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# Renormalized Entropy Solutions of Stochastic Scalar Conservation Laws with Boundary Condition

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**Abstract**

This paper is concerned with the renormalized stochastic entropy solutions of stochastic scalar conservation law forced by a multiplicative noise on a bounded domain with a non-homogeneous boundary condition. We first introduce a notion of renormalized stochastic entropy solution and then establish the existence and uniqueness of a renormalized stochastic entropy solutions for a general $L^1$-data. Our results allow us to give a positive answer to an open problem posed by Bauzet, Vallet and Wittbold in [4].

**Keywords**: Scalar conservation law; Renormalized entropy solutions; Itô's formula; $L^1$-theory.

**AMS subject classifications** (2010): 60H15, 60H40.

## 1 Introduction

Let $D$ be a bounded open set in $\mathbb{R}^N$ with boundary $\partial D$ in which we assume the boundary $\partial D$ is Lipschitz in case the space dimension $N > 1$. Let $T > 0$ be arbitrarily fixed. Set $Q = (0, T) \times D$ and $\Sigma = (0, T) \times \partial D$. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ be a given probability set-up. In this paper, we are interested in the first order stochastic conservation laws driven by a multiplicative noise of the following type

$$du - div(f(u))dt = h(u)dw(t), \quad \text{in } \Omega \times Q, \tag{1.1}$$

with initial condition

$$u(0, \cdot) = u_0(\cdot), \quad \text{in } D, \tag{1.2}$$

and boundary condition

$$u = a, \quad \text{on } \Sigma, \tag{1.3}$$

for a random scalar-valued function $u : (\omega, t, x) \in \Omega \times [0, T] \times D \mapsto u(\omega, t, x) =: u(t, x) \in \mathbb{R}$, where $f = (f_1, ..., f_N) : \mathbb{R} \to \mathbb{R}^N$ is a differentiable vector field standing for the flux, $h : \mathbb{R} \to \mathbb{R}$ is measurable and $w = \{w(t)\}_{0\le t\le T}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$.

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The initial data $u_0 : D \subset \mathbb{R}^N \to \mathbb{R}$ will be specified later and the boundary data $a : \Sigma \to \mathbb{R}$ is supposed to be measurable.

Problem (1.1)-(1.3) was studied recently by Kobayasi-Noboriguchi [16] and Lv-Wu [21] via kinetic solution approach and Kruzhkov's semi-entropy method, respectively. By introducing a notion of kinetic formulations in which the kinetic defect measures on the boundary of domain are turncated, Kobayasi-Noboriguchi [16] obtained the well-posedness of (1.1)-(1.3). Motivated by the deterministic case [1, 24], Lv-Wu [21] (see also [22]) introduced a notion of stochastic entropy solutions of (1.1)-(1.3) and obtained the existence and uniqueness of stochastic entropy solutions by utilising vanishing viscosity method and Kruzhkov's technique of doubling variables.

When $h = 0$, the deterministic problem (1.1)-(1.3) is well studied by many authors in the literature, see for example [1, 24] and references therein. The authors of [24] studied the problem (1.1)-(1.3) with $h = 0$ in the $L^1$-setting. In order to deal with unbounded solutions, they have defined a notion of renormalized entropy solution which generalizes the definition of entropy solutions introduced by Otto in [23] in the $L^\infty$ frame work. They have proved existence and uniqueness of such generalized solution in the case when $f$ is locally Lipschitz and the boundary data $a$ verifies the following condition: $f_{max}(a) \in L^1(\Sigma)$, where $f_{max}$ is the "maximal effective flux" defined by

$$f_{max}(a) = \{\sup |f(u)|, \quad u \in [-a^-, a^+]\}.$$

They gave an example to illustrate that the assumption $a \in L^1(\Sigma)$ is not enough in order to prove a priori estimates in $L^1(Q)$, and that the assumption should be $f_{max}(a) \in L^1(\Sigma)$. Furthermore, in [1], the authors revisited the problem (1.1)-(1.3) and introduced a notion of entropy solution to the problem (1.1)-(1.3) with $h = 0$. Following [1], an entropy solution of (1.1)-(1.3) is a function $u \in L^\infty(Q)$ satisfying

$$\begin{aligned} -\int_\Sigma \xi\omega^+(x,k,a(t,x))dSdt \quad &\le \int_Q \left[(u-k)^+\xi_t - \chi_{u>k}(f(u)-f(k))\cdot\nabla\xi\right]dxdt \\ &\quad + \int_D (u_0-k)^+\xi(0,\cdot)dx \quad \text{and} \qquad (1.4) \\ -\int_\Sigma \xi\omega^-(x,k,a(t,x))dSdt \quad &\le \int_Q \left[(k-u)^+\xi_t - \chi_{k>u}(f(k)-f(u))\cdot\nabla\xi\right]dxdt \\ &\quad + \int_D (k-u_0)^+\xi(0,\cdot)dx \qquad (1.5) \end{aligned}$$

for any $\xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, $\xi \ge 0$ and for all $k \in \mathbb{R}$, where

$$\begin{aligned} \omega^+(x,k,a) &:= \max_{k\le r,s\le a\vee k} |(f(r)-f(s))\cdot\vec{n}(x)| \\ \omega^-(x,k,a) &:= \max_{a\wedge k\le r,s\le k} |(f(r)-f(s))\cdot\vec{n}(x)| \end{aligned}$$

for any $k \in \mathbb{R}$, a.e. $x \in \partial D$, and $\vec{n}$ denoting the unit outer normal vector to $\partial D$. Here and in the sequel, $a \wedge k := \min\{a,k\}$ and $a \vee k := \max\{a,k\}$. It is remarked that the above definition of entropy solution is a natural extension of the definition of that given by Otto [23]. On the other hand, Carrillo-Wittbold [5] obtained the existence and uniqueness of renormalized entropy solutions of (1.1)-(1.3) with $h = 0 = a$.

Having a stochastic forcing term $h(u)dw(t)$ in Equation (1.1) is very natural for problem modeling arising in a wide variety of fields in physics, engineering, biology, just mention a few. The Cauchy problem of equation (1.1) with additive noise has been studied in [15] wherein Kim proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution. Moreover, a Kruzhkov-type method was used there to prove the uniqueness. Furthermore, in [25], Vallet and Wittbold extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. By utilising the vanishing viscosity method, Young measure techniques and Kruzhkov doubling variables technique, they managed to show the existence and uniqueness of the stochastic entropy solution.

On the other other, concerning the case of multiplicative noise, for Cauchy problem over the whole spatial space, Feng and Nualart in [11] introduced a notion of strong entropy solution in order to prove the uniqueness for the entropy solution. Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solution only in one space dimensional case. We would like to mention [2] where Bardos-le Roux-Nédélec firstly proved the well-posedness of the initial-boundary value problem for multidimensional scalar conservation laws. Moreover, Chen et al. [7] considered high space dimensional problem and they proved that the multi-dimensional stochastic problem is well-posedness by using a uniform spatial BV-bound. Following the idea of [11, 7], Lv et al. [19] considered the Cauchy problem of stochastic nonlocal conservation law. Bauzet et al.[3] proved a result of existence and uniqueness of the weak measure-valued entropy solution to the multi-dimensional Cauchy problem.

Using a kinetic formulation, Debussche and Vovelle [9] obtained a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus, (also see [16, 14]).

More recently, Bauzet et al. [4] studied the problem (1.1)-(1.3) with $a = 0$ (i.e., the homogeneous boundary condition). Under the assumptions that the flux function $f$ and $h$ satisfy the global Lipschitz condition, they obtained the existence and uniqueness of measure-valued solution to problem (1.1)-(1.3) with $a = 0$ in the $L^2$-setting. Lv et al. [20] extended the result of [4] to the stochastic nonlocal conservation law. Meanwhile, Bauzet et al. [4] posed ***an open problem***: whether there exists a renormalized stochastic entropy solution to problem (1.1)-(1.3) with $a = 0$. In the present paper, we aim to study this open problem and we end up with an affirmative answer.

Our object In this paper is the well posedness of renormalized stochastic entropy solutions of problem (1.1)-(1.3). Encouraged and inspired by the deterministic case, we first give a notion of renormalized stochastic entropy solution, and we then discuss the relation between the stochastic entropy solution with renormalized stochastic entropy solution. In the end, the existence and uniqueness of renormalized stochastic entropy solutions are established. We would like to point out that there are two big difficulties arisen here: one is how to get the limit of stochastic term in $L^1$-setting, and the other is how to deal with the stochastic term in proving the uniqueness. The solution to the former difficulty is that one can use the Itô isometry and the relevant convergence in probability. The method used to solve the second difficulty is the Fubini's Theorem and the technique of doubling variables, which is stimulated by [4]. There are probably three methods to deal with the stochastic term in proving the uniqueness so far. The first method is defining the stochastic strong entropy solution [11], which is used to control the noise-noise interaction. The second method is to use the regularity of viscous solution [4], which is only suitable to one dimensional Brownian motion. The third method is to use the kinetic formulation [9], which is suitable to cylindrical Brownian motion. Here we use a similar method to [4], but there is a significant difference. Noting that in paper [4], the authors established a comparison result for two solutions, one is stochastic entropy solution and the other is viscous solution. However, in this paper, we will establish a comparison result for two solutions (see Lemma 4.1), one is stochastic entropy solution and the other is renormalized stochastic entropy solution. Hence both solutions have little regularity. Fortunately, one should have a method to overcome it if one can clearly know how to get the stochastic entropy solution.

The paper is organized as follows. In Section 2, we introduce the notion of renormalized stochastic entropy solution for (1.1)-(1.3), then discuss the relationship between the stochastic entropy solution with renormalized stochastic entropy solution and lastly state out the main results. Section 3 is devoted to the proof of existence of renormalized stochastic entropy solution for (1.1)-(1.3). In Section 4, uniqueness of renormalized stochastic entropy solution for (1.1)-(1.3) is established by using Fubini's Theorem and the technique of doubling variables.

We end up this section by introducing some notations.

**Notations.** In general, if $G \subset \mathbb{R}^N$, $\mathcal{D}(G)$ denotes the restriction of functions $u \in \mathcal{D}(\mathbb{R}^N)$ to $G$ such that $support(u) \cap G$ is compact. The notation $\mathcal{D}^+(G)$ stands for the subset of non-negative elements of $\mathcal{D}(G)$. $\mathcal{M}(Q)$ denotes the space of functions measurable on $Q$.

For a given separable Banach space $X$, we denote by $N_w^2(0,T,X)$ the space of the predictable $X$-valued processes. This space is the space $L^2((0,T)\times\Omega, X)$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable $\sigma$-field (i.e. the $\sigma$-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $(s,t)\times A$ for any $A \in \mathcal{F}_s$, for $t > s > 0$).

Denote $\mathcal{E}^+$ the totality of non-negative convex functions $\eta$ in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \to x^+$ such that $\eta(x) = 0$ if $x \le 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then $\eta''$ has a compact support and $\eta$ and $\eta'$ are Lipschitz-continuous functions. $\mathcal{E}^-$ denotes the set $\{\breve{\eta} := \eta(-\cdot), \eta \in \mathcal{E}^+\}$ and $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$. Then, for convenience, denote

$$sgn_0^+(x) = 1 \text{ if } x > 0 \text{ and } 0 \text{ else}; \quad sgn_0^-(x) = -sgn_0^+(-x) \quad sgn_0 = sgn_0^+ + sgn_0^-,$$
$$F(a,b) = sgn_0(a-b)[f(a)-f(b)]; \quad F^{+(-)}(a,b) = sgn_0^{+(-)}(a-b)[f(a)-f(b)],$$
$$\text{and for any } \eta \in \mathcal{E}, \quad F^\eta(a,b) = \int_b^a \eta'(\sigma - b) f'(\sigma) d\sigma.$$

## 2 Entropy solution

The aim of this section is to give a definition of renormalized stochastic entropy solution. We then discuss the relationship between the stochastic entropy solution with renormalized stochastic entropy solution and present our main results. To this end, we first recall the definition of stochastic entropy solution.

In paper [21], the authors gave the following definition of stochastic entropy solution of (1.1)-(1.3). For convenience, for any function $u$ of $N_w^2(0,T;L^2(D))$, any real number $k$ and any regular function $\eta \in \mathcal{E}^+$, denote dP-a.s. in $\Omega$ by $\mu_{\eta,k}$, the distribution in $D$ defined by

$$\begin{aligned}
\varphi \mapsto \mu_{\eta,k}(\varphi) &= \int_D \eta(u_0 - k)\varphi(0)dx + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi dxdt \\
&\quad + \int_Q \eta'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\varphi dxdt \\
&\quad + \int_\Sigma \eta'(a-k)\varphi\omega^+(x,k,a(t,x))dSdt; \\
\varphi \mapsto \mu_{\breve{\eta},k}(\varphi) &= \int_D \breve{\eta}(u_0 - k)\varphi(0)dx + \int_Q \breve{\eta}(u-k)\partial_t\varphi - F^{\breve{\eta}}(u,k)\nabla\varphi dxdt \\
&\quad + \int_Q \breve{\eta}'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \breve{\eta}''(u-k)h^2(u)\varphi dxdt \\
&\quad + \int_\Sigma \breve{\eta}'(a-k)\varphi\omega^-(x,k,a(t,x))dSdt,
\end{aligned}$$

where $\omega^+(x,k,a(t,x))$ and $\omega^-(x,k,a(t,x))$ are defined as in the introduction. Based on this, we have the following

**Definition 2.1** *A function $u$ of $N_w^2(0,T;L^2(D))$ is an entropy solution of stochastic conservation law (1.1 ) with the initial condition $u_0 \in L^p(D)$ and boundary condition $a \in L^\infty(\Sigma)$, if $u \in L^2(0,T;L^2(\Omega;L^p(D)))$, $p = 2, 3, \cdots$ and*

$$\mu_{\eta,k}(\varphi) \geq 0, \qquad \mu_{\breve{\eta},k}(\varphi) \geq 0 \quad dP - a.s.,$$

*where $\varphi \in \mathcal{D}^+((0,T \times \mathbb{R}^N))$, $k \in \mathbb{R}$, $\eta \in \mathcal{E}^+$ and $\breve{\eta} \in \mathcal{E}^-$.*

We remark that for technical reasons, Bauzet et al. [4] gave a generalized notion of entropy solution. And then the uniqueness result implies the existence of entropy solution in sense of Definition 2.1. In fact, one can directly the existence of entropy solution in sense of Definition 2.1, for more details see [22]. Under the following assumptions

$(H_1)$: The flux function $f : \mathbb{R} \mapsto \mathbb{R}^N$ is of class $C^2$, its derivatives have at most polynomial growth, $f(0) = 0_{\mathbb{R}^N}$;

$(H_2)$: $h : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$;

$(H_3)$: $u_0 \in L^p(D)$, $p \geq 2$ and $a \in L^\infty(\Sigma)$,

Lv-Wu [21] obtained the existence and uniqueness of stochastic entropy solutions of (1.1)-(1.3) in sense of Definition 2.1.

In our setting, for a continuos flux function $f : \mathbb{R} \mapsto \mathbb{R}^N$ and for any measurable boundary data $a : \Sigma \mapsto \mathbb{R}$ with $\bar{f}(a,x) \in L^1(\Sigma)$ where $\bar{f} : \mathbb{R} \times \partial D \mapsto \mathbb{R}$ is defined by $\bar{f}(s,x) := \sup\{|f(r) \cdot \vec{n}(x)|, r \in [-s^-, s^+]\}$. Now we give the definition of renormalized stochastic entropy solution.

**Definition 2.2** *Let $a \in \mathcal{M}(\Sigma)$ with $\bar{f}(a,x) \in L^1(\Sigma)$ and $u_0 \in L^1(D)$. A function $u \in L^1(\Omega; L^1(Q)$ is said to be a renormalized stochastic entropy solution of the conservation law (1.1)-(1.3) if there exist some families of non-negative random measures $\mu_l := \mu_l(\omega;t,x)$ and $\nu_l := \nu_l(\omega;t,x)$ on $[0,T] \times \bar{D}$ such that*

$$\mathbb{E}\mu_l(\cdot;[0,T]\times\bar{D}) \to 0,\ \mathbb{E}\nu_{-l}(\cdot;[0,T]\times\bar{D}) \to 0,\ \text{as } l \to +\infty,$$

*and the following entropy inequalities hold: for all $k \in \mathbb{R}$, for all $l \geq k$, for any $\xi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$,*

$$\begin{aligned}
&\int_Q (u \wedge l - k)^+ \xi_t - \int_Q sgn_0^+(u \wedge l - k)[f(u \wedge l) - f(k)] \cdot \nabla\xi \\
&+ \int_Q sgn_0^+(u \wedge l - k) h(u \wedge l) \xi dx dw(t) + \frac{1}{2}\int_Q [1 - sgn_0^+(k - u \wedge l)] h^2(k)\xi \\
&+ \int_D (u_0 \wedge l - k)^+ \xi + \int_\Sigma sgn_0^+(a \wedge l - k)\xi\omega^+(x,k,a\wedge l) \\
\geq\ & -\langle \mu_l, \xi\rangle, \quad dP - a.s.,
\end{aligned}$$

*and for all $k \in \mathbb{R}$, for all $l \leq k$, for any $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$,*

$$\begin{aligned}
&\int_Q (k - u \vee l)^+ \xi_t - \int_Q sgn_0^+(k - u \vee l)[f(k) - f(u \vee l)] \cdot \nabla\xi \\
&+ \int_Q sgn_0^+(k - u \vee l) h(u \vee l) \xi dx dw(t) + \frac{1}{2}\int_Q [1 - sgn_0^+(u \vee l - k)] h^2(k)\xi \\
&+ \int_D (k - u_0 \wedge l)^+ \xi + \int_\Sigma sgn_0^+(k - a \vee l)\xi\omega^-(x,k,a\vee l) \\
\geq\ & -\langle \nu_l, \xi\rangle, \quad dP - a.s..
\end{aligned}$$

It is easy to see that the Definition 2.2 follows from Definition 2.1. In fact, by using the facts $\lim_{\delta\to 0}\eta_\delta(x) = x^+$, $\lim_{\delta\to 0}\eta'_\delta(x) = sgn_0^+(x)$ and $\lim_{\delta\to 0}\eta''_\delta(x-k) = \delta_x(k)$ ($\delta_x(k)$ denotes the Dirac delta function), we have

$$\begin{aligned}\lim_{\delta\to 0}\mu_{\eta_\delta,k}(\xi) &= \int_Q (u-k)^+\xi_t - \int_Q sgn_0^+(u-k)[f(u)-f(k)]\cdot\nabla\xi \\ &\quad + \int_Q sgn_0^+(u-k)h(u)\xi dxdw(t) + \frac{1}{2}\int_Q [1-sgn_0^+(k-u)]h^2(k)\xi \\ &\quad + \int_D (u_0-k)^+\xi + \int_\Sigma sgn_0^+(a-k)\xi\omega^+(x,k,a) \\ &=: -\tilde{\mu}_k(\xi).\end{aligned}$$

It follows from the Definition 2.1 that $\tilde{\mu}_k(\xi)\le 0$ almost surely. In addition, as in [5], we can also define like this.

**Definition 2.3** *Let $a\in\mathcal{M}(\Sigma)$ with $\bar{f}(a,x)\in L^1(\Sigma)$ and $u_0\in L^1(D)$. A function $u$ of $L^1(\Omega;L^1(Q)$ is said to be a renormalized stochastic entropy solution of conservation law (1.1)-(1.3) if for all $k,l\in\mathbb{R}$, for any $\xi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$, the functionals*

$$\begin{aligned}\mu_{k,l}(\xi) &= -\int_Q (u\wedge l-k)^+\xi_t + \int_Q sgn_0^+(u\wedge l-k)[f(u\wedge l)-f(k)]\cdot\nabla\xi \\ &\quad - \int_Q sgn_0^+(u\wedge l-k)h(u\wedge l)\xi dxdw(t) - \frac{1}{2}\int_Q [1-sgn_0^+(k-u\wedge l)]h^2(k)\xi \\ &\quad - \int_D (u_0\wedge l-k)^+\xi - \int_\Sigma sgn_0^+(a\wedge l-k)\xi\omega^+(x,k,a\wedge l)\quad dP-a.s., \\ \nu_{k,l}(\xi) &= -\int_Q (k-u\vee l)^+\xi_t + \int_Q sgn_0^+(k-u\vee l)[f(k)-f(u\vee l)]\cdot\nabla\xi \\ &\quad - \int_Q sgn_0^+(k-u\vee l)h(u\vee l)\xi dxdw(t) - \frac{1}{2}\int_Q [1-sgn_0^+(u\vee l-k)]h^2(k)\xi \\ &\quad - \int_D (k-u_0\wedge l)^+\xi - \int_\Sigma sgn_0^+(k-a\vee l)\xi\omega^-(x,k,a\vee l)\quad dP-a.s.\end{aligned}$$

*are random measure on $[0,T]\times\bar{D}$ satisfying*

$$\lim_{l\to+\infty}\mathbb{E}\mu_{k,l}^+(\cdot;[0,T]\times\bar{D}) = 0 \quad\text{and}\quad \lim_{l\to-\infty}\mathbb{E}\nu_{k,l}^+(\cdot;[0,T]\times\bar{D}) = 0\ \forall k\in\mathbb{R},$$

*where $\mu_{k,l}^+$ denotes the positive part of the random measure $\mu_{k,l}$.*

It is not difficult to prove that Definition 2.2 is equivalent to Definition 2.3 by using the following decomposition

$$\begin{aligned}\mu_{k,l}(\xi) &= \tilde{\mu}_k(\xi)-\tilde{\mu}_l(\xi) - \int_Q sgn_0^+(u-l)h(l)\xi dxdw(t) - \frac{1}{2}\int_Q [1-sgn_0^+(l-u)]h^2(l)dxdt \\ &\quad - \int_\Sigma [\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)]\xi,\quad dP-a.s.,\end{aligned}\tag{2.1}$$

where we used the facts that for $l>k$, $(u\wedge l-k)^+ = (u-k)^+-(u-l)^+$ and $sgn_0^+(u\wedge l-k)[f(u\wedge l)-f(k)] = sgn_0^+(u-k)[f(u)-f(k)] - sgn_0^+(u-l)[f(u)-f(l)]$. In other words, $\mu_l$ in Definition 2.2 is $\mu_{k,l}^+$ of Definition 2.3.

Next, we consider the equivalence between the renormalized stochastic entropy solutions and stochastic entropy solutions.

**Proposition 2.1** *If $u$ is a stochastic entropy solution in sense of Definition 2.1, then $u$ is a renormalized stochastic entropy solution in Definition 2.2.*

**Proof**. Let $u$ be the stochastic entropy solution in sense of Definition 2.1. Notice that

$$\begin{aligned} & \omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a) \\ = & \left(\max_{k\le r,s\le a\wedge l}-\max_{k\le r,s\le a}+\max_{l\le r,s\le a}\right)|(f(r)-f(s))\cdot\vec{n}(x)| \\ = & \begin{cases} 0, & \text{if } a\wedge l=a, \\ (\max_{k\le r,s\le l}-\max_{k\le r,s\le a}+\max_{l\le r,s\le a})\ge 0, & \text{if } a\wedge l=l. \end{cases} \end{aligned} \tag{2.2}$$

and the decomposition (2.1) yield

$$\begin{aligned} \mu_{k,l}(\xi) &= \tilde{\mu}_k(\xi)-\tilde{\mu}_l(\xi)-\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t)-\frac{1}{2}\int_Q[1-sgn_0^+(l-u)]h^2(l)dxdt \\ &\quad -\int_\Sigma[\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)]\xi \\ &\le -\tilde{\mu}_l(\xi)-\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t), \quad dP-a.s., \end{aligned}$$

where we used the fact that $\tilde{\mu}_k(\xi)\le 0$ almost surely, which is obtained from the definition of stochastic entropy solution.

Set

$$\langle\mu_l,\xi\rangle := -\tilde{\mu}_l(\xi)-\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t).$$

It remains to prove that $\mathbb{E}\mu_l(\omega;[0,T]\times\bar{D})\to 0$, as $l\to+\infty$. For any $l\ge 0$, for all $\sigma\in\mathcal{D}^+([0,T))$, we get

$$\begin{aligned} 0 &\le -\mathbb{E}\int_Q\sigma(t)d\tilde{\mu}_l \\ &= \mathbb{E}\int_Q(u-l)^+\sigma_t+\mathbb{E}\int_Q sgn_0^+(u-l)h(u)\sigma(t)dxdw(t) \\ &\quad +\mathbb{E}\int_D(u_0-l)^+\sigma(0)dx+\frac{1}{2}\mathbb{E}\int_Q[1-sgn_0^+(l-u)]h^2(l)dxdt \\ &\quad +\mathbb{E}\int_\Sigma\sigma(t)\omega^+(x,l,a) \\ &\to 0, \quad \text{as } l\to\infty. \end{aligned}$$

Therefore, $\mathbb{E}\mu_l(\omega;[0,T]\times\bar{D})\to 0$, as $l\to+\infty$. In a similar way, we one can prove the corresponding properties of $\nu_l$ and thus $u$ is a renormalized entropy solution of (1.1)-(1.3). The proof is complete. □

**Remark 2.1** *In the Definitions 2.2 and 2.3, we consider the convergence of $\mathbb{E}\mu_l(\omega;[0,T]\times\bar{D})$. The reason is as followings: from the proof of Proposition 2.1, one can prove that by using Itô isometry ($u$ is a stochastic entropy solution),*

$$\begin{aligned} \mathbb{E}\left[\int_Q sgn_0^+(u-l)h(u)dxdw(t)\right]^2 &\le \int_0^T\mathbb{E}\int_D\left[sgn_0^+(u-l)\right]^2h^2(u)dxdt \\ &\to 0, \quad \text{as } l\to\infty, \end{aligned}$$

*which implies that there exists a subsequence* $\{l_n\}_n$ *such that*

$$\int_Q sgn_0^+(u-l_n)h(u)dxdw(t) \to 0, \quad \text{as } n\to\infty, \quad dP-a.s..$$

*Therefore, we can not assume in the Definition 2.2 that*

$$\mu_l(\Omega;t,x)\to 0,\ \nu_{-l}(\Omega;t,x)\to 0,\ \text{as } l\to+\infty, \quad dP-a.s.,$$

**Remark 2.2** *Let* $u$ *be a stochastic renormalized entropy solution and* $f(u)\in L^1(Q)^N$. *Then it is not hard to get that* $\mu_{\eta,k}(\varphi)\geq 0$ *almost surely for* $\eta\in\mathcal{O}\subset\mathcal{E}^+$. *More precisely, we have the decomposition*

$$\begin{aligned}\mu_{k,l}(\xi) &= \tilde{\mu}_k(\xi)-\tilde{\mu}_l(\xi)-\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t)-\frac{1}{2}\int_Q[1-sgn_0^+(l-u)]h^2(l)dxdt\\ &\quad -\int_\Sigma[\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)]\xi, \quad dP-a.s..\end{aligned}$$

*As a consequence, for any* $\xi\in\mathcal{D}^+([0,T)\times\mathbb{R}^N)$

$$\begin{aligned}\tilde{\mu}_k(\xi) &\leq \langle\mu_l,\xi\rangle+\tilde{\mu}_l(\xi)+\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t)+\frac{1}{2}\int_Q[1-sgn_0^+(l-u)]h^2(l)dxdt\\ &\quad +\int_\Sigma[\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)]\xi, \quad dP-a.s..\end{aligned}$$

*But the Definition 2.2 shows that* $\mathbb{E}\langle\mu_l,\xi\rangle\to 0$, *as* $l\to+\infty$. *We take* $\{l_n\}_n$ *such that* $l_n\to\infty$ *as* $n\to\infty$, *and thus we have* $\mathbb{E}\langle\mu_{l_n},\xi\rangle\to 0$, *as* $n\to+\infty$, *which implies that there exists a subsequence (still denoted* $\{l_n\}_n$*) such that* $\langle\mu_{l_n},\xi\rangle\to 0$, *as* $n\to+\infty$ *almost surely. By using (2.2), it is easy to see that when* $l>\|a\|_{L^\infty}$,

$$\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)=0.$$

*Similarly, one can prove that there exists a subsequence (still denoted* $\{l_n\}_n$*) such that* $\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t)\to 0$ *as* $n\to\infty$ *almost surely. Now, we prove* $\tilde{\mu}_{l_n}(\xi)\to 0$ *as* $n\to\infty$ *almost surely. Notice that*

$$\begin{aligned}\tilde{\mu}_l(\xi) &= -\int_Q(u-l)^+\xi_t+\int_Q sgn_0^+(u-l)f(u)\cdot\nabla\xi-\int_D(u_0-l)^+\xi\\ &\quad -\int_Q sgn_0^+(u-l)f(l)\cdot\nabla\xi-\int_Q sgn_0^+(u-l)h(u)\xi dxdw(t)\\ &\quad +\frac{1}{2}\int_Q[1-sgn_0^+(k-u)]h^2(k)\xi-\int_\Sigma sgn_0^+(a-l)\xi\omega^+(x,l,a).\end{aligned}$$

*As* $u\in L^1(Q)$, $f(u)\in L^1(Q)^N$, $u_0\in L^1(D)$ *and* $|h(u)|\leq L|u|$ *(*$L$ *is the Lipschitz constant), the first three integrals on the right-hand side tend to 0 as* $l\to\infty$ *almost surely. When* $l>\|a\|_{L^\infty}$, $\omega^+(x,l,a)=0$ *and thus the last integral also tends to 0 as* $l\to\infty$ *almost surely. Moreover, either* $\{|f(l_n)|\}_n$ *is bounded for some sequence* $l_n\to\infty$, *and then* $\lim_{n\to\infty}\int_{u>l_n}|f(l_n)||\nabla\xi|dxdt=0$ *or* $\lim_{l\to\infty}|f(l)|=+\infty$. *In this case, there exists a sequence* $\{l_n\}_n$ *such that, for any* $n\in N$, $|f(l_n)|=\min_{l\in[n,\infty)}|f(l)|$. *For this choice of the sequence* $l_n$, *we have*

$$\int_{\{u>l_n\}}|f(l_n)||\nabla\xi|dxdt\leq\int_{\{u>l_n\}}|f(u)||\nabla\xi|dxdt\to 0, \quad \text{as } n\to\infty, \quad dP-a.s.$$

*as* $f(u) \in L^1(Q)^N$. *Therefore,* $\tilde{\mu}_k(\xi) \le 0$ *for all* $(k,\xi) \in \mathbb{R} \times \mathcal{D}^+([0,T) \times \mathbb{R}^N)$ *almost surely. That is*

$$P\left(\omega \in \Omega; \tilde{\mu}_k(\omega;\xi) \le 0\right) = 1, \quad \forall (k,\xi) \in \mathbb{R} \times \mathcal{D}^+([0,T) \times \mathbb{R}^N).$$

*Since* $\lim_{\delta\to 0} \mu_{\eta,k}(\xi) = -\tilde{\mu}_k(\xi)$, *using the properties of limit, there exists a family* $\eta \in \mathcal{O} \subset \mathcal{E}^+$ *such that*

$$P\left(\omega \in \Omega; \mu_{\eta,k}(\omega;\xi) \ge 0\right) = 1, \quad \forall (k,\xi) \in \mathbb{R} \times \mathcal{D}^+([0,T) \times \mathbb{R}^N).$$

The main result of this paper is:

**Theorem 2.1** *Let* $a \in \mathcal{M}(\Sigma)$ *with* $\bar{f}(a,x) \in L^1(\Sigma)$ *and* $u_0 \in L^1(D)$. *Under assumptions* $H_1 - H_2$ *there exists a unique renormalized stochastic entropy solution in sense of Definition 2.2.*

In order to obtain the uniqueness of the renormalized stochastic entropy solution in sense of Definition 2.2, we need the following proposition.

**Proposition 2.2** *([21]) Under assumptions* $H_1 - H_3$ *there exists a unique stochastic entropy solution in sense of Definition 2.1.*

*Moreover, if* $u_1, u_2$ *are entropy solutions of (1.1) corresponding to initial data* $u_{01}$, $u_{02} \in L^p(D)$ *and the boundary data* $a_1$, $a_2 \in L^\infty(\Sigma)$, *respectively, then for any* $t \in (0,T)$

$$\mathbb{E}\int_D |u_1 - u_2| \le \int_D |u_{01} - u_{02}|dx + \int_\Sigma \max_{min(a_1,a_2)\le r,s\le max(a_1,a_2)} |(f(r) - f(s)) \cdot \vec{n}(x)|.$$

## 3 Existence

In this section, we prove the first part of Theorem 2.1. That is the following Theorem.

**Theorem 3.1** *Let* $a \in \mathcal{M}(\Sigma)$ *with* $\bar{f}(a,x) \in L^1(\Sigma)$ *and* $u_0 \in L^1(D)$. *Under assumptions* $H_1 - H_2$ *there exists a renormalized stochastic entropy solution in sense of Definition 2.2.*

**Proof**. Let $a_n = (a \wedge n) \vee (-n)$, $u_0^n = (u_0 \wedge n) \vee (-n)$, and $u_n$ be the entropy solution of (1.1)-(1.3) with $(u_0^n, a_n)$. Then by Proposition 2.2, we have

$$\begin{aligned}\mathbb{E}\int_D |u_n - u_m| \ &\le \ \int_\Sigma \max_{min(a_n,a_m)\le r,s\le max(a_n,a_m)} |(f(r) - f(s)) \cdot \vec{n}(x)| \\ &\quad + \int_D |u_0^n - u_0^m|dx, \quad \forall t \in [0,T],\end{aligned}$$

which yields that $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega; C([0,T]; L^1(D)))$ and converges to some function $u$ in $L^1(\Omega; C([0,T]; L^1(D)))$. Moreover, by Proposition 2.1, we have for any $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$, $l \ge k$,

$$\begin{aligned}&\int_Q (u_n \wedge l - k)^+ \xi_t - \int_Q sgn_0^+(u_n \wedge l - k)[f(u_n \wedge l) - f(k)] \cdot \nabla\xi \\ &+ \int_Q sgn_0^+(u_n \wedge l - k)h(u_n \wedge l)\xi dxdw(t) + \frac{1}{2}\int_Q [1 - sgn_0^+(k - u_n \wedge l)]h^2(k)\xi \\ &+ \int_D (u_0^n \wedge l - k)^+\xi + \int_\Sigma sgn_0^+(a_n \wedge l - k)\xi\omega^+(x,k,a_n \wedge l) \\ \ge\ & -\langle \mu_l^n, \xi\rangle, \quad dP - a.s., \qquad (3.1)\end{aligned}$$

and for all $k \in \mathbb{R}$, for all $l \leq k$, for any $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$,

$$\begin{aligned}
& \int_Q (k - u_n \vee l)^+ \xi_t - \int_Q sgn_0^+(k - u_n \vee l)[f(k) - f(u_n \vee l)] \cdot \nabla \xi \\
& + \int_Q sgn_0^+(k - u_n \vee l) h(u_n \vee l) \xi dx dw(t) + \frac{1}{2} \int_Q [1 - sgn_0^+(u_n \vee l - k)] h^2(k) \xi \\
& + \int_D (k - u_0^n \wedge l)^+ \xi + \int_\Sigma sgn_0^+(k - a_n \vee l) \xi \omega^-(x, k, a_n \vee l) \\
\geq \; & -\langle \nu_l^n, \xi \rangle, \quad dP - a.s.. \qquad (3.2)
\end{aligned}$$

It is well-known that convergence in $r$-th order mean implies convergence in probability, where $r \geq 1$. Note that $u_n \to u$ in $L^1(\Omega; C([0,T]; L^1(D)))$ as $n \to \infty$, and thus we have $u_n$ converges in probability towards $u$, that is, for all $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|u_n - u| \geq \varepsilon) = 0,$$

which implies that there exists a subsequence of $\{u_n\}_n$ (still denoted $\{u_n\}_n$) such that $u_n$ converges towards $u$ almost surely. Therefore, we can assume that, $u_n$ and $u$ stay in the same interval. More precisely, $u_n < k < l$ and $u < k < l$, $k < u_n < l$ and $k < u < l$, $k < l < u_n$ and $k < l < u$ hold at the same time. It is easy to see that

$$sgn_0^+(u_n \wedge l - k) h(u_n \wedge l) = \begin{cases} 0, & u_n < k < l, \\ h(u_n), & k < u_n < l, \\ h(l), & k < l < u_n; \end{cases}$$

and

$$sgn_0^+(u \wedge l - k) h(u \wedge l) = \begin{cases} 0, & u < k < l, \\ h(u_n), & k < u < l, \\ h(l), & k < l < u. \end{cases}$$

By using Itô isometry, we have

$$\begin{aligned}
& \mathbb{E} \left[ \int_Q \xi \left( sgn_0^+(u_n \wedge l - k) h(u_n \wedge l) - sgn_0^+(u \wedge l - k) h(u \wedge l) \right) dx dw(t) \right]^2 \\
\leq \; & \int_0^T \mathbb{E} \left[ \int_D \xi \left( sgn_0^+(u_n \wedge l - k) h(u_n \wedge l) - sgn_0^+(u \wedge l - k) h(u \wedge l) \right) dx \right]^2 dt \\
\leq \; & \int_0^T \mathbb{E} \int_{x \in D; k < u_n, u < l} \xi^2 \left( h(u_n) - h(u) \right)^2 dx dt \\
\leq \; & C \int_0^T \mathbb{E} \int_D \xi |u_n - u| dx dt \to 0, \quad \text{as } n \to \infty, \qquad (3.3)
\end{aligned}$$

where $C$ depends on $k, l, \xi$ and the Lipschitz constant of $h$.

From the discussion of section 2, we know that if denote

$$\begin{aligned}
\tilde{\mu}_k^n(\xi) \;=\; & -\int_Q (u_n - k)^+ \xi_t + \int_Q sgn_0^+(u_n - k)[f(u_n) - f(k)] \cdot \nabla \xi \\
& - \int_Q sgn_0^+(u_n - k) h(u_n) \xi dx dw(t) - \frac{1}{2} \int_Q [1 - sgn_0^+(k - u_n)] h^2(k) \xi \\
& - \int_D (u_0^n - k)^+ \xi - \int_\Sigma sgn_0^+(a_n - k) \xi \omega^+(x, k, a_n),
\end{aligned}$$

then we have

$$\begin{aligned}
\mu^n_{k,l(}\xi) &:= \int_Q (u_n\wedge l-k)^+\xi_t - \int_Q sgn_0^+(u_n\wedge l-k)[f(u_n\wedge l)-f(k)]\cdot\nabla\xi \\
&+\int_Q sgn_0^+(u_n\wedge l-k)h(u_n\wedge l)\xi dxdw(t) + \frac{1}{2}\int_Q [1-sgn_0^+(k-u_n\wedge l)]h^2(k)\xi \\
&+\int_D (u_0^n\wedge l-k)^+\xi + \int_\Sigma sgn_0^+(a_n\wedge l-k)\xi\omega^+(x,k,a_n\wedge l) \\
&= -\tilde{\mu}^n_k(\xi)+\tilde{\mu}^n_l(\xi)+\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t)+\frac{1}{2}\int_Q[1-sgn_0^+(l-u)]h^2(l)dxdt \\
&+\int_\Sigma[\omega^+(x,k,a\wedge l)-\omega^+(x,k,a)+\omega^+(x,l,a)]\xi, \quad dP-a.s..
\end{aligned}$$

Due to $\tilde{\mu}^n_k(\xi)\le 0$ almost surely, we have

$$\mu^n_{k,l(}\xi)\ge\tilde{\mu}^n_l(\xi)+\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t),$$

where we have used the similar analysis to the proof of Proposition 2.1. Actually, we can take

$$\langle\mu^n_l,\xi\rangle := -\tilde{\mu}^n_l(\xi)-\int_Q sgn_0^+(u-l)h(l)\xi dxdw(t).$$

Hence, we have

$$\begin{aligned}
\mathbb{E}\mu^n_l(\Omega\times[0,T]\times\bar{D}) &\le \int_D(u_0^n-l)^+ + \int_\Sigma\omega^+(x,l.a_n) \\
&\le \int_D(u_0-l)^+ + \int_\Sigma\omega^+(x,l.a). \qquad (3.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}\nu^n_l(\Omega\times[0,T]\times\bar{D}) &\le \int_D(l-u_0^n)^+ + \int_\Sigma\omega^-(x,l.a_n) \\
&\le \int_D(l-u_0)^+ + \int_\Sigma\omega^-(x,l.a). \qquad (3.5)
\end{aligned}$$

By Lebesgue's theorem of dominated convergences, we can pass to the limit with $n$ in the right-hand side of inequalities (3.1) and (3.2). Moreover, from (3.4) and (3.5) it follows that $\mu^n_l$ and $\nu^n_l$ are bounded independently on $n$. Therefore, there exists a subsequence still denoted $\mu^n_l$ and a random measure $\mu_l$ on $\Omega\times[0,T]\times\bar{D}$ such that $\mu^n_l$ converges to $\mu_l$ with respect to the weak-topology on $L^1(\Omega;C([0,T];L^1(D)))$. Then, passing to the limit in the right-hand side of (3.1), we conclude that $u$ satisfies the renormalized entropy inequality of Definition 2.2. Moreover, since $u_n\to u$ in $L^1(\Omega;C([0,T];L^1(D)))$ as $n\to\infty$, we have thanks to (3.4)

$$\mathbb{E}\mu_l(\Omega\times[0,T]\times\bar{D})\le\liminf_{n\to\infty}\mathbb{E}\mu^n_l(\Omega\times[0,T]\times\bar{D})\le\int_D(u_0-l)^+ + \int_\Sigma\omega^+(x,l.a).$$

Furthermore,

$$\begin{aligned}
\lim_{l\to\infty}\mathbb{E}\mu_l(\Omega\times[0,T]\times\bar{D}) &\le \lim_{l\to\infty}\liminf_{n\to\infty}\mathbb{E}\mu^n_l(\Omega\times[0,T]\times\bar{D}) \\
&\le \lim_{l\to\infty}\Big(\int_D(u_0-l)^+ + \int_\Sigma\omega^+(x,l.a)\Big) \\
&= 0.
\end{aligned}$$

Arguing similarly, we prove that $\nu^n_l$ has the similar properties to $\mu^n_l$. That is, we obtain the existence of renormalized entropy solution to problem (1.1)-(1.3). The proof is complete. □

## 4 Uniqueness

In present section, we complete the proof of Theorem 2.1. As said in Introduction, there are three methods to deal with the stochastic term. The method we used here is similar to that of [4].

In order to use the method of [4], we first consider the following problem

$$\begin{cases} du^\varepsilon - [\varepsilon\Delta u^\varepsilon + div(f(u^\varepsilon))]dt = h(u^\varepsilon)dw(t) & \text{in } Q, \\ u_\varepsilon(0,x) = u_0^\varepsilon(x) & \text{in } D, \\ u^\varepsilon = a^\varepsilon & \text{on } \Sigma, \end{cases} \tag{4.1}$$

where we assume that $a^\varepsilon \in C^\infty(\Sigma)$, $\|a^\varepsilon\|_{C^1} \leq \|a\|_{L^\infty}$ and $a^\varepsilon \to a$ in $L^\infty(\Sigma)$. Moreover, $a^\varepsilon$ is the trace on $\Sigma$ of a function $U \in C([0,T]\times\bar{D})$ such that $\partial_t U \in C^{\gamma,0}([0,T]\times D)$, $\Delta U \in C^{\gamma,0}([0,T]\times D)$, $U(t,\cdot) \in W^{2,p}(D)$ for some $\gamma \in (0,1)$ and for any $p > 1$. Following [21], problem (4.1) admits a unique solution $u^\varepsilon \in L^\infty([0,T], L^p(\Omega\times D)) \cap N_w^2(0,T,H^1(D))$ satisfying

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|_{L^p(D)}^p + \varepsilon\int_0^T\int_D |\nabla u^\varepsilon|^2 dxds \leq C, \tag{4.2}$$

where $p \geq 2$ and $C$ does not depend on $\varepsilon$. By using Young measure theory, we prove $u^\varepsilon$ converges an "entropy process" denoted by $u$ in [21].

The following comparison result plays a crucial role in proof of Theorem 2.1.

**Lemma 4.1** *Let $u_{01} \in L^1(D)$, $a_1 \in \mathcal{M}(\Sigma)$, $\bar{f}(a_1,\cdot) \in L^1(\Sigma)$ and $u_{02} \in L^p(D)$, $a_1 \in L^\infty(\Sigma)$. Let $u_1$ be the stochastic renormalized entropy solution of problem (1.1); $u_2$ be the stochastic entropy solution of (1.1). Then for any $\xi \in \mathcal{D}^+([0,T\times\mathbb{R}^N)$, for any $l > \|a_2\|_{L^\infty(\Sigma)}$,*

$$\begin{aligned} &-\mathbb{E}\langle\mu_l,\xi\rangle - \int_\Sigma \omega^+(x,a_2,a_1\wedge l)\xi \\ \leq\ & \mathbb{E}\int_Q (u_1\wedge l - u_2)^+\xi_t + \int_D (u_{01}\wedge l - u_{02})^+\xi(0) \\ &-\mathbb{E}\int_Q sgn_0^+(u_1\wedge l - u_2)(f(u_1\wedge l) - f(u_2))\cdot\nabla\xi. \end{aligned}$$

**Proof.** As usual we use Kruzhkov's technique of doubling variables [17, 18] in order to prove the comparison result. We choose two pairs of variables $(t,x)$ and $(s,y)$ and consider $u_1$ as a function of $(t,x)\in Q$ and $u_2$ as a function of $(s,y)\in Q$. For any $r>0$, let $\{B_i^r\}_{i=0,\cdots,m_r}$ be a covering of $\bar{D}$ satisfying $B_0^r \cap \partial D = \emptyset$, and such that, for each $i \geq 1$, $B_i^r$ is a ball of diameter $\leq r$, contained in some larger ball $\tilde{B}_i^r$ with $\tilde{B}_i^r \cap \partial D$ is part of the graph of a Lipschitz function. Let $\{\phi_i^r\}_{i=0,\cdots,m_r}$ denote a partition of unity subordinate to the covering $\{B_i^r\}_i$. Let $\varphi \in \mathcal{D}^+((0,T)\times\mathbb{R}^N)$.

Now, let $i \in \{1,\cdots,m_r\}$ be fixed in the following. For simplicity, we omit the dependence on $r$ and $i$ and simply set $\phi = \phi_i^r$ and $B = B_i^r$. We choose a sequence of mollifiers $(\rho_n)_n$ in $\mathbb{R}^N$ such that $x \mapsto \rho_n(x-y) \in \mathcal{D}$ for all $y \in B$. $\sigma_n(x) = \int_D \rho_n(x-y)dy$ is an increasing sequence for all $x \in B$ and $\sigma_n(x) = 1$ for all $x \in B$ with $dist(x,\mathbb{R}^N\setminus D) > \frac{c}{n}$ for some $c = c(i,r)$ depending on $B = B_i^r$. Let $(\varrho_m)_m$ denote a sequence of mollifiers in $\mathbb{R}$ with $supp\varrho_m \subset (-\frac{2}{m},0)$.

Define the test function

$$\zeta_{m,n}(t,x,s,y) = \varphi(s,y)\phi(y)\rho_n(y-x)\varrho_m(t-s)$$

Note that, for $m,n$ sufficiently large

$$\begin{aligned} &(t,x) \mapsto \zeta_{m,n}(t,x,s,y) \in \mathcal{D}((0,T)\times\mathbb{R}^N), \quad \text{for any } (s,y)\in Q, \\ &(s,y) \mapsto \zeta_{m,n}(t,x,s,y) \in \mathcal{D}(Q), \quad \text{for any } (t,x)\in Q. \end{aligned}$$

Let $u_2^\varepsilon(s,y)$ be the solution of (4.1) with initial data $u_{02}^\varepsilon$ and boundary data $a_2^\varepsilon$, and $\eta_\delta \in \mathcal{E}^+$ satisfying $\eta_\delta(\cdot) \mapsto (\cdot)^+$ and $\eta'_\delta(\cdot) \mapsto sgn_0^+(\cdot)$ as $\delta \to 0$. Here we assume that the limit of $u_2^\varepsilon(s,y)$ is the stochastic entropy solution $u_2(s,y)$ of (1.1). Then taking $\varphi = \zeta_{m,n}(t,x,s,y)$ in Definition 2.2, for a. e. $(t,x) \in Q$, we have

$$
\begin{aligned}
& -\langle \mu_l, \zeta_{m,n}\rangle - \int_\Sigma sgn_0^+(a_1 \wedge l - k)\zeta_{m,n}\omega^+(x,k,a_1 \wedge l) \\
\leq\ & \int_Q (u_1 \wedge l - k)^+ (\zeta_{m,n})_t - \int_Q sgn_0^+(u_1 \wedge l - k)[f(u_1 \wedge l) - f(k)] \cdot \nabla_x \zeta_{m,n} \\
& + \int_Q sgn_0^+(u_1 \wedge l - k) h(u_1 \wedge l)\zeta_{m,n} dx dw(t) + \frac{1}{2}\int_Q [1 - sgn_0^+(k - u_1 \wedge l)] h^2(k)\zeta_{m,n} \\
& + \int_D (u_{01} \wedge l - k)^+ \zeta_{m,n}(0,x,s,y), \quad dP - a.s.,
\end{aligned}
$$

In order to keep pace with the Definition of stochastic entropy solution, we need to rewrite the above inequality. By using the facts $\lim_{\delta\to 0} \eta_\delta(x) = x^+$, $\lim_{\delta\to 0} \eta'_\delta(x) = sgn_0^+(x)$ and $\lim_{\delta\to 0} \eta''_\delta(x-k) = \delta_x(k)$ again, we can rewrite the above inequality as

$$
\begin{aligned}
& -\langle \mu_l, \zeta_{m,n}\rangle - \lim_{\delta\to 0}\int_\Sigma \eta'_\delta(a_1 \wedge l - k)\zeta_{m,n}\omega^+(x,k,a_1 \wedge l) \\
\leq\ & \lim_{\delta\to 0}\left\{ \int_Q \eta_\delta(u_1 \wedge l - k)(\zeta_{m,n})_t - \int_Q \eta'_\delta(u_1 \wedge l - k)[f(u_1 \wedge l) - f(k)] \cdot \nabla_x \zeta_{m,n} \right. \\
& + \int_Q \eta'_\delta(u_1 \wedge l - k) h(u_1 \wedge l)\zeta_{m,n} dx dw(t) + \frac{1}{2}\eta''_\delta(u_1 \wedge l - k) h^2(k)\zeta_{m,n} \\
& \left. + \int_D \eta_\delta(u_{01} \wedge l - k)\zeta_{m,n}(0,x,s,y) \right\}, \quad dP - a.s.,
\end{aligned}
$$

Since $u_1$ is a renormalized stochastic entropy solution, it is easy to see that Fibula's theorem can be applied to the above inequality. In other words, the above inequality is bounded uniformly with respect to $\delta$.

Multiplying the above inequality by $\varrho_r(k - u_2^\varepsilon)$ and integrating in $k$ and $(t,x)$ over $\mathbb{R}$ and $Q$, respectively, and taking expectation, we have

$$
\begin{aligned}
0 \quad \leq\ & \lim_{\delta\to 0}\left\{ \mathbb{E}\int_Q \int_{\mathbb{R}} \int_D \eta_\delta(u_{01} \wedge l - k)\zeta_{m,n}(0,x,s,y) dx \varrho_r(k - u_2^\varepsilon) dk dy ds \right. \\
& + \mathbb{E}\int_Q \int_Q \int_{\mathbb{R}} \eta_\delta(u_1 \wedge l - k)\varphi\phi\rho_n \partial_t \varrho_m(t-s)\varrho_r(k - u_2^\varepsilon) dk dx dt dy ds \\
& - \mathbb{E}\int_Q \int_Q \int_{\mathbb{R}} F^{\eta_\delta}(u_1 \wedge l, k)\varphi\phi\varrho_m \cdot \nabla_x \rho_n(y-x)\varrho_r(k - u_2^\varepsilon) dk dx dt dy ds \\
& + \frac{1}{2}\mathbb{E}\int_Q \int_Q \int_{\mathbb{R}} h^2(u_1 \wedge l)\eta''_\delta(u_1 \wedge l - k)\zeta_{m,n}\varrho_r(k - u_2^\varepsilon) dk dx dt dy ds \\
& + \mathbb{E}\int_Q \int_Q \int_{\mathbb{R}} \eta'_\delta(u_1 \wedge l - k) h(u_1 \wedge l)\zeta_{m,n} dx dw(t)\varrho_l(k - u_2^\varepsilon) dk dy ds \\
& + \mathbb{E}\int_Q \int_{\mathbb{R}} \int_\Sigma \eta'_\delta(a_1 \wedge l - k)\zeta_{m,n}\omega^+(x,k,a_1 \wedge l) dS dt \varrho_r(k - u_2^\varepsilon) dk dy ds \\
& \left. + \int_Q \int_{\mathbb{R}} \langle \mu_l, \zeta_{m,n}\rangle \varrho_r(k - u_2^\varepsilon) dk dy ds \right\} \\
:=\ & \lim_{\delta\to 0}\{I_1 + I_2 + \cdots + I_7\}.
\end{aligned}
$$

As $u_2^\varepsilon$ is a viscous solution, the Itô formula applied to $\int_D \eta_\delta(k - u_2^\varepsilon)\zeta_{m,n} dy$ yields that for a.e.

$(t,x)\in Q$

$$
\begin{aligned}
0 \quad \le \quad & \int_D \eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}(t,x,0,y)dy+\int_Q \eta_\delta(k-u_2^\varepsilon)(\zeta_{m,n})_s dyds\\
& -\varepsilon\int_Q \eta_\delta(k-u_2^\varepsilon)\Delta u_2^\varepsilon \zeta_{m,n}dyds-\int_Q F^{\check{\eta}_\delta}(k,u_2^\varepsilon)\cdot\nabla_y\zeta_{m,n}dyds\\
& +\frac{1}{2}\int_Q \eta_\delta''(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds-\int_Q \eta_\delta'(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s),
\end{aligned}
$$

where we used the fact that for any fixed $(t,x)\in Q$, $\zeta_{m,n}(t,x,s,y)\in\mathcal{D}(Q)$. Meanwhile, using (4.2), it is easy to verify the above inequality is bounded uniformly with respect to $\varepsilon$. And thus Fubini's theorem can be applied to the above inequality.

Multiplying the above inequality by $\varrho_r(u_1\wedge l-k)$ and integrating in $k$ over $\mathbb{R}$ and in $(t,x)$ over $Q$, respectively, and taking expectation, we have

$$
\begin{aligned}
0 \quad \le \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D \eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}(t,x,0,y)\varrho_r(u_1\wedge l-k)dkdydxdt\\
& +\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta(k-u_2^\varepsilon)(\partial_s\varphi\varrho_m+\varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_r(u_1\wedge l-k)dkdxdt\\
& -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta'(k-u_2^\varepsilon)\Delta_y u_2^\varepsilon\zeta_{m,n}dyds\varrho_r(u_1\wedge l-k)dkdxdt\\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q F^{\check{\eta}_\delta}(u_2^\varepsilon,k)\cdot\nabla_y\zeta_{m,n}dyds\varrho_r(u_1\wedge l-k)dkdxdt\\
& +\frac{1}{2}\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta''(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds\varrho_r(u_1\wedge l-k)dkdxdt\\
& -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta'(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s)\varrho_r(u_1\wedge l-k)dkdxdt\\
:= \quad & J_1+J_2+\cdots+J_6.
\end{aligned}
$$

Noting that $\varrho_m(t)=0$, $t\in[0,T]$, we have

$$
\begin{aligned}
I_1+J_1 \quad = \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u_1\wedge l-k)\zeta_{m,n}(0,x,s,y)\varrho_r(k-u_2^\varepsilon)d\alpha dkdydxds\\
= \quad & \mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_0^1 \eta_\delta(u_1\wedge l-k)\varphi\phi\rho_n\varrho_m(-s)\varrho_r(k-u_2^\varepsilon)d\alpha dkdydxds.
\end{aligned}
$$

And thus we have

$$
\lim_{m,n,\varepsilon,r,\delta}(I_1+J_1)=\int_D(u_{01}\wedge l-u_{02})^+\varphi(0,x)\phi(x)dx.
$$

Due to $u_1\in L^1(\Omega;L^1Q)$, $u_{01},\,u_{02}\in L^2(D)$ and the compact support of $\zeta_{m,n}$, we know that the convergences in above inequality hold, see [3] for the similar proof.

By using the fact $\partial_t \varrho_m(t-s) + \partial_s \varrho_m(t-s) = 0$ and changing variable technique, we get

$$\begin{aligned}
I_2 + J_2 &= \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}} \eta_\delta(u_1\wedge l - k)\varphi\phi\rho_n\partial_t\varrho_m(t-s)\varrho_r(k-u_2^\varepsilon)dkdxdtdyds \\
&\quad + \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta(k-u_2^\varepsilon)(\partial_s\varphi\varrho_m + \varphi\partial_s\varrho_m)\phi\rho_n dyds\varrho_r(u_1\wedge l-k)dkdxdt \\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta(k-u_2^\varepsilon)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_r(u_1\wedge l-k)dkdxdt \\
&\quad + \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}} \eta_\delta(u_1\wedge l - u_2^\varepsilon-\tau)\varphi\phi\rho_n\partial_t\varrho_m(t-s)\varrho_r(\tau)d\tau dxdtdyds \\
&\quad + \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta(u_1\wedge l - u_2^\varepsilon-\tau)\varphi\phi\rho_n\partial_s\varrho_m(t-s)dyds\varrho_r(\tau)d\tau dxdt \\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q \eta_\delta(k-u_2^\varepsilon)\partial_s\varphi\varrho_m\phi\rho_n dyds\varrho_r(u_1\wedge l-k)dkdxdt.
\end{aligned}$$

Therefore,

$$\lim_{r,m,\delta,\varepsilon}(I_2+J_2) = \mathbb{E}\int_Q (u_1(t,x)\wedge l - u_2(t,x))^+\partial_t\varphi(t,x)\phi(x)dxdt.$$

By using again the fact that for any fixed $(t,x)\in Q$, $\zeta_{m,n}(t,x,s,y)\in\mathcal{D}(Q)$ and Hölder inequality, we obtain

$$\begin{aligned}
J_3 &= -\varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta'(k-u_2^\varepsilon)\Delta_y u_2^\varepsilon\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&= \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \left(\Delta_y\eta_\delta(k-u_2^\varepsilon) - \eta_\delta'(k-u_2^\varepsilon)|\nabla u_2^\varepsilon|^2\right)\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\le \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \Delta_y\eta_\delta(k-u_2^\varepsilon)\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&= \varepsilon\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta(k-u_2^\varepsilon)\Delta_y\zeta_{m,n}dyds\varrho_l(u_1-k)d\alpha dkdxdt \\
&\to_{l,\delta} \varepsilon\mathbb{E}\int_Q\int_Q\int_0^1 (u_1-u_2^\varepsilon)^+\Delta_y\zeta_{m,n}dydsdxdt \\
&\le \varepsilon\mathbb{E}\int_Q\int_Q\int_0^1 |u_1|\Delta_y\zeta_{m,n}dydsdxdt + \varepsilon\mathbb{E}\sup_{0\le t\le T}\|u_2^\varepsilon\|_{L^2(D)} \\
&\quad \times\int_Q\int_0^T\int_0^1\left[\int_D\left(\Delta_y(\varphi(s,y)\phi(y)\rho_n(y-x))\right)^2dy\right]^{\frac{1}{2}}\varrho_m d\alpha dsdxdt \\
&\to_\varepsilon 0,
\end{aligned}$$

where we used $\mathbb{E}\sup_{0\le t\le T}\|u_2^\varepsilon\|_{L^2(D)}$ is uniformly bounded for $\varepsilon>0$, see (4.2).

Noting that $\nabla_x \rho_m(y-x) + \nabla_y \rho_m(y-x) = 0$, we have

$$\begin{aligned}
I_3 + J_4 \quad &= \quad -\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}} F^{\eta_\delta}(u_1\wedge l, k)\varphi\phi\varrho_m \cdot \nabla_x \rho_n(y-x)\varrho_r(k-u_2^\varepsilon)dkdxdtdyds \\
&\quad\; -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q F^{\check{\eta}_\delta}(u_2^\varepsilon, k)\cdot(\rho_n\nabla_y(\varphi\phi) + \varphi\phi\nabla_y\rho_n)\varrho_m dyds\varrho_r(u_1\wedge l-k)dkdxdt \\
\to_r &\quad -\mathbb{E}\int_Q\int_Q F^{\eta_\delta}(u_1\wedge l, u_2^\varepsilon)\varphi\phi\varrho_m\cdot\nabla_x\rho_n(y-x)dxdtdyds \\
&\quad -\mathbb{E}\int_Q\int_Q F^{\eta_\delta}(u_1\wedge l, u_2^\varepsilon)\varphi\phi\varrho_m\cdot\nabla_y\rho_n(y-x)dxdtdyds \\
&\quad -\mathbb{E}\int_Q\int_Q F^{\eta_\delta}(u_1\wedge l, u_2^\varepsilon)\cdot\rho_n\nabla_y(\varphi\phi)\varrho_m dydsdxdt \\
\to_{m,\delta,\varepsilon,n} &\quad -\mathbb{E}\int_Q F^+(u_1\wedge l, u_2)\nabla(\varphi(t,x)\phi(x))dxdt.
\end{aligned}$$

$$\begin{aligned}
I_4 + J_5 \quad &= \quad \frac{1}{2}\mathbb{E}\int_Q\int_Q\int_{\mathbb{R}} h^2(u_1\wedge l)\eta_\delta''(u_1\wedge l-k)\zeta_{m,n}\varrho_r(k-u_2^\varepsilon)dkdxdtdyds \\
&\quad +\frac{1}{2}\mathbb{E}\int_Q\int_{\mathbb{R}}\int_Q\int_0^1 \eta_\delta''(k-u_2^\varepsilon)h^2(u_2^\varepsilon)\zeta_{m,n}dyds\varrho_r(u_1\wedge l-k)d\alpha dkdxdt \\
\to_{r,m} &\quad \frac{1}{2}\mathbb{E}\int_Q\int_D \eta_\delta''(u_1\wedge l-u_2^\varepsilon)\left(h^2(u_1\wedge l)+h^2(u_2^\varepsilon)\right)\varphi(t,y)\phi(y)\rho_n(y-x)dydxdt.
\end{aligned}$$

Now, we come to the estimate of most interesting part, the stochastic integrals. Since $\alpha(t) = \varrho_r(u_1(t,x)\wedge l - k)$ is predictable and if one denotes

$$\beta(s) = \int_D \eta_\delta'(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dy,$$

we have that

$$\mathbb{E}\left[\alpha(t)\int_t^T \beta(s)dw(s)\right] = \mathbb{E}\left[\alpha(t)\int_0^T \beta(s)dw(s)\right] - \mathbb{E}\left[\alpha(t)\int_0^t \beta(s)dw(s)\right] = 0$$

because that

$$\mathbb{E}\left[\alpha(t)\int_0^T \beta(s)dw(s)\right] = \mathbb{E}\left[\alpha(t)\mathbb{E}\left(\int_0^T \beta(s)dw(s)|\mathcal{F}_t\right)\right] = \mathbb{E}\left[\alpha(t)\int_0^t \beta(s)dw(s)\right].$$

Similarly, let $\alpha\left(s-\frac{2}{m}\right) = \varrho_r(k-u_2^\varepsilon)$ and

$$\beta(t) = \int_D \eta_\delta'(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dx,$$

then we get that

$$\mathbb{E}\int_Q\int_{\mathbb{R}}\alpha\left(s-\frac{2}{m}\right)\int_0^T \beta(t)dw(t)dkdyds = \int_Q\int_{\mathbb{R}}\mathbb{E}\alpha\left(s-\frac{2}{m}\right)\int_{\left(s-\frac{2}{m}\right)^+}^s \beta(t)dw(t)dkdyds = 0.$$

Thus, we have

$$\begin{aligned}
I_5+J_6 &= \mathbb{E}\int_Q\int_Q\int_{\mathbb{R}}\eta'_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\varrho_r(k-u_2^\varepsilon)dkdyds\\
&\quad -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_D\int_t^T\eta'_\delta(k-u_2^\varepsilon)h(u_2^\varepsilon)\zeta_{m,n}dydw(s)\varrho_l(u_1\wedge l-k)dkdxdt\\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\eta'_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\varrho_l(k-u_2^\varepsilon)dkdyds\\
&= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\eta'_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\\
&\quad\times\left[\varrho_r(k-u_2^\varepsilon(s,y))-\varrho_r\left(k-u_2^\varepsilon\left(s-\frac{2}{m},y\right)\right)\right]dkdyds
\end{aligned}$$

As $du_2^\varepsilon=[\varepsilon\Delta u_2^\varepsilon+div(f(u_2^\varepsilon))]dt+h(u_2^\varepsilon)dw(t):=A_\varepsilon dt+h(u_2^\varepsilon)dw(t)$, by Itô formula, we arrive that

$$\begin{aligned}
&\varrho_r(k-u_2^\varepsilon(s,y))-\varrho_r\left(k-u_2^\varepsilon\left(s-\frac{2}{m},y\right)\right)\\
=&-\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_r(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\\
&-\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_r(k-u_2^\varepsilon(\sigma,y))h(u_2^\varepsilon(\sigma,y))dw(\sigma)\\
&+\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho''_r(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma\\
=&-\frac{\partial}{\partial k}\left\{\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho_r(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\right.\\
&+\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho_r(k-u_2^\varepsilon(\sigma,y))h(u_2^\varepsilon(\sigma,y))dw(\sigma)\\
&\left.-\frac{1}{2}\int_{\left(s-\frac{2}{m}\right)^+}^s\varrho'_r(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma\right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_5+J_6 &= -\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{\left(s-\frac{2}{m}\right)^+}^s\int_D\eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\\
&\quad\times\{\cdots\}dkdyds\\
&:= L_1+L_2+L_3.
\end{aligned}$$

Let us evaluate the limits of $L_1,L_2$ and $L_3$. Following [12, 13], we know that the solution $u_2^\varepsilon$ of (3.1) will belong to $L^p(D)$ if $u_{02}^\varepsilon\in L^p(D)$. We assume that $u_{02}^\varepsilon\in C^\infty(D)$ and $u_{02}^\varepsilon$ converges to $u_{02}$ in $L^2(D)$. Thus the solution $u_2^\varepsilon\in L^p(D)$, $\forall p\geq 2$. By using the properties of the heat kernel, one can prove that $u_2^\varepsilon\in W^{2,p}(D)$, see [8, 10, 26]. That is, $A_\varepsilon\in L^p(D)$. The proof of this part is similar to that of [4]. We first consider $L_1$:

$$
\begin{aligned}
|L_1| \quad &\le \quad \int_Q\int_{\mathbb{R}}\int_D \left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^s \eta_\delta''(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dw(t)\right)^2\right]^{\frac{1}{2}} \\
&\quad \times\left[\mathbb{E}\left(\int_{\left(s-\frac{2}{m}\right)^+}^s \varrho_r(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)d\sigma\right)^2\right]^{\frac{1}{2}} dkdxdyds \\
&\le \quad \int_Q\int_{\mathbb{R}}\int_D \rho_n\varphi\phi\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s \left(\eta_\delta''(u_1\wedge l-k)\varrho_m(t-s)h(u_1\wedge l)\right)^2dt\right]^{\frac{1}{2}} \\
&\quad \times\frac{2}{\sqrt{m}}\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s \left(\varrho_r(k-u_2^\varepsilon(\sigma,y))A_\varepsilon(\sigma,y)\right)^2d\sigma\right]^{\frac{1}{2}} dkdxdyds \\
&\le \quad Cr\sqrt{m}\int_Q\int_{\mathbb{R}}\int_D \rho_n\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s \left(\eta_\delta''(u_1\wedge l-k)h(u_1\wedge l)\right)^2dt\right]^{\frac{1}{2}} \\
&\quad \times\left[\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s 1_{\{-\frac{2}{r}\le k-u_2^\varepsilon(\sigma,y)\le 0\}}A_\varepsilon^2(\sigma,y)d\sigma\right]^{\frac{1}{2}} dkdxdyds \\
&\le \quad \frac{Cr\sqrt{m}}{\delta^2}\int_Q\int_{\mathbb{R}}\int_D \rho_n\left\{\int_x\int_y\int_{\left(s-\frac{2}{m}\right)^+}^s 1_{\{u_1\wedge l-\delta\le k\le u_1\wedge l\}}h^2(u_1\wedge l)dt\right. \\
&\quad \left.\times\int_{\left(s-\frac{2}{m}\right)^+}^s 1_{\{-\frac{2}{r}\le k-u_2^\varepsilon(\sigma,y)\le 0\}}A_\varepsilon^2(\sigma,y)d\sigma dP_xdP_y\right\}^{\frac{1}{2}} dkdydxds \\
&\le \quad \frac{Cr\sqrt{m}}{\delta^2}\int_Q\int_{\mathbb{R}}\int_D \rho_n\left\{\int_x\int_y\int_{\left(s-\frac{2}{m}\right)^+}^s\int_{\left(s-\frac{2}{m}\right)^+}^s 1_{\{u_1\wedge l-\delta\le k\le u_1\wedge l\}}\right. \\
&\quad \left.\times\left[|u_2^\varepsilon|^2+\delta^2+\frac{4}{r^2}\right]1_{\{-\frac{2}{r}\le k-u_2^\varepsilon(\sigma,y)\le 0\}}A_\varepsilon^2(\sigma,y)d\sigma dtdP_xdP_y\right\}^{\frac{1}{2}} dkdydxds \\
&\le \quad \frac{Cr}{\delta}\int_Q\int_D\rho_n\left\{\int_y\int_{\left(s-\frac{2}{m}\right)^+}^s\left[|u_2^\varepsilon|^2+\delta^2+\frac{4}{r^2}\right]A_\varepsilon^2(\sigma,y)d\sigma dtdP_y\right\}^{\frac{1}{2}} dydxds \\
&\le \quad \frac{Cr}{\delta}\int_Q\left\{\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s\left[|u_2^\varepsilon|^2+\delta^2+\frac{4}{r^2}\right]^2d\sigma\right\}^{\frac{1}{2}} dyds \\
&\quad +\frac{Cr}{\delta}\int_Q\left\{\mathbb{E}\int_{\left(s-\frac{2}{m}\right)^+}^s A_\varepsilon^4(\sigma,y)d\sigma\right\}^{\frac{1}{2}} dyds \\
&\le \quad \frac{Cr}{\delta\sqrt{m}}\int_Q\left\{\mathbb{E}\sup_{\left(s-\frac{2}{m}\right)^+\le t\le s}\left[|u_2^\varepsilon(t,y)|^2+\delta^2+\frac{4}{r^2}\right]^2\right\}^{\frac{1}{2}} dyds \\
&\quad +\frac{Cr}{\delta\sqrt{m}}\int_Q\left\{\mathbb{E}\sup_{\left(s-\frac{2}{m}\right)^+\le t\le s}A_\varepsilon^4(t,y)\right\}^{\frac{1}{2}} dyds \\
&\to_m \quad 0,
\end{aligned}
$$

where we have used the facts $\mathbb{E}X=\int_x X(\omega,x)dP_x$ and

$$
h^2(u_1\wedge l)=|h(u_1\wedge l)-h(0)|^2\le L^2|u_1\wedge l|^2\le L^2\left[|u_2^\varepsilon|^2+\delta^2+\frac{4}{r^2}\right], \tag{4.3}
$$

thanks to the condition $(H_2)$.

Similarly, by using (4.3), we get

$$\begin{aligned}
|L_3| &\le \frac{1}{2}\Big|\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D \eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\\
&\quad\times\int_{(s-\frac{2}{m})^+}^{s}\varrho'_r(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma dkdyds\Big|\\
&\le \frac{1}{2}\int_Q\int_D\int_{\mathbb{R}}\left[\mathbb{E}\left(\int_{(s-\frac{2}{m})^+}^{s}\eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dw(t)\right)^2\right]^{\frac{1}{2}}\\
&\quad\times\left[\mathbb{E}\left(\int_{(s-\frac{2}{m})^+}^{s}\varrho'_r(k-u_2^\varepsilon(\sigma,y))h^2(u_2^\varepsilon(\sigma,y))d\sigma\right)^2\right]^{\frac{1}{2}}dkdxdyds\\
&\le \int_Q\int_D\int_{\mathbb{R}}\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s}\left(\eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}\right)^2dt\right]^{\frac{1}{2}}\\
&\quad\times\frac{C}{\sqrt{m}}\left[\mathbb{E}\int_{(s-\frac{2}{m})^+}^{s}\left(\varrho'_r(k-u_2^\varepsilon(\sigma,y))\right)^2h^4(u_2^\varepsilon(\sigma,y))d\sigma\right]^{\frac{1}{2}}dkdxdyds\\
&\le C\sqrt{m}\int_Q\int_D\int_{\mathbb{R}}\rho_n(y-x)\left\{\int_x\int_y\int_{(s-\frac{2}{m})^+}^{s}\int_{(s-\frac{2}{m})^+}^{s}\frac{1}{\delta^2}1_{\{u_1\wedge l-\delta\le k\le u_1\wedge l\}}\right.\\
&\quad\left.\times h^2(u_1\wedge l)r^2 1_{\{-\frac{2}{r}\le k-u_2^\varepsilon(\sigma,y)\le 0\}}h^4(u_2^\varepsilon(\sigma,y))dtd\sigma dP_xdP_y\right\}^{\frac{1}{2}}dkdxdyds\\
&\le \frac{Cr}{\delta^2}\int_Q\left\{\int_y\int_{(s-\frac{2}{m})^+}^{s}\left[|u_2^\varepsilon|^2+\delta^2+\frac{4}{r^2}\right]h^4(u_2^\varepsilon(\sigma,y))d\sigma dP_y\right\}^{\frac{1}{2}}dyds\\
&\le \frac{Cr}{\delta\sqrt{m}}\int_Q\left\{\mathbb{E}\sup_{(s-\frac{2}{m})^+\le t\le s}\left[|u_2^\varepsilon(t,y)|^2+\delta^2+\frac{4}{r^2}\right]^2\right\}^{\frac{1}{2}}dyds\\
&\quad+\frac{Cr}{\delta\sqrt{m}}\int_Q\left\{\mathbb{E}\sup_{(s-\frac{2}{m})^+\le t\le s}h^8(t,y)\right\}^{\frac{1}{2}}dyds\\
&\to_m 0,
\end{aligned}$$

where we used the facts that $u_2^\varepsilon\in L^p(D)$, $p\ge 2$. Thanks to Fibula's theorem and the properties of Itô integral, we have

$$\begin{aligned}
\lim_m(L_1+L_2+L_3) &= -\lim_m\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D\eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}dxdw(t)\\
&\quad\times\int_{(s-\frac{2}{m})^+}^{s}\varrho_r(k-u_2^\varepsilon(\sigma,y))h(u_2^\varepsilon(\sigma,y))dw(\sigma)dkdyds\\
&= -\lim_m\mathbb{E}\int_Q\int_{\mathbb{R}}\int_{(s-\frac{2}{m})^+}^{s}\int_D\eta''_\delta(u_1\wedge l-k)h(u_1\wedge l)\zeta_{m,n}\\
&\quad\times\varrho_r(k-u_2^\varepsilon(t,y))h(u_2^\varepsilon(t,y))dtdxdkdyds\\
&\to_r -\mathbb{E}\int_Q\int_D\eta''_\delta(u_1\wedge l-u_2^\varepsilon(t,y))h(u_1\wedge l)\varphi(t,y)\phi\rho_n h(u_2^\varepsilon)d\alpha dtdxdy
\end{aligned}$$

Therefore, we get

$$\begin{aligned}\lim_{m,r}(I_4+J_5+I_5+J_6) &= \frac{1}{2}\mathbb{E}\int_Q\int_D \eta''_\delta(u_1\wedge l-u_2^\varepsilon)\left(h^2(u_1\wedge l)-2h(u_1\wedge l)h(u_2^\varepsilon)+h^2(u_2^\varepsilon)\right)\\ &\quad \times\varphi(t,y)\phi(y)\rho_n(y-x)dydxdt\\ &= \frac{1}{2}\mathbb{E}\int_Q\int_D \eta''_\delta(u_1\wedge l-u_2^\varepsilon)\\ &\quad \times\left(h(u_1\wedge l)-h(u_2^\varepsilon)\right)^2\varphi(t,y)\phi(y)\rho_n(y-x)dyd\alpha dxdt\\ &\to_\delta 0,\end{aligned}$$

and thus

$$\lim_\delta\lim_{m,r} I_4+J_5+I_5+J_6\le 0.$$

Lastly, we consider $I_6$ and $I_7$. By the assumptions of $a_2^\varepsilon$, we have

$$\begin{aligned} I_6 &= \mathbb{E}\int_Q\int_{\mathbb{R}}\int_\Sigma \eta'_\delta(a_1\wedge l-k)\zeta_{m,n}\omega^+(x,k,a_1\wedge l)dSdt\varrho_r(k-u_2^\varepsilon)dkdyds\\ &\to_{m,n,r,\varepsilon,\delta} \int_\Sigma sgn_0^+(a_1\wedge l-a_2)\varphi\phi\omega^+(x,a_2,a_1\wedge l)dSdt.\end{aligned}$$

It is easy to see that

$$\lim_\delta\lim_{m,n,r} I_7=\langle\mu_l,\varphi\phi\rangle.$$

Combining all estimates yield

$$\begin{aligned}-\mathbb{E}\langle\mu_l,\varphi\phi\rangle &\le \mathbb{E}\int_D(u_{01}\wedge l-u_{02})^+\varphi(0,x)\phi(x)dx\\ &\quad +\mathbb{E}\int_Q(u_1\wedge l-u_2)^+\partial_t\varphi(t,x)\phi(x)dxdt\\ &\quad -\mathbb{E}\int_Q F^+(u_1\wedge l,u_2)\nabla(\varphi(t,x)\phi(x))dxdt\\ &\quad +\int_\Sigma sgn_0^+(a_1\wedge l-a_2)\varphi\phi\omega^+(x,a_2,a_1\wedge l)dSdt.\end{aligned}$$

Similar to the above discussion, for any $k\in\mathbb{R}$, one can prove exactly that $u_1,u_2$ satisfy the following local comparison principle: for any $\xi\in\mathcal{D}^+(Q)$,

$$\begin{aligned}-\mathbb{E}\langle\mu_l,\xi\rangle &\le \mathbb{E}\int_Q(u_1\wedge l-u_2)^+\xi_t+\int_D(u_{01}\wedge l-u_{02})^+\xi(0)\\ &\quad -\mathbb{E}\int_Q sgn_0^+(u_1\wedge l-u_2)(f(u_1\wedge l)-f(u_2))\cdot\nabla\xi.\end{aligned}$$

In particular, the above inequality holds with $\xi=\varphi\phi_0^r$.

Summing over $i=0,1,\cdots,m_r$, taking into account the local inequality for $i=0$, we find, for any $\xi\in\mathcal{D}([0,T)\times\mathbb{R}^N)$,

$$\begin{aligned}-\mathbb{E}\langle\mu_l,\xi\rangle &\le \mathbb{E}\int_D(u_{01}\wedge l-u_{02})^+\xi(0,x)dx\\ &\quad +\mathbb{E}\int_Q(u_1\wedge l-u_2)^+\partial_t\xi(t,x)dxdt\\ &\quad +\mathbb{E}\int_Q sgn_0^+(u_1\wedge l-u_2)(f(u_1\wedge l)-f(u_2))\cdot\nabla\xi\\ &\quad +\int_\Sigma \xi sgn_0^+(a_1\wedge l-a_2)\omega^+(x,a_2,a_1\wedge l)dSdt.\end{aligned}$$

The proof of Lemma 4.1 is complete. □

Next, we consider the second half. Similarly, as $u_1$ is a renormalized stochastic entropy solution, using the other half of Definition 2.2, and applying the Itô formula to $\int_D \eta_\delta(u_2^\varepsilon - k)$, we have for any $l \leq -\|a_2\|_{L^\infty}$,

$$\begin{aligned} -\mathbb{E}\langle \nu_l, \xi\rangle \quad \leq \quad & \mathbb{E}\int_D (u_{02} - u_{01} \vee l)^+ \xi(0,x)dx \\ & + \mathbb{E}\int_Q (u_2 - u_1 \vee l)^+ \partial_t \xi(t,x)dxdt \\ & + \mathbb{E}\int_Q sgn_0^+(u_2 - u_1 \vee l)(f(u_2) - f(u_1 \vee l)) \cdot \nabla\xi \\ & + \int_\Sigma \xi sgn_0^+(a_2 - a_1 \vee l)\omega^+(x, a_2, a_1 \vee l)dSdt. \end{aligned} \tag{4.4}$$

**Proof of Theorem 2.1**. Now we complete the proof of Theorem 2.1. Let $v$ be a renormalized stochastic entropy solution of (1.1)-(1.3) and $u_n$ be defined as in Theorem 3.1. Then by Lemma 4.1, we have for any $\xi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$ and for any $l \geq -\|a_2\|_{L^\infty}$,

$$\begin{aligned} -\mathbb{E}\langle \mu_l^n, \xi\rangle \quad \leq \quad & \mathbb{E}\int_D (v_0 \wedge l - u_{0n})^+ \xi(0,x)dx \\ & + \mathbb{E}\int_Q (v \wedge l - u_n)^+ \partial_t \xi(t,x)dxdt \\ & + \mathbb{E}\int_Q sgn_0^+(v \wedge l - u_n)(f(v \wedge l) - f(u_n)) \cdot \nabla\xi \\ & + \int_\Sigma \xi sgn_0^+(a \wedge l - a_n)\omega^+(x, a_n, a \wedge l)dSdt. \end{aligned} \tag{4.5}$$

And by (4.4), we have $\xi \in \mathcal{D}^+([0,T)\times\mathbb{R}^N)$ and for any $l \leq -\|a_2\|_{L^\infty}$,

$$\begin{aligned} -\mathbb{E}\langle \nu_l^n, \xi\rangle \quad \leq \quad & \mathbb{E}\int_D (u_{0n} - v_0 \vee l)^+ \xi(0,x)dx \\ & + \mathbb{E}\int_Q (u_n - v \vee l)^+ \partial_t \xi(t,x)dxdt \\ & + \mathbb{E}\int_Q sgn_0^+(u_n - v \vee l)(f(u_n) - f(v \vee l)) \cdot \nabla\xi \\ & + \int_\Sigma \xi sgn_0^+(a_n - a \vee l)\omega^+(x, a_n, a \vee l)dSdt. \end{aligned} \tag{4.6}$$

Summing (4.5) and (4.6), letting $n \to \infty$, similar to the proof of [21, Theorem 3.1], we get $v = \lim_{n\to\infty} u_n = u$. This completes the proof of Theorem 2.1. □

**Remark 4.1** (A good example) *In the proofs of Theorem 3.1 and Lemma 4.1, we use the Lipschitz condition of $h(u)$, see (3.3) and the estimates of $L_1$ and $L_3$. A good example can make the proofs simpler. Let $h(u) = sinu$ or $cosu$, then $h$ satisfies the condition $(H_2)$. What's more, $h(u)$ is bounded uniformly with respect to $u$, that is,*

$$|h(u_n) - h(u)|^2 \leq 2|u_n - u|.$$

*In other words, the proofs of (3.3), $L_1$ and $L_3$ will be simpler.*

*In addition, it is not difficult to find our method is also suitable to the whole space, that is, $D = \mathbb{R}^N$.*

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