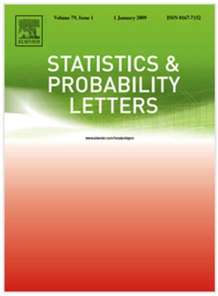
Paper:
Qiao, H. & Wu, J. (2016). Characterizing the path-independence of the Girsanov transformation for non-Lipschitz SDEs with jumps. *Statistics & Probability Letters, 119*, 326-333.
http://dx.doi.org/10.1016/j.spl.2016.09.001

Accepted Manuscript



Characterising the path-independence of the Girsanov transformation for non-Lipschitz SDEs with jumps

Huijie Qiao, Jiang-Lun Wu

PII: S0167-7152(16)30166-3
DOI: http://dx.doi.org/10.1016/j.spl.2016.09.001
Reference: STAPRO 7699

To appear in: *Statistics and Probability Letters*

Received date: 3 March 2016
Revised date: 2 September 2016
Accepted date: 2 September 2016

CHARACTERISING THE PATH-INDEPENDENCE OF THE GIRSANOV TRANSFORMATION FOR NON-LIPSCHITZ SDES WITH JUMPS*

HUIJIE QIAO[1] AND JIANG-LUN WU[2]

1. Department of Mathematics, Southeast University
Nanjing, Jiangsu 211189, China
hjqiaogean@seu.edu.cn
2. Department of Mathematics, Swansea University
Singleton Park, Swansea SA28PP, UK
j.l.wu@swansea.ac.uk

Abstract. In the paper, by virtue of the Girsanov transformation, we derive a link of a class of (time-inhomogeneous) non-Lipschitz stochastic differential equations (SDEs) with jumps to a class of semi-linear partial integro-differential equations (PIDEs) of parabolic type, in such a manner that these obtained PIDEs characterize the path-independence property of the density process of Girsanov transformation for the non-Lipschitz SDEs with jumps.

1. Introduction

The object of this paper is to establish a link of a class of (time-inhomogeneous) non-Lipschitz stochastic differential equations (SDEs) with jumps to semilinear partial integro-differential equations (PIDEs) of parabolic type, by virtue of the Girsanov transformation for the SDEs. The class of SDEs with jumps we are concerned with was investigated by the first author in [8]. Our result gives a characterisation of the path-independent property for the density process of the Girsanov transformation for the SDEs with jumps. Such a link was considered in [13] where the simple case of one dimensional Itô SDEs to the celebrated (generalised) Burgers equation was derived. Furthermore, the multi-dimensional SDEs driven by Brownian motions on $\mathbb{R}^d$ as well as on connected complete manifolds were carried out in [12] and the characterisation of the path-independence of the associated Girsanov transformation was governed by a Burgers-KPZ type nonlinear parabolic equation along with the drift coefficients being of gradient form.

To extend such a link for SDEs with jumps is not straightforward, as there is no analogous version of the Girsanov transformation (i.e., the drift transformation) for the SDEs with jumps modeled generally by Poisson random measures and the associated compensated martingale measures, see, e.g., [11]. One has to work with somehow a modified formulation towards the Girsanov transformation for SDEs with jumps. Such a realisation was succeeded by the first author in [9], which is the starting point of the present paper.

AMS Subject Classification(2010): 60H10; 35Q53.

Keywords: non-Lipschitz stochastic differential equations with jumps, the Girsanov transformation, semi-linear partial integro-differential equation of parabolic type.

*This work was partly supported by NSF of China (No. 11001051, 11371352).

On the other hand, recently, there is an increasing interest in studying partial integro-differential equations (PIDEs) from a diverse need. Let us just mention, in particular, two recent papers in mathematical finance [1, 5] where very interestingly PIDEs are linked with SDEs with jumps via the Feynman-Kac formula. The link we derive in this paper is different from theirs. It would be interesting to explore our derivation to cover more general SDEs with jumps, such as those equations arising in [1, 5] and to compare the corresponding PIDEs. We intend to address this consideration elsewhere in the future.

2. The characterisation theorem on $\mathbb{R}^d$

2.1. The characterization theorem for SDEs with continuous diffusions on $\mathbb{R}^d$.

Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geqslant 0})$ be a complete, filtered probability space. Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a finite dimensional normed space with its Borel σ-algebra $\mathscr{U}$. Let ν be a σ-finite measure defined on $(\mathbb{U}, \mathscr{U})$. We fix $\mathbb{U}_0 \in \mathscr{U}$ with $\nu(\mathbb{U} \setminus \mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2 \,\nu(\mathrm{d}u) < \infty$. Furthermore, let $\lambda : [0, \infty) \times \mathbb{U} \to (0, 1]$ be a given measurable function. Following e.g. [3, 4], there exists an integer-valued $(\mathcal{F}_t)_{t\geqslant 0}$-Poisson random measure $N_\lambda(\mathrm{d}t, \mathrm{d}u)$ on $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geqslant 0})$ with intensity $\mathbb{E}(N_\lambda(\mathrm{d}t, \mathrm{d}u)) = \lambda(t, u)\mathrm{d}t\nu(\mathrm{d}u)$. Denote

$$\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u) := N_\lambda(\mathrm{d}t, \mathrm{d}u) - \lambda(t, u)\mathrm{d}t\nu(\mathrm{d}u)$$

that is, $\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u)$ stands for the compensated $(\mathcal{F}_t)_{t\geqslant 0}$-predictable martingale measure of $N_\lambda(\mathrm{d}t, \mathrm{d}u)$.

Next, let $T > 0$ be arbitrarily fixed, and we consider the following SDE with jumps on $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t + \int_{\mathbb{U}_0} f(t, X_{t-}, u)\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u), & t \in (0, T], \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases} \tag{1}$$

where (B_t) is a d-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion, which is independent of N_λ. The coefficients $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $f : [0, T] \times \mathbb{R}^d \times \mathbb{U}_0 \mapsto \mathbb{R}^d$ are all Borel measurable.

Remark 2.1. *Since in the sequel the density process of the Girsanov transformation is not related with "big" jumps, we only consider "small" jumps in Eq.(1).*

Assume:

($\mathbf{H}_1$) There exists $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$

$$2\langle x - y, b(t, x) - b(t, y)\rangle + \|\sigma(t, x) - \sigma(t, y)\|^2 \leqslant \lambda_0 |x - y|^2 \kappa(|x - y|),$$

where κ is a positive continuous function, bounded on $[1, \infty)$ and satisfying

$$\lim_{x\downarrow 0} \frac{\kappa(x)}{\log x^{-1}} = \delta < \infty.$$

($\mathbf{H}_2$) There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in [0, T]$

$$|b(t, x)|^2 + \|\sigma(t, x)\|^2 \leqslant \lambda_1(1 + |x|)^2.$$

($\mathbf{H}_3$) $b(t, x)$ is continuous in x and there exists $\lambda_2 > 0$ such that

$$\langle\sigma(t, x)h, h\rangle \geqslant \sqrt{\lambda_2}|h|^2, \qquad t \in [0, T], \quad x, h \in \mathbb{R}^d. \tag{2}$$

$(\mathbf{H}_f)$ For all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$,

$$\int_{\mathbb{U}_0} \left|f(t,x,u) - f(t,y,u)\right|^2 \nu(\mathrm{d}u) \leqslant 2|\lambda_0||x-y|^2\kappa(|x-y|)$$

and for $q = 2$ and 4

$$\int_{\mathbb{U}_0} |f(t,x,u)|^q \, \nu(\mathrm{d}u) \leqslant \lambda_1(1+|x|)^q.$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$, $|\cdot|$ the length of a vector in $\mathbb{R}^d$ and $\|\cdot\|$ the Hilbert-Schmit norm from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Remark 2.2. *In* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_f)$, $\kappa(x)$ *can be taken as*

$$\kappa(x) = \begin{cases} \log x^{-1}, & 0 < x \leqslant \eta, \\ \log \eta^{-1} - 1 + \eta/x, & x > \eta, \end{cases}$$

for $0 < \eta < 1/e$. *And Condition (2) then assures that for any* $(t, x) \in [0, T] \times \mathbb{R}^d$, *the inverse of* $\sigma(t, x)$ *exists and is bounded.*

Under $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ and $(\mathbf{H}_f)$, it is well known that there exists a unique strong solution to Eq.(1) (cf. [11, Theorem 170, p.140]). This solution will be denoted by X_t. In the following, we define the support of a random vector ([6]) and then present a result about the support of X_t under the above assumptions.

Definition 2.3. *The support of a random vector* Y *is defined as*

$$supp(Y) := \{x \in \mathbb{R}^d | (\mathbb{P} \circ Y^{-1})(B(x,r)) > 0, \ \textit{for all } r > 0\}$$

where $B(x, r) := \{y \in \mathbb{R}^d | |y - x| < r\}$, *the open ball centered at* x *with radius* r.

Lemma 2.4. *Under* $(\mathbf{H}_1)$*-*$(\mathbf{H}_3)$ *and* $(\mathbf{H}_f)$, $supp(X_t) = \mathbb{R}^d$ *for* $t \in [0, T]$.

Proof. Since it is easy to see $supp(X_t) \subset \mathbb{R}^d$, we only prove $supp(X_t) \supset \mathbb{R}^d$. Moreover, from Definition 2.3, we only need to show that for any $x \in \mathbb{R}^d$ and $r > 0$,

$$\mathbb{P}\{|X_t - x| < r\} > 0,$$

or equivalently,

$$\mathbb{P}\{|X_t - x| \geqslant r\} < 1.$$

By the same method to that in [8, Proposition 2.4], one can prove the above result. $\square$

To apply the Girsanov transformation, we assume further the following

$(\mathbf{H}_{b,\sigma,\lambda})$

$$(i)\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2 \mathrm{d}s\right\}\right] < \infty,$$

$$(ii) \int_0^T \int_{\mathbb{U}_0} \left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s < \infty,$$

where $\sigma(s, X_s)^{-1}$ stands for the inverse of $\sigma(s, X_s)$.

Remark 2.5. *If b is bounded, then Condition (i) is satisfied. And for Condition (ii), for example, take $\mathbb{U}_0 = \{u \in \mathbb{U} | \|u\|_{\mathbb{U}} < 1\}$ and for any $(t,u) \in [0,T] \times \mathbb{U}$*

$$\lambda(t,u) = \begin{cases} 1, & 0 \leqslant \|u\|_{\mathbb{U}} \leqslant \delta, \\ \|u\|_{\mathbb{U}}^2, & \delta < \|u\|_{\mathbb{U}} < 1, \\ \delta^2, & 1 \leqslant \|u\|_{\mathbb{U}}, \end{cases}$$

where $0 < \delta < 1$ is a constant. Thus, one can justify that Condition (ii) is fulfilled.

Set

$$\begin{aligned} \Lambda_t : &= \exp\Big\{ -\int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t \left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2 \mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{U}_0} \log\lambda(s,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u) \\ &\quad - \int_0^t \int_{\mathbb{U}_0} \Big(\big(\log\lambda(s,u)\big)\lambda(s,u) + \big(1-\lambda(s,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}s\Big\}, \\ M_t : &= -\int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}B_s\rangle + \int_0^t \int_{\mathbb{U}_0} \frac{1-\lambda(s,u)}{\lambda(s,u)}\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u), \end{aligned}$$

and then (Λ_t) is the Doléans-Dade exponential of (M_t). Under $(\mathbf{H}_{b,\sigma,\lambda})$, (M_t) is a locally square integrable martingale. Moreover, $M_t - M_{t-} > -1$ a.s. and

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{\frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T \Big\}\Big] \\ =& \mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2 \mathrm{d}s \\ &\quad + \int_0^T \int_{\mathbb{U}_0} \Big(\frac{1-\lambda(s,u)}{\lambda(s,u)}\Big)^2 \lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s\Big\}\Big] \\ <& \infty, \end{aligned}$$

where M^c and M^d are continuous and purely discontinuous martingale parts of (M_t), respectively. Thus, it follows from [7, Theorem 6] that (Λ_t) is an exponential martingale. Define a measure $\tilde{\mathbb{P}}$ via

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \Lambda_T.$$

By the Girsanov theorem for Brownian motions and random measures, one can obtain that under the measure $\tilde{\mathbb{P}}$ the system (1) is transformed into the following

$$\mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}\tilde{B}_t + \int_{\mathbb{U}_0} f(t,X_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

where

$$\tilde{B}_t := B_t + \int_0^t \sigma^{-1}(s,X_s)b(s,X_s)\mathrm{d}s, \quad \tilde{N}(\mathrm{d}t,\mathrm{d}u) := N_\lambda(\mathrm{d}t,\mathrm{d}u) - \mathrm{d}t\nu(\mathrm{d}u).$$

Next, we set

$$\begin{aligned} Y_t &:= -\log\Lambda_t \\ &= \int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t \left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2 \mathrm{d}s \end{aligned}$$

$$+\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)$$
$$+\int_0^t\int_{\mathbb{U}_0}\Big(\big(\log\lambda(s,u)\big)\lambda(s,u)+\big(1-\lambda(s,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}s.$$

Clearly, (Y_t) is a one-dimensional stochastic process with the following stochastic differential form

$$\begin{aligned}\mathrm{d}Y_t &= \langle\sigma^{-1}(t,X_t)b(t,X_t),\mathrm{d}B_t\rangle+\frac{1}{2}\left|\sigma^{-1}(t,X_t)b(t,X_t)\right|^2\mathrm{d}t\\ &\quad+\int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)+\int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}t.\end{aligned}$$

Now, we state and prove the first result of this paper.

Theorem 2.6. *Let $v:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$\begin{aligned}v(t,X_t) &= v(0,x_0)+\int_0^t\int_{\mathbb{U}_0}\Big(\big(\log\lambda(s,u)\big)\lambda(s,u)+\big(1-\lambda(s,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}s\\ &\quad+\frac{1}{2}\int_0^t\left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2\mathrm{d}s+\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\langle\sigma^{-1}(s,X_s)b(s,X_s),\mathrm{d}B_s\rangle, \qquad (3)\end{aligned}$$

equivalently,

$$Y_t=v(t,X_t)-v(0,x_0),\quad t\in[0,T]$$

holds if and only if

$$b(t,x) = (\sigma\sigma^*\nabla v)(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^d, \qquad (4)$$
$$\lambda(t,u) = \exp\{v(t,x+f(t,x,u))-v(t,x)\},\qquad (t,x,u)\in[0,T]\times\mathbb{R}^d\times\mathbb{U}_0, \qquad (5)$$

and v satisfies the following time-reversed partial integro-differential equation (PIDE),

$$\begin{aligned}\frac{\partial}{\partial t}v(t,x) &= -\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x)-\frac{1}{2}|\sigma^*\nabla v|^2(t,x)-\int_{\mathbb{U}_0}\Big[e^{v(t,x+f(t,x,u))-v(t,x)}-1\\ &\quad-\langle f(t,x,u),\nabla v(t,x)\rangle e^{v(t,x+f(t,x,u))-v(t,x)}\Big]\nu(\mathrm{d}u), \qquad (6)\end{aligned}$$

where $\sigma^*(t,x)$ stands for the transposed matrix of $\sigma(t,x)$, ∇ and ∇^2 stand for the gradient and Hessian operators with respect to the second variable, respectively.

Proof. Firstly, we prove necessity. On one hand, there exists a $C^{1,2}$-function $v(t,x)$ such that $v(t,X_t)$ satisfies Eq.(3), i.e.

$$\begin{aligned}\mathrm{d}v(t,X_t) &= \left[\frac{1}{2}\left|\sigma^{-1}(t,X_t)b(t,X_t)\right|^2+\int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u)+\big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t\\ &\quad+\int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u)+\langle\sigma^{-1}(t,X_t)b(t,X_t),\mathrm{d}B_t\rangle. \qquad (7)\end{aligned}$$

It is clear from (7) that $v(t, X_t)$ is a càdlàg semimartingale with a predictable finite variation part. On the other hand, note that X_t solves Eq.(1) and $v(t, x)$ is a $C^{1,2}$-function. By applying the Itô formula to the composition process $v(t, X_t)$, one obtains the following

$$
\begin{aligned}
\mathrm{d}v(t, X_t) &= \frac{\partial}{\partial t}v(t, X_t)\mathrm{d}t + \langle b, \nabla v\rangle(t, X_t)\mathrm{d}t + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, X_t)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}\Big[v(t, X_{t-} + f(t, X_{t-}, u)) - v(t, X_{t-}) \\
&\qquad -\langle f(t, X_{t-}, u), \nabla v(t, X_{t-})\rangle\Big]\lambda(t, u)\nu(\mathrm{d}u)\mathrm{d}t \\
&\quad + \int_{\mathbb{U}_0}[v(t, X_{t-} + f(t, X_{t-}, u)) - v(t, X_{t-})]\,\tilde{N}_\lambda(\mathrm{d}t, \mathrm{d}u) \\
&\quad +\langle(\sigma^*\nabla v)(t, X_t), \mathrm{d}B_t\rangle.
\end{aligned}
\tag{8}
$$

Thus, (8) is another decomposition of the semimartingale $v(t, X_t)$. By uniqueness for decomposition of the semimartingale, it holds that for $t \in [0, T]$,

$$
\begin{aligned}
\sigma^{-1}(t, X_t)b(t, X_t) &= (\sigma^*\nabla v)(t, X_t), \\
\log\lambda(t, u) &= v(t, X_{t-} + f(t, X_{t-}, u)) - v(t, X_{t-}), \quad u \in \mathbb{U}_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left|\sigma^{-1}(t, X_t)b(t, X_t)\right|^2 + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t, u)\big)\lambda(t, u) + \big(1 - \lambda(t, u)\big)\Big)\nu(\mathrm{d}u) \\
= \;& \frac{\partial}{\partial t}v(t, X_t) + \langle b, \nabla v\rangle(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, X_t) \\
&+ \int_{\mathbb{U}_0}\Big[v(t, X_{t-} + f(t, X_{t-}, u)) - v(t, X_{t-}) \\
&\qquad -\langle f(t, X_{t-}, u), \nabla v(t, X_{t-})\rangle\Big]\lambda(t, u)\nu(\mathrm{d}u), \quad a.s..
\end{aligned}
$$

Based on Lemma 2.4, X_t runs through $\mathbb{R}^d$. So, we have that

$$
\sigma^{-1}(t, x)b(t, x) = (\sigma^*\nabla v)(t, x), \qquad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{9}
$$

$$
\log\lambda(t, u) = v(t, x + f(t, x, u)) - v(t, x), \quad (t, x, u) \in [0, T] \times \mathbb{R}^d \times \mathbb{U}_0, \tag{10}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left|\sigma^{-1}(t, x)b(t, x)\right|^2 + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t, u)\big)\lambda(t, u) + \big(1 - \lambda(t, u)\big)\Big)\nu(\mathrm{d}u) \\
= \;& \frac{\partial}{\partial t}v(t, x) + \langle b, \nabla v\rangle(t, x) + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, x) \\
&+ \int_{\mathbb{U}_0}\Big[v(t, x + f(t, x, u)) - v(t, x) \\
&\qquad -\langle f(t, x, u), \nabla v(t, x)\rangle\Big]\lambda(t, u)\nu(\mathrm{d}u).
\end{aligned}
\tag{11}
$$

It is easy to see that (9) and (10) correspond to (4) and (5), respectively, which together with (11) further yields the PIDE (6).

Next, let us show sufficiency. Assume that there exists a $C^{1,2}$-function $v(t,x)$ satisfying (4), (5) and (6). For the composition process $v(t,X_t)$, the Itô formula admits us to get (8). Combining (4), (5) and (6) with (8), we have

$$\begin{aligned} \mathrm{d}v(t,X_t) &= \left[\frac{1}{2}\left|\sigma^{-1}(t,X_t)b(t,X_t)\right|^2 + \int_{\mathbb{U}_0}\Big(\big(\log\lambda(t,u)\big)\lambda(t,u) + \big(1-\lambda(t,u)\big)\Big)\nu(\mathrm{d}u)\right]\mathrm{d}t \\ &\quad + \int_{\mathbb{U}_0}\log\lambda(t,u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u) + \langle\sigma^{-1}(t,X_t)b(t,X_t),\mathrm{d}B_t\rangle. \end{aligned}$$

The proof is completed. □

The above theorem gives a necessary and sufficient condition, and hence a characterisation of path-independence for the density Λ_t of the Girsanov transformation for SDEs with jumps in terms of a PIDE. Namely, we establish a bridge from Eq.(1) to a PIDE in the form of (6).

Remark 2.7. *Let $f(t,x,u)=0$, then Eq.(1) has no jumps. In Theorem 2.6, by (5), we know that $\lambda(t,u)=1$ for $u\in\mathbb{U}_0$. Thus, Eq.(3) becomes*

$$v(t,X_t) = v(0,x_0) + \int_0^t \langle\sigma^{-1}(s,X_s)b(s,X_s),\mathrm{d}B_s\rangle + \frac{1}{2}\int_0^t\left|\sigma^{-1}(s,X_s)b(s,X_s)\right|^2\mathrm{d}s.$$

By Theorem 2.6, the above equation holds if and only if (4) and the following equation are right,

$$\frac{\partial}{\partial t}v(t,x) = -\frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t,x) - \frac{1}{2}|\sigma^*\nabla v|^2(t,x).$$

This is exactly Theorem 2.1 in [12]. *Hence, our result is more general to allow SDEs having jumps.*

For the simplest case that $d=1$, let us look at Eq.(6). By the Hopf-Cole transformation $w(t,x):=e^{v(t,x)}$ or reciprocally $v(t,x)=\log w(t,x)$, Eq.(6) becomes

$$\begin{aligned} \frac{\partial}{\partial t}w(t,x) &= -\frac{1}{2}[\sigma^2\frac{\partial^2}{\partial x^2}w](t,x) - \int_{\mathbb{U}_0}\Big[w(t,x+f(t,x,u)) - w(t,x) \\ &\qquad -f(t,x,u)\frac{\partial}{\partial x}w(t,x)\frac{w(t,x+f(t,x,u))}{w(t,x)}\Big]\nu(\mathrm{d}u). \end{aligned}$$

The above equation is a usual PIDE.

Assume that $f(t,x,u)$ is independent of t, i.e. $f(t,x,u)=f(x,u)$. Set $f(x,u)=:y$ and $\nu(\mathrm{d}f^{-1}(x,\cdot)(y)) =: \frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}$, where $K(x)$ is a positive function and $0<\alpha(x)<2$. Then the above equation can be written as

$$\begin{aligned} \frac{\partial}{\partial t}w(t,x) &= -\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](t,x) - \int_{\mathbb{R}\setminus\{0\}}\Big[w(t,x+y)-w(t,x) \\ &\qquad -y\frac{\partial}{\partial x}w(t,x)\frac{w(t,x+y)}{w(t,x)}\Big]\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}. \end{aligned} \tag{12}$$

For $0<\alpha(x)<1$, Eq.(12) has the following form

$$\frac{\partial}{\partial t}w(t,x) = -\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](t,x) - \int_{\mathbb{R}\setminus\{0\}}\left[w(t,x+y)-w(t,x)\right]\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}$$

$$
\begin{aligned}
& +\frac{\frac{\partial}{\partial x}w(t,x)}{w(t,x)}\int_{\mathbb{R}\setminus\{0\}} w(t,x+y)y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}} \\
= & -\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](t,x) - \left[-(-\Delta)^{\frac{\alpha(x)}{2}}w\right](t,x) \\
& +\frac{\frac{\partial}{\partial x}w(t,x)}{w(t,x)}\int_{\mathbb{R}\setminus\{0\}} w(t,x+y)y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}.
\end{aligned}
$$

For $1 < \alpha(x) < 2$, Eq.(12) has the other form

$$
\begin{aligned}
\frac{\partial}{\partial t}w(t,x) = & -\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](t,x) - \int_{\mathbb{R}\setminus\{0\}}\left[w(t,x+y)-w(t,x)-y\frac{\partial}{\partial x}w(t,x)\right]\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}} \\
& +\frac{\frac{\partial}{\partial x}w(t,x)}{w(t,x)}\int_{\mathbb{R}\setminus\{0\}}\Big[w(t,x+y)-w(t,x)\Big]y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}} \\
= & -\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](t,x) - \left[-(-\Delta)^{\frac{\alpha(x)}{2}}w\right](t,x) \\
& +\frac{\frac{\partial}{\partial x}w(t,x)}{w(t,x)}\int_{\mathbb{R}\setminus\{0\}}\Big[w(t,x+y)-w(t,x)\Big]y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}.
\end{aligned}
$$

If we assume further that $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$, the two above parabolic PIDEs then change into

$$
\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](x) + \left[-(-\Delta)^{\frac{\alpha(x)}{2}}w\right](x) - \frac{\frac{\partial}{\partial x}w(x)}{w(x)}\int_{\mathbb{R}\setminus\{0\}} w(x+y)y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}} = 0,
$$

for $0 < \alpha(x) < 1$, and

$$
\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial x^2}w\right](x) + \left[-(-\Delta)^{\frac{\alpha(x)}{2}}w\right](x) - \frac{\frac{\partial}{\partial x}w(x)}{w(x)}\int_{\mathbb{R}\setminus\{0\}}\Big[w(x+y)-w(x)\Big]y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}} = 0,
$$

for $1 < \alpha(x) < 2$. In particular, for $\alpha(x) = \alpha$, the two equations contain first-order, second-order and fractional derivatives.

2.2. **The characterization theorem for SDEs without diffusion coefficient term.** Consider Eq.(1) with $\sigma(t,x) = 0$, i.e.

$$
\left\{\begin{array}{l}
\mathrm{d}\bar{X}_t = b(t,\bar{X}_t)\mathrm{d}t + \int_{\mathbb{U}_0} f(t,\bar{X}_{t-},u)\tilde{N}_\lambda(\mathrm{d}t,\mathrm{d}u), \quad t\in[0,T], \\
\bar{X}_0 = \bar{x}_0.
\end{array}\right. \tag{13}
$$

Since Eq.(13) is driven by a purely jump process, there are something different from the above derivations. Let us explicate this as follows. By [11, Theorem 170, p.140], when b, f satisfy $(\mathbf{H}_1)$-$(\mathbf{H}_2)$ and $(\mathbf{H}_f)$, Eq.(13) has a pathwise unique strong solution denoted by $\bar{X}_t$. We assume:

$(\mathbf{H}_\lambda)$

$$
\int_0^T\int_{\mathbb{U}_0}\left(\frac{1-\lambda(s,u)}{\lambda(s,u)}\right)^2\lambda(s,u)\nu(\mathrm{d}u)\mathrm{d}s < \infty.
$$

Set

$$
\bar{\Lambda}_t := \exp\Big\{-\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)
$$

$$-\int_0^t\int_{\mathbb{U}_0}\Big(\big(\log\lambda(s,u)\big)\lambda(s,u)+\big(1-\lambda(s,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}s\Big\},$$

then by similar derivations above, $\bar{\Lambda}_t$ is an exponential martingale. Define a probability measure $\bar{\mathbb{P}}$ via

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}}=\bar{\Lambda}_T.$$

Under $\bar{\mathbb{P}}$, by the Girsanov theorem for random measures, the system (13) is then transformed into the following

$$\mathrm{d}\bar{X}_t=b(t,\bar{X}_t)\mathrm{d}t+\int_{\mathbb{U}_0}f(t,\bar{X}_{t-},u)\tilde{N}(\mathrm{d}t,\mathrm{d}u).$$

Note that the drift term still exists in the above equation.

Now, we turn to the path-independence of $\bar{\Lambda}_t$. By similar arguments as in the proof of Theorem 2.6, we obtain the following result

Theorem 2.8. *Let $\bar{v}:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$\begin{aligned}\bar{v}(t,\bar{X}_t) &= \bar{v}(0,\bar{x}_0)+\int_0^t\int_{\mathbb{U}_0}\log\lambda(s,u)\tilde{N}_\lambda(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\int_0^t\int_{\mathbb{U}_0}\Big(\big(\log\lambda(s,u)\big)\lambda(s,u)+\big(1-\lambda(s,u)\big)\Big)\nu(\mathrm{d}u)\mathrm{d}s,\end{aligned}\tag{14}$$

holds if and only if

$$\lambda(t,u)=\exp\{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)\},\quad (t,x,u)\in[0,T]\times\mathbb{R}^d\times\mathbb{U}_0,\tag{15}$$

and $\bar{v}$ satisfies the following time-reversed equation,

$$\begin{aligned}\frac{\partial}{\partial t}\bar{v}(t,x) &= -\langle b,\nabla\bar{v}\rangle(t,x)-\int_{\mathbb{U}_0}\Big[e^{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)}-1\\ &\quad-\langle f(t,x,u),\nabla\bar{v}(t,x)\rangle e^{\bar{v}(t,x+f(t,x,u))-\bar{v}(t,x)}\Big]\nu(\mathrm{d}u).\end{aligned}\tag{16}$$

We analysis Eq.(16) for the special case of $d=1$. By the Hopf-Cole transformation $\bar{v}(t,x)=\log\bar{w}(t,x)$, we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\bar{w}(t,x) &= -\left[b\frac{\partial}{\partial x}\bar{w}\right](t,x)-\int_{\mathbb{U}_0}\Big[\bar{w}(t,x+f(t,x,u))-\bar{w}(t,x)\\ &\quad-f(t,x,u)\frac{\partial}{\partial x}\bar{w}(t,x)\frac{\bar{w}(t,x+f(t,x,u))}{\bar{w}(t,x)}\Big]\nu(\mathrm{d}u).\end{aligned}$$

This equation is an integro-differential equation. Assume that $b(t,x)=b(x)$, $f(t,x,u)=f(x,u)=:y$ and $\nu(\mathrm{d}f^{-1}(x,\cdot)(y))=:\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}$. By similar deduction to above, the above equation becomes

$$\left[b\frac{\partial}{\partial x}\bar{w}\right](x)+\left[-(-\Delta)^{\frac{\alpha(x)}{2}}\bar{w}\right](x)-\frac{\frac{\partial}{\partial x}\bar{w}(x)}{\bar{w}(x)}\int_{\mathbb{R}\setminus\{0\}}\bar{w}(x+y)y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}=0,$$

for $0<\alpha(x)<1$, and

$$\left[b\frac{\partial}{\partial x}\bar{w}\right](x)+\left[-(-\Delta)^{\frac{\alpha(x)}{2}}\bar{w}\right](x)-\frac{\frac{\partial}{\partial x}\bar{w}(x)}{\bar{w}(x)}\int_{\mathbb{R}\setminus\{0\}}\Big[\bar{w}(x+y)-\bar{w}(x)\Big]y\frac{K(x)\mathrm{d}y}{|y|^{1+\alpha(x)}}=0,$$

for $1 < \alpha(x) < 2$.

Acknowledgements

We thank the referees for their careful reading of our manuscript and for the detailed comments, which have led to improve the presentation of the paper.

References

[1] B. Hambly, M. Mariapragassam and Ch. Reisinger: A forward equation for barrier options under the Brunick&Shreve Markovian projection, arXiv:1411.3618 [q-fin.MF].

[2] C. Dellacherie and P. A. Meyer: *Probabilities and Potential B: Theory of Martingales*. North-Holland, Amsterdam/New York/Oxford, 1982.

[3] N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland/Kodanska, Amsterdam/Tokyo, 1989.

[4] J. Jacod and A.N. Shiryaev: *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, 1987.

[5] A. Matoussi, W. Sabbagh and C. Zhou: The obstacle problem for semilinear parabolic partial integro-differential equations, arXiv:1307.0875v3 [math.PR]

[6] K. R. Parthasarathy: *Probability measures on metric spaces*. AMS Chelsea Publishing, 2005.

[7] P. E. Protter and K. Shimbo: No arbitrage and general semimartingales. http://legacy.orie.cornell.edu/protter/WebPapers/nagirsanov8.pdf.

[8] H. Qiao: Exponential ergodicity for SDEs with jumps and non-Lipschitz coefficients, *J. Theor. Probab.*, 27 (2014) 137-152.

[9] H. Qiao and J. Duan: Nonlinear filtering of stochastic dynamical systems with Lévy noises, *Advances in Applied Probability,* 47 (2015) 902-918.

[10] H. Qiao and X. Zhang: Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps, *Stochastic Processes and their Applications,* 118 (2008) 2254-2268.

[11] R. Situ: *Theory of Stochastic Differential Equations with Jumps and Applications*. Series: Mathematical and Analytical Techniques with Applications to Engineering. Springer-Verlag, Berlin, Heidelberg, NewYork, 2005.

[12] A. Truman, F. -Y. Wang, J.-L. Wu, W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics*, 55 (2012) 1971-1976.

[13] J.-L. Wu and W. Yang: On stochastic differential equations and a generalised Burgers equation, pp 425-435 in *Stochastic Analysis and Its Applications to Finance: Essays in Honor of Prof. Jia-An Yan (eds. T S Zhang, X Y Zhou)*, Interdisciplinary Mathematical Sciences, Vol. 13, World Scientific, Singapore, 2012.