact subset of $(\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+})$. Up to a translation in $\mathbb{R}^N$, we can thus assume that the function

$$p \in (\tfrac{N+\alpha}{N}, \tfrac{N+\alpha}{(N-2)_+}) \mapsto \int_{B_1} |u|^{\frac{2Np}{N+\alpha}}$$

is locally bounded away from 0.

We assume now $(p_n)_{n \in \mathbb{N}}$ to be a sequence in the interval $(\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+})$ that converges to $p_* \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{(N-2)_+})$. Since the sequence $(u_{p_n})_{n \in \mathbb{N}}$ is

bounded in the space $H^1(\mathbb{R}^N)$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ diverging to infinity and $u_* \in H^1(\mathbb{R}^N)$ such that the subsequence $(u_{p_{n_k}})_{k\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to u_*. Moreover, we have by the Rellich–Kondrachov compactness theorem

$$\int_{B_1} |u_*|^{\frac{2Np_*}{N+\alpha}} = \lim_{k\to\infty} \int_{B_1} |u_{p_{n_k}}|^{\frac{2Np}{N+\alpha}} > 0.$$

Thus $u_* \neq 0$ and u_* satisfies

$$-\Delta u_* + u_* = (I_\alpha * |u|^{p_*})|u|^{p_*-2}u.$$

We have thus

$$c_{0,p_*} \leq \mathcal{A}_{p_*}(u_*) = \Big(\frac{1}{2} - \frac{1}{2p_*}\Big) \int_{\mathbb{R}^N} |\nabla u_*|^2 + |u_*|^2$$

$$\leq \liminf_{k\to\infty} \Big(\frac{1}{2} - \frac{1}{2p_{n_k}}\Big) \int_{\mathbb{R}^N} |\nabla u_{p_{n_k}}|^2 + |u_{p_{n_k}}|^2$$

$$= \liminf_{k\to\infty} \mathcal{A}_{p_{n_k}}(u_{p_{n_k}}) = \liminf_{k\to\infty} c_{0,p_{n_k}}.$$

Since the sequence $(p_n)_{n\in\mathbb{N}}$ is arbitrary, this proves the lower semicontinuity. $\square$

3. Proof of the main theorem

Proof of theorem 1. We shall successively construct a family of solutions of $(\mathcal{C}_2)$, prove that neither the positive nor the negative part of this family goes to 0, and show that the negative part and the positive part cannot diverge from each other as $p \to 2$. The theorem will then follow from a classical weak convergence and local compactness argument.

Claim 1. *There exists a family* $(u_p)_{p\in(2,\frac{N+\alpha}{N-2})}$ *in* $H^1(\mathbb{R}^N)$ *such that for each* $p \in (2, \frac{N+\alpha}{N-2})$, *the function* u_p *changes sign and*

$$-\Delta u_p + u_p = (I_\alpha * |u_p|^p)|u_p|^{p-2}u_p.$$

Moreover

$$\limsup_{p\to 2} \mathcal{A}_p(u_p) \leq c_{\mathrm{nod},2}$$

and

$$\limsup_{p\to 2} \int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \leq 4\, c_{\mathrm{nod},2}.$$

The claim implies the uniform boundedness in $H^1(\mathbb{R}^N)$ of the solutions u_p since $c_{\mathrm{nod},2} \leq c_{\mathrm{odd},2} < 2c_{0,2} < \infty$ [14].

Proof of the claim. The existence of a function $u_p \in H^1(\mathbb{R}^N)$ that changes sign and satisfies Choquard equation $(\mathcal{C}_p)$ has been proved in [14, theorem 2]. Moreover,

$$\Big(\frac{1}{2} - \frac{1}{2p}\Big) \int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 = \mathcal{A}_p(u_p) = c_{\mathrm{nod},p}.$$

It remains to obtain some upper asymptotics on $c_{\mathrm{nod},p}$ as $p \searrow 2$.

Let $w \in \mathcal{N}_{2,\mathrm{nod}}$ and define $w_p = t_{+,p}^{1/p} w^+ - t_{-,p}^{1/p} w^-$, where $(t_{+,p}, t_{-,p}) \in [0,\infty)^2$ is the unique maximizer of the concave function

$$(t_+, t_-) \in [0, \infty)^2 \mapsto E_p(t_+, t_-) = \mathcal{A}_p(t_+^{\frac{1}{p}} w^+ - t_-^{\frac{1}{p}} w^-)$$

$$= \frac{t_+^{\frac{2}{p}}}{2} \int_{\mathbb{R}^N} |\nabla w^+|^2 + |w^+|^2 + \frac{t_-^{\frac{2}{p}}}{2} \int_{\mathbb{R}^N} |\nabla w^-|^2 + |w^-|^2$$

$$- \frac{1}{2p} \int_{\mathbb{R}^N} |I_{\alpha/2} * (t_+ |w^+|^p + t_- |w^-|^p)|^2.$$

Since $(t_{+,p}, t_{-,p}) \in (0, \infty)$, we have $w_p \in \mathcal{N}_{\mathrm{nod},p}$ and

$$c_{\mathrm{nod},p} \le \mathcal{A}_p(w_p).$$

Since $E_p(t_+, t_-) \to -\infty$ as $(t_+, t_-) \to \infty$ uniformly in p in bounded sets and since $E_p \to E_2$ as $p \to 2$ uniformly over compact subsets of $[0, \infty)^2$, we have $t_{\pm,p} \to 1$ as $p \to 2$. Therefore

$$\lim_{p \to 2} \mathcal{A}_p(w_p) = \mathcal{A}_2(w).$$

Since the function $w \in \mathcal{N}_{\mathrm{nod},2}$ is arbitrary, we deduce that

$$\limsup_{p \to 2} c_{\mathrm{nod},p} \le c_{\mathrm{nod},2},$$

from which the upper bounds of the claim follow. $\diamond$

Claim 2.

$$\liminf_{p \to 2} \int_{\mathbb{R}^N} |\nabla u_p^\pm|^2 + |u_p^\pm|^2 = \liminf_{p \to 2} \int_{\mathbb{R}^N} (I_\alpha * |u_p|^p) |u_p^\pm|^p > 0.$$

Proof of the claim. We first compute by the Hardy–Littlewood–Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_p|^p) |u_p|^p \le C_3 \left(\int_{\mathbb{R}^N} |u_p|^{\frac{2Np}{N+\alpha}} \right)^{1 + \frac{\alpha}{N}}$$

$$\le C_4 \left(\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \right)^p,$$

where the constant C_4 can be taken independently of $p \in (2, \frac{N+\alpha}{N-2})$ once p remains bounded. It follows then that

$$\liminf_{p \to 2} C_4 \left(\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \right)^{p-1} \ge 1,$$

and thus,

$$\liminf_{p \to 2} \int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 > 0.$$

We assume now that there is a sequence $(p_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_{p_n}^-|^2 + |u_{p_n}^-|^2 = 0. \tag{3.1}$$

For $p \in (2, \frac{N+\alpha}{(N-2)_+})$ we define the renormalised negative part

$$v_p = \frac{u_p^-}{\|u_p^-\|_{H^1(\mathbb{R}^N)}}.$$

We first observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_p|^p) |v_p|^p = 1. \tag{3.2}$$

By [14, lemma 3.6], for every $\beta \in (\alpha, N)$ there exist $C_5, C_6 > 0$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_p|^p) |v_p|^p \leq C_5 \Big(\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \int_{\mathbb{R}^N} |\nabla v_p|^2 + |v_p|^2 \Big)^{\frac{1}{2}}$$

$$\times \Big(\sup_{a \in \mathbb{R}^N} \int_{B_R(a)} |u_p|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v_p|^{\frac{2Np}{N+\alpha}} \Big)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})}$$

$$+ \frac{C_6}{R^{\beta-\alpha}} \Big(\int_{\mathbb{R}^N} |\nabla u_p|^2 + |u_p|^2 \int_{\mathbb{R}^N} |\nabla v_p|^2 + |v_p|^2 \Big)^{\frac{p}{2}}$$

The constants come from the Hardy–Littlewood–Sobolev inequality and from the Sobolev inequality; they can thus be taken to be uniform as $p \to 2$. Since u_p and v_p remain bounded in $H^1(\mathbb{R}^N)$ as $p \to 2$, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_p|^p) |v_p|^p$$

$$\leq C_7 \Big(\sup_{a \in \mathbb{R}^N} \int_{B_R(a)} |u_p|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v_p|^{\frac{2Np}{N+\alpha}} \Big)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})} + \frac{C_8}{R^{\beta-\alpha}}.$$

In view of (3.2), there exists $R > 0$ such that

$$\liminf_{p \to 2} \sup_{a \in \mathbb{R}^N} \Big(\int_{B_R(a)} |u_p|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v_p|^{\frac{2Np}{N+\alpha}} \Big) > 0.$$

In particular, there exists a sequence of vectors $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^N$ and a sequence of real numbers $(p_n)_{n \in \mathbb{N}}$ in $(2, \frac{N+\alpha}{N-2})$ converging to 2 such that

$$\liminf_{n \to \infty} \Big(\int_{B_R(a_n)} |u_{p_n}|^{\frac{2Np_n}{N+\alpha}} \int_{B_R(a_n)} |v_{p_n}|^{\frac{2Np_n}{N+\alpha}} \Big) > 0. \qquad (3.3)$$

There exists thus a subsequence $(n_k)_{k \in \mathbb{N}}$ such that the sequences of functions $(u_{p_{n_k}}(\cdot - a_{n_k}))_{k \in \mathbb{N}}$ and $(v_{p_{n_k}}(\cdot - a_{n_k}))_{k \in \mathbb{N}}$ both converge weakly in the space $H^1(\mathbb{R}^N)$ to some functions u and $v \in H^1(\mathbb{R}^N)$.

By our contradiction assumption (3.1) and the Rellich–Kondrachov compactness theorem, we have $u \geq 0$. By the classical Rellich–Kondrachov compactness theorem, it follows from (3.3) that

$$\int_{B_R} |u|^{\frac{2Np}{N+\alpha}} > 0 \qquad \text{and} \qquad \int_{B_R} |v|^{\frac{2Np}{N+\alpha}} > 0. \qquad (3.4)$$

We also observe that by definition of v_p,

$$\{ x \in \mathbb{R}^N \; : \; v_p(x) < 0 \} \subseteq \{ x \in \mathbb{R}^N \; : \; u_p(x) \leq 0 \},$$

so that by the Rellich–Kondrachov theorem, we have

$$\{ x \in \mathbb{R}^N \; : \; v(x) < 0 \} \subseteq \{ x \in \mathbb{R}^N \; : \; u(x) \leq 0 \}. \qquad (3.5)$$

Since by the classical Rellich–Kondrachov compactness theorem, the sequence $(|u_{p_{n_k}}(\cdot - a_{n_k})|^{p_n})_{k \in \mathbb{N}}$ converges locally in measure to $|u|^2$ and is bounded in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$, it converges weakly to $|u|^2$ in the space $L^{2N/(N+\alpha)}(\mathbb{R}^N)$ [2, proposition 4.7.12; 33, proposition 5.4.7]. In view of the Hardy–Littlewood–Sobolev inequality (1.1) and the continuity of bounded linear operators for the weak topology, the sequence $(I_\alpha * |u_{p_{n_k}}(\cdot - a_{n_k})|^{p_{n_k}})_{k \in \mathbb{N}}$ converges weakly

to $I_\alpha * |u|^2$ in $L^{2N/(N-\alpha)}(\mathbb{R}^N)$. Since $((|u_{p_{n_k}}|^{p_{n_k}-2}u_{p_{n_k}})(\cdot-a_{n_k}))_{k\in\mathbb{N}}$ converges to u in $L^2_{\text{loc}}(\mathbb{R}^N)$, we conclude that

$$(I_\alpha * |u_{p_{n_k}}(\cdot - a_{n_k})|^{p_{n_k}})(|u_{p_{n_k}}|^{p_{n_k}-2}u_{p_{n_k}})(\cdot - a_{n_k}) \to (I_\alpha * |u|^2)u$$

in $L^{2N/(2N-\alpha)}(\mathbb{R}^N)$, as $k \to \infty$. By construction of the function u_p, we deduce from that the function $u \in H^1(\mathbb{R}^N)$ is a weak solution of the problem

$$-\Delta u + u = (I_\alpha * |u|^2)u.$$

By the classical bootstrap method for subcritical semilinear elliptic problems applied to the Choquard equation (see for example [7, lemma A.1; 26, proposition 4.1]), u is smooth. Since $u \geq 0$, by the strong maximum principle we have either $u = 0$ or $u > 0$, in contradiction with (3.4) and (3.5). The claim is thus proved by contradiction.   ◇

Claim 3. *There exists $R > 0$ such that*

$$\limsup_{p\to 2} \sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u_p^+|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |u_p^-|^{\frac{2Np}{N+\alpha}} > 0.$$

Proof of the claim. We assume by contradiction that for every $R > 0$,

$$\lim_{p\to 2} \sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u_p^+|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |u_p^-|^{\frac{2Np}{N+\alpha}} = 0. \tag{3.6}$$

In view of [14, lemma 3.6] and since the sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ are both bounded in $H^1(\mathbb{R}^N)$, we have, as in the proof of claim 2, for every $\beta \in (\alpha, N)$ and $R > 0$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_p^+|^p)|u_p^-|^p$$

$$\leq C_9 \Big(\sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u_p^+|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |u_p^-|^{\frac{2Np}{N+\alpha}} \Big)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})} + \frac{C_{10}}{R^{\beta-\alpha}}.$$

By our assumption (3.6), we have thus

$$\lim_{p\to 2} \int_{\mathbb{R}^N} (I_\alpha * |u_p^+|^p)|u_p^-|^p = 0. \tag{3.7}$$

We define now the pair $(t_{p,+}, t_{p,-}) \in (0, \infty)^2$ by the condition that $t_{p,\pm} u_p^\pm \in \mathcal{N}_{0,p}$, or equivalently,

$$t_{p,\pm}^{2p-2} = \frac{\int_{\mathbb{R}^N} |\nabla u_p^\pm|^2 + |u_p^\pm|^2}{\int_{\mathbb{R}^N} (I_\alpha * |u_p^\pm|^p)|u_p^\pm|^p} = \frac{\int_{\mathbb{R}^N} (I_\alpha * |u_p|^p)|u_p^\pm|^p}{\int_{\mathbb{R}^N} (I_\alpha * |u_p^\pm|^p)|u_p^\pm|^p} = 1 + o(1)$$

as $p \to 2$, in view of claim 2 and of (3.7). Since the family u_p remains bounded in $H^1(\mathbb{R}^N)$, we have

$$\lim_{p\to 2} \mathcal{A}_p(t_{p,+}u_p^+ + t_{p,-}u_p^-) - \mathcal{A}_p(u_p) = 0.$$

In view of the identity

$$\mathcal{A}_p(t_{p,+}u_p^+ + t_{p,-}u_p^-)$$

$$= \mathcal{A}_p(t_{p,+}u_p^+) + \mathcal{A}_p(t_{p,-}u_p^-) - \frac{t_{p,+}^p t_{p,-}^p}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_p^+|^p)|u_p^-|^p$$

and by (3.7), we conclude that

$$\liminf_{p\to 2} \mathcal{A}_p(u_p) \geq 2 \liminf_{p\to 2} c_{0,p}.$$

By claim 1 and by proposition 2.2, this implies that

$$c_{\mathrm{nod},2} \geq 2c_{0,2},$$

in contradiction with proposition 2.1. $\diamond$

We are now in a position to conclude the proof. Up to a translation, there exist $R > 0$ and a sequence $(p_n)_{n\in\mathbb{N}}$ in $(2, \frac{N+\alpha}{N-2})$ such that $p_n \searrow 2$ as $n \to \infty$,

$$\liminf_{n\to\infty} \int_{B_R} |u_{p_n}^\pm|^{\frac{2Np}{N+\alpha}} \geq 0$$

and the sequence $(u_{p_n})_{n\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to some function $u \in H^1(\mathbb{R}^N)$.

As in the proof of claim 2, by the weak convergence and by the classical Rellich–Kondrachov compactness theorem, we have $\mathcal{A}_2'(u) = 0$ and $u^\pm \neq 0$, whence $u \in \mathcal{N}_{2,\mathrm{nod}}$. We also have by the weak lower semicontinuity of the norm,

$$\lim_{n\to\infty} \mathcal{A}_{p_n}(u_{p_n}) = \lim_{n\to\infty} \Big(\frac{1}{2} - \frac{1}{2p_n}\Big) \int_{\mathbb{R}^N} |\nabla u_{p_n}|^2 + |u_{p_n}|^2$$
$$\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 = \mathcal{A}_2(u),$$

and thus $\mathcal{A}_2(u) = c_{\mathrm{nod},2}$. $\square$

In claim 2, the study of the renormalised negative part to prevent vanishing is reminiscent of the idea of taking the renormalised approximate solution to bypass the Ambrosetti–Rabinowitz superlinearity condition [22, 29].

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