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A new framework for large strain electromechanics based on convex multi-variable strain energies: conservation laws and hyperbolicity and extension to electro-magnetomechanics.

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Abstract

This work is the third on a series of papers by Gil and Ortigosa [1, 2] on the development of a new computational framework for the analysis of Electro Active Polymers, where the concept of polyconvexity [3] is extended to the case of electro-magneto-mechanical energy functionals. Specifically, four key novelties are incorporated in this paper. Firstly, a new set of first order hyperbolic equations is presented in the context of nonlinear electro-magneto-elasticity, including conservation laws for all the fields of the extended set of arguments which determine the convex multi-variable nature of the internal energy. Secondly, the one-to-one and invertible relationship between this extended set and its associated entropy conjugate set enables the definition of a generalised convex entropy function, resulting in the symmetrisation of the system when expressed in terms of the entropy variables. Thirdly, this paper shows that, after careful analysis of the eigenvalue structure of the system, the definition of multi-variable convexity in [1] leads to positive definiteness of the electro-magneto-acoustic tensor. Therefore, multi-variable convexity ensures the satisfaction of the Legendre-Hadamard condition, hence showing that the speeds of propagation of acoustic and electro-magnetic waves in the neighbourhood of a stationary point are real. Finally, under a characteristic experimental set-up for electrostrictive dielectric elastomers, a study of the material stability of convex and non-convex multi-variable constitutive

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models is carried out.

Keywords: Dielectric elastomers, Magneto Active Polymers, Nonlinear electro-elasticity, Conservation laws, Polyconvexity, Hyperbolicity

1. Introduction

Dielectric elastomers [4–7] and piezoelectric polymers have become increasingly relevant due to their outstanding capabilities as artificial actuators [4–8], energy harvesters and as power generators [9]. Another type of smart materials with potential applications within the field of artificial actuators is that of magneto-active elastomers [10].

In the present manuscript, a general computational framework for the analysis of electro-magneto-mechanical interactions is presented. The framework can be particularised to more simplified scenarios, where only electromechanical or magnetomechanical interactions are considered. The present manuscript is the third of a series of papers written by the authors [1, 2].

Despite the enormous interest of the experimental [4–8] and computational scientific communities [10–22], the definition of suitable constitutive models for both electro-active and magneto-active materials is still at its early stages. As customary in nonlinear continuum mechanics and as dictated by thermodynamics, the constitutive behaviour of the material is encoded in an energy functional which typically depends upon appropriate strain measures, Lagrangian electric and magnetic variables and, if dissipative effects are considered, appropriate internal variables. Several authors have proposed alternative representations of the energy functional in terms of electromechanical or magnetomechanical invariants [12–17]. However, some restrictions need to be imposed on the resulting invariant representation if a physically admissible behaviour is expected to occur. Bustamante and Merodio [23] considered classical constitutive inequalities, namely the Baker-Ericksen inequality, the pressure-compression inequality, the traction-extension inequality and the ordered forces inequality. The objective was to study under what conditions a specific invariant representation of the energy functional for magneto-sensitive elastomers can violate the aforementioned inequalities for very specific deformation scenarios.

The most well accepted constitutive inequality is ellipticity, also known as the Legendre-Hadamard condition [3, 24]. This inequality has important physical implications. In particular, it guarantees positive definiteness of the

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generalised electromechanical acoustic tensor (or electro-magneto-mechanical acoustic tensor in the more general scenario of electro-magneto-elasticity) [25] and hence, existence of real wave speeds in the material in the vicinity of an equilibrium configuration. Moreover, this condition is intimately related to material stability [26] of the constitutive equations. Several authors have also studied under what conditions positive definiteness of the generalised electromechanical acoustic tensor and hence material stability is compromised for a specific invariant representation of the energy functional [25, 27, 28].

In nonlinear elasticity, polyconvexity [3, 24, 26, 26, 29–46] of the (strain) energy functional, namely, convexity with respect to the components of the deformation gradient tensor $\boldsymbol{F}$, the components of its adjoint or Co-factor $\boldsymbol{H}$ and its determinant J, automatically implies the satisfaction of the Legendre-Hadamard condition.

In Reference [1], a new electro-magneto-kinematic variable set is introduced including the aforementioned kinematical entities, namely $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and the Lagrangian electric displacement field $\boldsymbol{D}_0$, the Lagrangian magnetic induction $\boldsymbol{B}_0$, and two additional spatial or Eulerian vectors $\boldsymbol{d}$ and $\boldsymbol{b}$. The latter entities, namely $\boldsymbol{d}$ and $\boldsymbol{b}$ are computed as the product between the deformation gradient tensor and the Lagrangian electric displacement or magnetic induction, respectively. The resulting energy functional is called the internal energy and as presented in Reference [1], its convexity with respect to the elements of the new extended set permits an extension of the Legendre-Hadamard condition [24], not only to the entire range of deformations but also to any electric and magnetic fields.

It should be emphasised that the focus of this paper is on material stability and not on the existence of minimisers. The latter would also require the study of the sequentially weak lower semicontinuity and the coercivity of the energy functional [3, 40]. This is the fundamental reason behind the use of the more appropriate term multi-variable convexity for the general context of electro-magneto-elasticity as opposed to polyconvexity in the particular case of nonlinear elasticity.

The present manuscript provides a physically meaningful interpretation of the proposed definition of multi-variable convexity of the internal energy as a result of the strong relationship between the Legendre-Hadamard condition (satisfied by convex multi-variable functionals) and the hyperbolicity of the dynamic equations. More specifically, the Legendre-Hadamard condition is intimately related to the nature of the arguments of multi-variable convexity, which must be expressed in terms of first order hyperbolic equations. A

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system of first order hyperbolic conservation laws is presented for each of the arguments of the extended set of variables which define multi-variable convexity, namely $\{\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}\}$. First order conservation laws have already been presented for $\boldsymbol{F}$, $\boldsymbol{H}$ and J in References [36, 42, 47–52]. The set of Maxwell equations is also comprised of a first order hyperbolic conservation laws for $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$ [53]. Two completely new conservation laws for the variables $\boldsymbol{d}$ and $\boldsymbol{b}$ are presented in the present manuscript.

The extended set of variables $\{\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}\}$ enables a new set of entropy conjugate variables $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J, \boldsymbol{\Sigma}_{\boldsymbol{D}_0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_0}, \boldsymbol{\Sigma}_{\boldsymbol{d}}, \boldsymbol{\Sigma}_{\boldsymbol{b}}\}$ [54] to be introduced. Multi-variable convexity of the internal energy guarantees that the relationship between both sets of variables is one-to-one and invertible. Crucially, in line with the pioneering work of Hughes *et al.* [55] in the context of Computational Fluid Dynamics and later Bonet *et al.* [49] in the context of nonlinear Computational Solid Dynamics, the new definition of multi-variable convexity enables, for the first time, a generalised convex entropy function to be defined in the context of nonlinear electro-magneto-mechanics.

This innovative idea facilitates the transformation of the system of conservation laws in electro-magneto-mechanics into a symmetric set of hyperbolic equations when expressed in terms of the entropy conjugates of the conservation variables. The physical implications of this symmetrisation are extremely relevant, as existence of real travelling waves (hyperbolicity) is automatically satisfied [55]. Furthermore, the eigenvalue structure [25, 26] of the full new set of hyperbolic equations is studied in this paper. In this case, it can be proved that multi-variable convexity entails positive definiteness of the electro-magneto-mechanical acoustic tensor [25, 26].

From the computational standpoint, the new definition of multi-variable convexity has been shown very useful for the development of new extended Hu-Washizu type of mixed variational principles [1, 2] in the context of nonlinear electromechanics, which can help overcoming the inherent difficulties of displacement-potential based formulations. An extension of these new Hu-Wahizu type of mixed variational principles to the field of magneto-elasticity would be straightforward. In addition to these new Hu-Washizu type of mixed variational principles, the new set of conservations laws proposed in this paper opens up very interesting possibilities to use stabilised based formulations, in line with those published in References [36, 42, 48, 49, 52, 56] in the context of nonlinear elasticity.

This paper is organised as follows. Section 2 succinctly revises fundamental concepts of large strain kinematics with the help of a tensor cross product

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notation initially introduced by de Boer [57] and further developed by Bonet *et al.* [41]. Section 3 revisits the definition of multi-variable convexity introduced in Reference [1] for nonlinear electro-magneto-mechanics. Furthermore, a new extended set of first order hyperbolic conservation laws is derived for the first time in the context of nonlinear electro-magneto-mechanics. In particular, conservation laws are presented for all the elements of the new extended set of arguments, namely $\{\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}\}$. Section 4 presents the eigenvalue structure associated with the set of first order hyperbolic equations, proving the existence of real eigenvalues (wave speeds) for constitutive models complying with the new definition of multi-variable convexity. Section 5 defines the generalised convex entropy function and its associated flux and presents the symmetric set of conservation laws expressed in terms of the new set of extended entropy variables. Finally, Section 6 focusses on the analysis of material stability of convex and non-convex multi-variable constitutive models suitable for the description of electrostrictive dielectric elastomers [4–7].

Five appendices have been included for the sake of completeness. Appendix A presents the algebra associated to a tensor cross product operation introduced by de Boer [57] and further exploited by Bonet *et al.* [41, 58]. This tensor cross product operation proves very convenient when dealing with the derivatives of the Co-factor and hence will feature heavily in the convex multi-variable framework under study. Appendix B presents complementary information necessary to Section 4. Appendix C includes a re-expression of the relevant constitutive tensors emerging in nonlinear electro-magneto-mechanics in terms of the algebraically more convenient components of the extended Hessian operator of the convex multi-variable representation of the internal energy. Appendix D includes the proof of a theorem relevant to the eigenvalue structure of the set of hyperbolic equations in nonlinear electro-magneto-mechanics, developed in Section 4. Finally, Appendix E includes some algebraic manipulations which facilitate the derivation of explicit expressions for the speed of propagation of acoustic waves for simple convex multi-variable electromechanical constitutive models.

2. Motion and deformation

Let us consider the motion of a continuum that could represent an electro and/or a magneto active material. Let this continuum in its initial or material configuration be defined by a domain V of boundary ∂V with outward unit

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normal $\boldsymbol{N}$. After the motion, the continuum occupies a spatial configuration defined by a domain v of boundary ∂v with outward unit normal $\boldsymbol{n}$.

The motion of the continuum V is defined by a pseudo-time t dependent mapping field $\boldsymbol{\phi}$ which links a material particle from material configuration $\boldsymbol{X}$ to spatial configuration $\boldsymbol{x}$ according to $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t)$. The two-point deformation gradient tensor or fibre-map $\boldsymbol{F}$, which relates a fibre of differential length from the material configuration $d\boldsymbol{X}$ to the spatial configuration $d\boldsymbol{x}$, namely $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$, is defined as the material gradient $\boldsymbol{\nabla}_0$ of the spatial configuration [3, 24, 29–31, 59–63]

$$\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x} = \frac{\partial \boldsymbol{\phi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}. \tag{1}$$

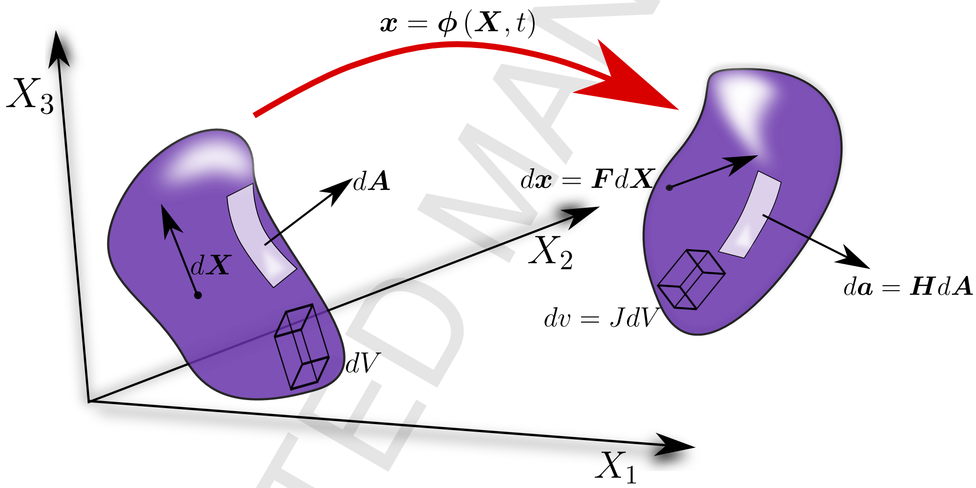

Figure 1: Deformation mapping of a continuum. Associated kinematics magnitudes: $\boldsymbol{F}, \boldsymbol{H}, J$.

In addition, $J = \det \boldsymbol{F}$ represents the Jacobian or volume-map of the deformation, which relates differential volume elements in the material configuration dV and the spatial configuration dv as $dv = JdV$. Finally, the element area vector is mapped from initial $d\boldsymbol{A}$ (colinear with $\boldsymbol{N}$) to final $d\boldsymbol{a}$ (colinear with $\boldsymbol{n}$) configuration by means of the two-point Co-factor or adjoint tensor $\boldsymbol{H}$ as $d\boldsymbol{a} = \boldsymbol{H}d\boldsymbol{A}$, which is related to the deformation gradient

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tensor via the so-called Nanson's rule [59]

$$\boldsymbol{H} = J\boldsymbol{F}^{-T}. \tag{2}$$

Figure 1 depicts the deformation process as well the three kinematic maps, that is, $\boldsymbol{F}$, $\boldsymbol{H}$ and J. With the help of the definition of the tensor cross product operation $\times$ introduced in [57] and rediscovered and utilised in the context of computational nonlinear continuum mechanics in [41, 58], it is possible to re-write the area $\boldsymbol{H}$ and volume J maps in tensor and indicial notation as follows

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{F} \times \boldsymbol{F}; \qquad H_{iI} = \frac{1}{2}\mathcal{E}_{ijk}\mathcal{E}_{IJK}F_{jJ}F_{kK}; \tag{3a}$$

$$J = \frac{1}{3}\boldsymbol{H} : \boldsymbol{F}; \qquad J = \frac{1}{3}H_{iI}F_{iI}, \tag{3b}$$

where $\mathcal{E}_{ijk}$ (or $\mathcal{E}_{IJK}$) symbolises the third order alternating tensor components[2] and the use of repeated indices implies summation, unless otherwise stated.

In addition, throughout the paper, the symbol $(\cdot)$ is used to indicate the scalar product or contraction of a single index $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$; the symbol $(:)$ is used to indicate double contraction of two indices $\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij}$; the symbol $(\times)$ is used to indicate the cross product between vectors $[\boldsymbol{a} \times \boldsymbol{b}]_i = \mathcal{E}_{ijk}a_j b_k$ and the symbol $(\otimes)$ is used to indicate the outer or dyadic product $[\boldsymbol{a} \otimes \boldsymbol{b}]_{ij} = a_i b_j$.

For a smooth deformation mapping $\boldsymbol{\phi}$, it can be shown from equations (1) and (3) that the material divergence of the Co-factor $\boldsymbol{H}$ as well as the material CURL of the deformation gradient $\boldsymbol{F}$ vanish, that is [49]

$$\mathrm{DIV}\boldsymbol{H} = \boldsymbol{0}; \qquad \mathrm{CURL}\boldsymbol{F} = \boldsymbol{0}, \tag{4a}$$

where the material CURL and DIV of a second order tensor $\boldsymbol{A}$ are defined as

$$[\mathrm{CURL}\boldsymbol{A}]_{iI} = \mathcal{E}_{IJK}\frac{\partial A_{iK}}{\partial X_J}; \qquad (\mathrm{DIV}\boldsymbol{A})_i = \frac{\partial A_{iI}}{\partial X_I}. \tag{5}$$

In the context of the conservation laws that will be developed in the following Sections, equations (4) represent two sets of involutions [49] that need to be satisfied by the conservation variables.

[2] Lower case indices $\{i, j, k\}$ will be used to represent the spatial configuration whereas capital case indices $\{I, J, K\}$ will be used to represent the material description.

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3. Conservation laws in convex multi-variable nonlinear electro-magneto-mechanics

In this Section, the set of first order hyperbolic equations in large deformations and large electric/magnetic fields scenarios will be presented.

3.1. General remarks

The aim of this Section is, following the work in Reference [49] in the particular context of nonlinear elasticity, to express the equations of electro-magneto-mechanics in the form of a set of global conservation laws for a set of conservation variables $\boldsymbol{\mathcal{U}} = [\mathcal{U}_\alpha]$ with fluxes $\boldsymbol{\mathcal{F}} = [\mathcal{F}_{\alpha I}]$ and source terms $\boldsymbol{\mathcal{S}} = [\mathcal{S}_\alpha]$, where $\alpha = 1, \dots, n$ represents the set of unknowns and $I = 1, 2, 3$ the reference coordinates. Global conservation laws are generally expressed as [49]

$$\frac{d}{dt}\int_V \boldsymbol{\mathcal{U}}\, dV + \int_{\partial V} \boldsymbol{\mathcal{F}}\, d\boldsymbol{A} = \int_V \boldsymbol{\mathcal{S}}\, dV. \tag{6}$$

For smooth functions, this integral expression is equivalent to the set of first order differential equations

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \mathrm{DIV}\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{S}}. \tag{7}$$

In addition, for discontinuous solutions, the integral conservation laws also lead to the following jump conditions across a discontinuity surface with normal $\boldsymbol{N}$ moving with normal speed U [51, 64, 65]

$$U[\![\boldsymbol{\mathcal{U}}]\!] = [\![\boldsymbol{\mathcal{F}}]\!]\boldsymbol{N}, \tag{8}$$

where the notation $[\![\phi]\!] = (\phi_+ - \phi_-)$ is used to denote the jump of a variable across a moving discontinuity surface. It is often convenient to express equation (7) in the quasi-linear form,

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \boldsymbol{\mathcal{A}}_I \frac{\partial \boldsymbol{\mathcal{U}}}{\partial X_I} = \boldsymbol{\mathcal{S}}; \qquad \boldsymbol{\mathcal{A}}_I = [\mathcal{A}_{\alpha\beta}]_I = \frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial \boldsymbol{\mathcal{U}}}, \tag{9}$$

where $\boldsymbol{\mathcal{F}}_I$ denotes the columns of the flux matrix and, in general, the square $n\times n$ matrices $\boldsymbol{\mathcal{A}}_I$ (with $\alpha, \beta = 1, \dots, n$) will not be symmetric. Alternatively, the flux matrix $\boldsymbol{\mathcal{F}}$ defined in (7) can be expressed as

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_I \otimes \boldsymbol{E}_I; \qquad \boldsymbol{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad \boldsymbol{E}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \boldsymbol{E}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

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3.2. Conservation of mass and momentum

For Lagrangian formulations, conservation of mass is simply stated as

$$\frac{d}{dt}\int_V \rho_0 \, dV = 0. \tag{11}$$

The above equation (11) simply implies that the initial density of the material, namely ρ_0, is constant and therefore does not need to be considered as part of the vector of problem unknowns $\boldsymbol{\mathcal{U}}$.

Global conservation of linear momentum $\boldsymbol{p} = \rho_0 \boldsymbol{v}$ is established for any arbitrary Lagrangian volume V as

$$\frac{d}{dt}\int_V \boldsymbol{p} \, dV - \int_{\partial V} \boldsymbol{t}_0 \, dA = \int_V \boldsymbol{f}_0 \, dV, \tag{12}$$

where $\boldsymbol{f}_0$ represents a body force per unit undeformed volume and $\boldsymbol{t}_0$ the traction vector associated with the initial normal $\boldsymbol{N}$. Assuming the existence of a first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ such that the traction vector associated with the initial area with normal $\boldsymbol{N}$ is $\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{N}$, the equivalent local equilibrium equation and jump condition can be written as

$$\frac{\partial \boldsymbol{p}}{\partial t} - \mathbf{DIV}\boldsymbol{P} = \boldsymbol{f}_0; \qquad U[\![\boldsymbol{p}]\!] = -[\![\boldsymbol{P}]\!]\boldsymbol{N}. \tag{13}$$

3.3. Conservation of electric charge and Maxwell equations

In a Lagrangian setting, conservation of electric charge is simply stated as

$$\frac{d}{dt}\int_V \rho_0^e \, dV = 0. \tag{14}$$

Analogously to the equation of conservation of mass in (11), equation (14) implies that the initial electric density charge of the material ρ_0^e is constant and therefore does not need to be considered as part of the vector of problem unknowns $\boldsymbol{\mathcal{U}}$. The Faraday and Ampère laws for an arbitrary Lagrangian volume can be established as

$$\frac{d}{dt}\int_V \boldsymbol{B}_0 \, dV - \int_{\partial V} \boldsymbol{E}_0 \times \, d\boldsymbol{A} = \boldsymbol{0}; \tag{15a}$$

$$\frac{d}{dt}\int_V \boldsymbol{D}_0 \, dV + \int_{\partial V} \boldsymbol{H}_0 \times \, d\boldsymbol{A} = -\int_V \boldsymbol{J}_0 \, dV, \tag{15b}$$

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where $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$ denote the material form of the electric displacement field and the magnetic induction vectors. $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ denote the electric and magnetic fields, respectively, and $\boldsymbol{J}_0$ is the volume current density in the reference or material configuration. The equivalent local differential equations and jump conditions are

$$\frac{\partial \boldsymbol{B}_0}{\partial t} + \text{CURL}\boldsymbol{E}_0 = \boldsymbol{0}; \qquad U[\![\boldsymbol{B}_0]\!] = -[\![\boldsymbol{E}_0]\!] \times \boldsymbol{N}; \tag{16a}$$

$$\frac{\partial \boldsymbol{D}_0}{\partial t} - \text{CURL}\boldsymbol{H}_0 = -\boldsymbol{J}_0; \qquad U[\![\boldsymbol{D}_0]\!] = [\![\boldsymbol{H}_0]\!] \times \boldsymbol{N}. \tag{16b}$$

Maxwell equations are completed with an additional set of partial differential equations including the Gauss law and the Gauss law for magnetism, which can be expressed as

$$\text{DIV}\boldsymbol{D}_0 = \rho_0^e; \qquad \text{DIV}\boldsymbol{B}_0 = 0. \tag{17}$$

There exists a clear similitude between equations in (17) and equations in (4). In fact, both Gauss laws in (17) can be seen as an additional set of (electro-magnetic) involutions [53] to those described in equation (4) which also need to be satisfied by the conservation variables $\boldsymbol{\mathcal{U}}$.

3.4. The internal energy in nonlinear electro-elasticity

Equations (12), (15a) and (15b) establish a balance between the evolution of the conservation variables $\boldsymbol{p}$, $\boldsymbol{B}_0$ and $\boldsymbol{D}_0$ and their associated fluxes, related to the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the material form of the electric and magnetic fields, namely $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. In addition, the latter variables $\{\boldsymbol{P}, \boldsymbol{E}_0, \boldsymbol{H}_0\}$ are determined based upon an appropriate constitutive law.

In the case of reversible electro-magneto-mechanics, where thermal effects and any other possible state variables (i.e. accumulated plastic deformation, electrical relaxation, etc) are disregarded, constitutive laws can be derived from the internal energy density e per unit of undeformed volume, defined as

$$e = e(\boldsymbol{\nabla}_0 \boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0). \tag{18}$$

The first principle of thermodynamics (under the assumption of no dissipative effects) yields the following expression for the time derivative of the internal energy $e(\boldsymbol{\nabla}_0 \boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0)$

$$\dot{e} = \boldsymbol{P} : \boldsymbol{\nabla}_0 \boldsymbol{v} + \boldsymbol{E}_0 \cdot \dot{\boldsymbol{D}}_0 + \boldsymbol{H}_0 \cdot \dot{\boldsymbol{B}}_0. \tag{19}$$

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Combination of equations (18) and (19) establishes the necessary constitutive relationships which enable the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the material form of the electric and magnetic fields, namely $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ respectively, to be defined as

$$\boldsymbol{P} = \frac{\partial e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0)}{\partial \boldsymbol{\nabla}_0\boldsymbol{x}}; \quad \boldsymbol{E}_0 = \frac{\partial e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0)}{\partial \boldsymbol{D}_0}; \quad \boldsymbol{H}_0 = \frac{\partial e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0)}{\partial \boldsymbol{B}_0}. \tag{20}$$

3.5. The convex multi-variable internal energy in nonlinear electro-magneto-elasticity

Constitutive equations must satisfy appropriate restrictions if a physically admissible behaviour is expected to occur. The most well accepted constitutive restriction, namely the Legendre-Hadamard condition [24], ensures the existence of real travelling wave speeds in the vicinity of a stationary point [26]. Typically, suitable convex requirements on the internal energy e (18) can lead to the fulfilment of the Legendre-Hadamard condition. In the field of nonlinear elasticity, it is well known [3] that convexity of the internal (strain) energy $e(\boldsymbol{\nabla}_0\boldsymbol{x})$ with respect to $\boldsymbol{\nabla}_0\boldsymbol{x}$ complies with the Legendre-Hadamard condition. However, this is too stringent of a restriction, as it precludes other physical behaviours such as buckling [3]. Analogously, in the more general context of electro-magnetomechanics, convexity with respect to $\{\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0\}$ should also be discarded as a plausible convex restriction on e (18).

A definition of the internal energy compliant with the Legendre-Hadamard condition which does not exclude buckling can be achieved via a relaxation of the convexity requirements on $e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{B}_0, \boldsymbol{D}_0)$. In the context of nonlinear elasticity, quasiconvexity [66] and polyconvexity [3, 24, 30] are successful examples of relaxation. In a previous work [2], the authors have postulated an extension of polyconvexity to the field of electromechanics (where the more appropriate term multi-variable convexity has been adopted) and hinted its immediate extension to the more general context of electro-magneto-mechanics. In Reference [2], the internal energy $e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0)$ is defined as convex multi-variable if it can be expressed as

$$e\left(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B}_0\right) = W\left(\mathcal{V}\right), \tag{21}$$

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where W represents a convex muti-variable functional in terms of the extended set of arguments $\mathcal{V}$, defined as

$$\mathcal{V} = \{\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}\}, \tag{22}$$

with the spatial vectors $\boldsymbol{d}$ and $\boldsymbol{b}$ defined as

$$\boldsymbol{d} = \boldsymbol{F}\boldsymbol{D}_0; \qquad \boldsymbol{b} = \boldsymbol{F}\boldsymbol{B}_0. \tag{23}$$

3.5.1. Work conjugate variables in convex multi-variable electro-magneto-mechanics

The definition of multi-variable convexity in equation (21) enables the introduction of a set of work conjugate variables $\Sigma_{\mathcal{V}} = \{\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J, \boldsymbol{\Sigma_{D_0}}, \boldsymbol{\Sigma_{B_0}}, \boldsymbol{\Sigma_d}, \boldsymbol{\Sigma_b}\}$ to those in $\mathcal{V}$ (22) defined as

$$\begin{aligned}
\boldsymbol{\Sigma_F} &= \frac{\partial W}{\partial \boldsymbol{F}}; \quad \boldsymbol{\Sigma_H} = \frac{\partial W}{\partial \boldsymbol{H}}; \quad \Sigma_J = \frac{\partial W}{\partial J};\\
\boldsymbol{\Sigma_{D_0}} &= \frac{\partial W}{\partial \boldsymbol{D}_0}; \quad \boldsymbol{\Sigma_{B_0}} = \frac{\partial W}{\partial \boldsymbol{B}_0};\\
\boldsymbol{\Sigma_d} &= \frac{\partial W}{\partial \boldsymbol{d}}; \quad \boldsymbol{\Sigma_b} = \frac{\partial W}{\partial \boldsymbol{b}}.
\end{aligned} \tag{24}$$

Using the properties of the tensor cross product operation $\times$ in Reference [41, 49], it is possible to obtain the time derivative of the internal energy e (18) now expressed in terms of the set $\mathcal{V}$ (22) as

$$\begin{aligned}
\dot{e}(\boldsymbol{F}, \boldsymbol{D}_0, \boldsymbol{B}_0) &= \dot{W}(\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{d}, \boldsymbol{B}_0, \boldsymbol{b})\\
&= \boldsymbol{\Sigma_F} : \dot{\boldsymbol{F}} + \boldsymbol{\Sigma_H} : \dot{\boldsymbol{H}} + \Sigma_J \dot{J}\\
&+ \boldsymbol{\Sigma_{D_0}} \cdot \dot{\boldsymbol{D}}_0 + \boldsymbol{\Sigma_{B_0}} \cdot \dot{\boldsymbol{B}}_0\\
&+ \boldsymbol{\Sigma_d} \cdot \dot{\boldsymbol{d}} + \boldsymbol{\Sigma_b} \cdot \dot{\boldsymbol{b}}\\
&= (\boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H} + \boldsymbol{\Sigma_d} \otimes \boldsymbol{D}_0 + \boldsymbol{\Sigma_b} \otimes \boldsymbol{B}_0) : \boldsymbol{\nabla}_0 \boldsymbol{v}\\
&+ \left(\boldsymbol{\Sigma_{D_0}} + \boldsymbol{F}^T \boldsymbol{\Sigma_d}\right) \cdot \dot{\boldsymbol{D}}_0 + \left(\boldsymbol{\Sigma_{B_0}} + \boldsymbol{F}^T \boldsymbol{\Sigma_b}\right) \cdot \dot{\boldsymbol{B}}_0.
\end{aligned} \tag{25}$$

Comparison of equations (25) against (19) enables the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the material form of the electric and magnetic fields, namely $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ respectively, to be expressed in terms of the elements of

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both sets $\mathcal{V}$ and $\Sigma_{\mathcal{V}}$ as

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H} + \boldsymbol{\Sigma_d} \otimes \boldsymbol{D}_0 + \boldsymbol{\Sigma_b} \otimes \boldsymbol{B}_0; && \text{(26a)}\\
\boldsymbol{E}_0 &= \boldsymbol{\Sigma}_{\boldsymbol{D}_0} + \boldsymbol{F}^T \boldsymbol{\Sigma_d}; && \text{(26b)}\\
\boldsymbol{H}_0 &= \boldsymbol{\Sigma}_{\boldsymbol{B}_0} + \boldsymbol{F}^T \boldsymbol{\Sigma_b}. && \text{(26c)}
\end{aligned}$$

Finally, a convex multi-variable representation of the internal energy as that in equation (21) ensures that the extended Hessian operator of the extended internal energy, namely $[\mathbb{H}_W]$ (21) is positive definite, with $[\mathbb{H}_W]$ defined as

$$[\mathbb{H}_W] = \begin{bmatrix}
W_{\boldsymbol{FF}} & W_{\boldsymbol{FH}} & W_{\boldsymbol{F}J} & W_{\boldsymbol{FD}_0} & W_{\boldsymbol{FB}_0} & W_{\boldsymbol{Fd}} & W_{\boldsymbol{Fb}} \\
W_{\boldsymbol{HF}} & W_{\boldsymbol{HH}} & W_{\boldsymbol{H}J} & W_{\boldsymbol{HD}_0} & W_{\boldsymbol{HB}_0} & W_{\boldsymbol{Hd}} & W_{\boldsymbol{Hb}} \\
W_{J\boldsymbol{F}} & W_{J\boldsymbol{H}} & W_{JJ} & W_{J\boldsymbol{D}_0} & W_{J\boldsymbol{B}_0} & W_{J\boldsymbol{d}} & W_{J\boldsymbol{b}} \\
W_{\boldsymbol{D}_0\boldsymbol{F}} & W_{\boldsymbol{D}_0\boldsymbol{H}} & W_{\boldsymbol{D}_0J} & W_{\boldsymbol{D}_0\boldsymbol{D}_0} & W_{\boldsymbol{D}_0\boldsymbol{B}_0} & W_{\boldsymbol{D}_0\boldsymbol{d}} & W_{\boldsymbol{D}_0\boldsymbol{b}} \\
W_{\boldsymbol{B}_0\boldsymbol{F}} & W_{\boldsymbol{B}_0\boldsymbol{H}} & W_{\boldsymbol{B}_0J} & W_{\boldsymbol{B}_0\boldsymbol{D}_0} & W_{\boldsymbol{B}_0\boldsymbol{B}_0} & W_{\boldsymbol{B}_0\boldsymbol{d}} & W_{\boldsymbol{B}_0\boldsymbol{b}} \\
W_{\boldsymbol{dF}} & W_{\boldsymbol{dH}} & W_{\boldsymbol{d}J} & W_{\boldsymbol{dD}_0} & W_{\boldsymbol{dB}_0} & W_{\boldsymbol{dd}} & W_{\boldsymbol{db}} \\
W_{\boldsymbol{bF}} & W_{\boldsymbol{bH}} & W_{\boldsymbol{b}J} & W_{\boldsymbol{bD}_0} & W_{\boldsymbol{bB}_0} & W_{\boldsymbol{bd}} & W_{\boldsymbol{bb}}
\end{bmatrix}. \tag{27}$$

3.5.2. A simple convex multi-variable electro-magneto-mechanical constitutive model

As an example, a simple energy functional which complies with the definition of multi-variable convexity in equation (21) can be defined as

$$W_1 = \mu_1 II_{\boldsymbol{F}} + \mu_2 II_{\boldsymbol{H}} + f(J) + \frac{1}{2\varepsilon_1} II_{\boldsymbol{D}_0} + \frac{1}{2\varepsilon_2} II_{\boldsymbol{d}} + \frac{1}{2\hat{\mu}_1} II_{\boldsymbol{B}_0} + \frac{1}{2\hat{\mu}_2} II_{\boldsymbol{b}}, \tag{28}$$

where $II_{(\bullet)}$ denotes the squared of the Euclidean norm for vector entities and the Frobenius norm for second order tensors. A possible definition of the (convex) function $f(J)$ in equation (28) could be [41]

$$f(J) = -2(\mu_1 + 2\mu_2) \ln J + \frac{\kappa}{2} (J-1)^2 . \tag{29}$$

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Moreover, multi-variable convexity requires positiveness of all the material parameters in equations (28) and (29). Among those, μ_1, μ_2 and κ have unit of stress, namely $[N/m^2]$, ε_1 and ε_2, units of electric permittivity, namely $[N/V^2]$ and $\hat{\mu}_1$ and $\hat{\mu}_2$ units of magnetic permeability, namely $[N/A^2]$. These material parameters are related to the shear modulus μ, the first Lamé parameter $\hat{\lambda}$, the relative electric permittivity ε_r and the relative magnetic permeability $\hat{\mu}_r$ of the material in the reference configuration as

$$\mu = 2\mu_1 + 2\mu_2; \qquad \hat{\lambda} = \kappa + 4\mu_2; \qquad \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} = \frac{1}{\varepsilon_r \varepsilon_0}; \qquad \frac{1}{\hat{\mu}_1} + \frac{1}{\hat{\mu}_2} = \frac{1}{\hat{\mu}_r \hat{\mu}_0}, \tag{30}$$

where ε_0 and $\hat{\mu}_0$ denote the electric permittivity and magnetic permeability of the vacuum, respectively, with $\varepsilon_0 = 8.854 \times 10^{-12}\, N/V^2$ and $\hat{\mu}_0 = 1.256 \times 10^{-6}\, N/A^2$.

The work conjugates in (24) for the internal energy functional in (28) are obtained as

$$\begin{aligned} &\boldsymbol{\Sigma}_{\boldsymbol{F}} = 2\mu_1 \boldsymbol{F}; \quad \boldsymbol{\Sigma}_{\boldsymbol{H}} = 2\mu_2 \boldsymbol{H}; \quad \Sigma_J = f'(J); \\ &\boldsymbol{\Sigma}_{\boldsymbol{D}_0} = \frac{1}{\varepsilon_2} \boldsymbol{D}_0; \quad \boldsymbol{\Sigma}_{\boldsymbol{d}} = \frac{1}{\varepsilon_1} \boldsymbol{d}; \\ &\boldsymbol{\Sigma}_{\boldsymbol{B}_0} = \frac{1}{\varepsilon_2} \boldsymbol{B}_0; \quad \boldsymbol{\Sigma}_{\boldsymbol{b}} = \frac{1}{\hat{\mu}_2} \boldsymbol{b}. \end{aligned} \tag{31}$$

The first Piola-Kirchhoff stress tensor and the material electric and magnetic fields for this constitutive model are obtained according to equation (26) as

$$\begin{aligned} \boldsymbol{P} =& 2\mu_1 \boldsymbol{F} + 2\mu_2 \boldsymbol{H} \times \boldsymbol{F} + f'(J)\boldsymbol{H} + \frac{1}{\varepsilon_2} \boldsymbol{d} \otimes \boldsymbol{D}_0 + \frac{1}{\hat{\mu}_2} \boldsymbol{b} \otimes \boldsymbol{B}_0; \\ \boldsymbol{E}_0 =& \frac{1}{\varepsilon_1} \boldsymbol{D}_0 + \frac{1}{\varepsilon_2} \boldsymbol{F}^T \boldsymbol{d}; \qquad \boldsymbol{H}_0 = \frac{1}{\hat{\mu}_1} \boldsymbol{B}_0 + \frac{1}{\hat{\mu}_2} \boldsymbol{F}^T \boldsymbol{b}. \end{aligned} \tag{32}$$

Finally, the Hessian operator $[\mathbb{H}_W]$ in above equation (27) adopts the following diagonal positive definite representation,

$$[\mathbb{H}_W] = \begin{bmatrix} 2\mu_1 \boldsymbol{\mathcal{I}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mu_2 \boldsymbol{\mathcal{I}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & f''(J) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\varepsilon_1} \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\varepsilon_2} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\hat{\mu}_1} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\hat{\mu}_2} \boldsymbol{I} \end{bmatrix}, \tag{33}$$

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where $\boldsymbol{I}$ is the second order identity tensor and $\boldsymbol{\mathcal{I}}$, the fourth order identity tensor, i.e. $\mathcal{I}_{iIjJ} = \delta_{ij}\delta_{IJ}$.

3.6. Additional set of conservation laws in nonlinear electro-magneto-mechanics

The definition of multi-variable convexity in equation (28) represents a suitable relaxation of the convexity criteria on the internal energy $e(\boldsymbol{\nabla}_0\boldsymbol{x}, \boldsymbol{D}_0, \boldsymbol{B_0})$ which is compatible with the Legendre-Hadamard condition (refer to Reference [1]) and which does not preclude buckling. This relaxation relies on the definition of the internal energy in terms of the extended set of variables $\mathcal{V}$ (22). Remarkably, each of the elements of the extended set $\mathcal{V}$ can be expressed as a first order conservation law, which enables to identify all the variables in $\mathcal{V}$ as conservation variables. For instance, equations (15a) and (15b) represent a set of conservation laws in terms of the conservation variables $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$. Moreover, the authors in Reference [36, 42, 48–52] have presented the conservation laws associated to the purely kinematic entities of the set $\mathcal{V}$ (22), namely $\{\boldsymbol{F}, \boldsymbol{H}, J\}$. The objective of this section is to briefly recall the conservation laws for $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and then, to present two completely new conservation laws for the remaining variables in $\mathcal{V}$ (22), namely $\boldsymbol{d}$ and $\boldsymbol{b}$.

3.6.1. Conservation of fibre, area and volume maps

The global form of the conservation laws associated to the kinematical entities of $\mathcal{V}$ (22), namely $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ can be expressed as (refer to [49])

$$\frac{d}{dt}\int_V \boldsymbol{F}\, dV - \int_{\partial V} \frac{1}{\rho_0}\boldsymbol{p} \otimes\, d\boldsymbol{A} = \boldsymbol{0}; \tag{34a}$$

$$\frac{d}{dt}\int_V \boldsymbol{H}\, dV - \int_{\partial V} \boldsymbol{F} \times \left(\frac{1}{\rho_0}\boldsymbol{p} \otimes d\boldsymbol{A}\right) = \boldsymbol{0}; \tag{34b}$$

$$\frac{d}{dt}\int_V J\, dV - \int_{\partial V} \boldsymbol{H} : \left(\frac{1}{\rho_0}\boldsymbol{p} \otimes d\boldsymbol{A}\right) = 0. \tag{34c}$$

The corresponding local differential equations and jump conditions to

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those in equation (34) are

$$\frac{\partial \boldsymbol{F}}{\partial t} - \text{DIV}\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{I}\right) = \boldsymbol{0}; \quad U\llbracket\boldsymbol{F}\rrbracket = -\left\llbracket\frac{1}{\rho_0}\boldsymbol{p}\right\rrbracket\otimes\boldsymbol{N}; \tag{35a}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} - \text{CURL}\left(\frac{1}{\rho_0}\boldsymbol{p}\times\boldsymbol{F}\right) = \boldsymbol{0}; \quad U\llbracket\boldsymbol{H}\rrbracket = -\boldsymbol{F}\times\left(\left\llbracket\frac{1}{\rho_0}\boldsymbol{p}\right\rrbracket\otimes\boldsymbol{N}\right); \tag{35b}$$

$$\frac{\partial J}{\partial t} - \text{DIV}\left(\frac{1}{\rho_0}\boldsymbol{H}^T\boldsymbol{p}\right) = 0; \quad U\llbracket J\rrbracket = -\boldsymbol{H}:\left(\left\llbracket\frac{1}{\rho_0}\boldsymbol{p}\right\rrbracket\otimes\boldsymbol{N}\right). \tag{35c}$$

3.7. *Conservation of* $\boldsymbol{d}$

In order to evaluate global and local conservation laws for the electromechanical variable $\boldsymbol{d}$ $(23)_a$, let us recall its definition $\boldsymbol{d} = \boldsymbol{F}\boldsymbol{D}_0$. In the regions of the domain V where the fields are smooth, it is possible to obtain

$$\begin{aligned}\frac{\partial \boldsymbol{d}}{\partial t} = \frac{\partial \boldsymbol{F}}{\partial t}\boldsymbol{D}_0 + \boldsymbol{F}\frac{\partial \boldsymbol{D}_0}{\partial t} &= \boldsymbol{\nabla}_0\left(\frac{1}{\rho_0}\boldsymbol{p}\right)\boldsymbol{D}_0 + \boldsymbol{F}\left(\text{CURL}\boldsymbol{H}_0 - \boldsymbol{J}_0\right)\\ &= \text{DIV}\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{D}_0\right) - \left(\text{DIV}\boldsymbol{D}_0\right)\left(\frac{1}{\rho_0}\boldsymbol{p}\right) + \boldsymbol{F}\left(\text{CURL}\boldsymbol{H}_0 - \boldsymbol{J}_0\right),\end{aligned} \tag{36}$$

where the local conservations laws for $\boldsymbol{F}$ and $\boldsymbol{D}_0$ in equations (35a) and (16b), respectively, have been conveniently substituted in above (36). Use of the involution equation for $\boldsymbol{D}_0$ $(17)_a$ into above equation (36) yields

$$\frac{\partial \boldsymbol{d}}{\partial t} = \text{DIV}\left(\boldsymbol{v}\otimes\boldsymbol{D}_0\right) - \frac{\rho_0^e}{\rho_0}\boldsymbol{p} + \boldsymbol{F}\left(\text{CURL}\boldsymbol{H}_0 - \boldsymbol{J}_0\right). \tag{37}$$

Let us now focus on the term $\boldsymbol{F}\text{CURL}\boldsymbol{H}_0$ in above equation (37). Consideration of the *curl-free* involution equation for $\boldsymbol{F}$ in (4) enables us to re-write this term as

$$\left(\boldsymbol{F}\text{CURL}\boldsymbol{H}_0\right)_i = F_{iI}\mathcal{E}_{IJK}\frac{\partial\left(\boldsymbol{H}_0\right)_K}{\partial X_J} = -\frac{\partial}{\partial X_J}\left(\mathcal{E}_{JIK}F_{iI}\left(\boldsymbol{H}_0\right)_K\right), \tag{38}$$

and hence,

$$\boldsymbol{F}\text{CURL}\boldsymbol{H}_0 = -\text{DIV}\left(\boldsymbol{F}\times\boldsymbol{H}_0\right). \tag{39}$$

Substitution of the above expression (39) into equation (37) finally yields the local conservation law for the variable $\boldsymbol{d}$ as

$$\frac{\partial \boldsymbol{d}}{\partial t} - \text{DIV}\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{D}_0 - \boldsymbol{F}\times\boldsymbol{H}_0\right) = -\boldsymbol{F}\boldsymbol{J}_0 - \frac{\rho_0^e}{\rho_0}\boldsymbol{p}. \tag{40}$$

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With the purpose of deriving the correct jump conditions for the variable $\boldsymbol{d}$ across a discontinuity surface with unit normal $\boldsymbol{N}$ and moving with velocity U, let us note first that the jump of any product operation $(\bullet)$ of two conservation variables $\boldsymbol{a}$ and $\boldsymbol{b}$ across the discontinuity can be obtained as

$$[\![\boldsymbol{a} \bullet \boldsymbol{b}]\!] = \bar{\boldsymbol{a}} \bullet [\![\boldsymbol{b}]\!] + [\![\boldsymbol{a}]\!] \bullet \bar{\boldsymbol{b}}, \tag{41}$$

where $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{b}}$ denote the average values of both variables at both sides of the discontinuity, i.e, $\bar{\boldsymbol{a}} = \frac{1}{2}\left(\boldsymbol{a}^+ + \boldsymbol{a}^-\right)$. Hence, above equation (41) can be applied in order to compute the jump of the variable $\boldsymbol{d}$ as

$$[\![\boldsymbol{d}]\!] = [\![\boldsymbol{F}\boldsymbol{D}_0]\!] = \bar{\boldsymbol{F}}[\![\boldsymbol{D}_0]\!] + [\![\boldsymbol{F}]\!]\bar{\boldsymbol{D}}_0. \tag{42}$$

Introduction of the jump conditions for both $\boldsymbol{D}_0$ (16b) and $\boldsymbol{F}$ (35a) into (42) gives

$$U[\![\boldsymbol{d}]\!] = \bar{\boldsymbol{F}}\left([\![\boldsymbol{H}_0]\!] \times \boldsymbol{N}\right) - \left([\![\frac{1}{\rho_0}\boldsymbol{p}]\!] \otimes \boldsymbol{N}\right)\bar{\boldsymbol{D}}_0. \tag{43}$$

Moreover, note that the second term of the right hand side of above equation (43) can be alternatively written as

$$\begin{aligned}
\left([\![\frac{1}{\rho_0}\boldsymbol{p}]\!] \otimes \boldsymbol{N}\right)\bar{\boldsymbol{D}}_0 &= [\![\left(\frac{1}{\rho_0}\boldsymbol{p}\right)(\boldsymbol{D}_0 \cdot \boldsymbol{N})]\!] - \left(\frac{1}{\rho_0}\bar{\boldsymbol{p}}\right)[\![\boldsymbol{D}_0 \cdot \boldsymbol{N}]\!] \\
&= [\![\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{D}_0]\!]\boldsymbol{N} - \left(\frac{1}{\rho_0}\bar{\boldsymbol{p}}\right)\left([\![\boldsymbol{D}_0]\!] \cdot \boldsymbol{N}\right) \\
&= [\![\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{D}_0]\!]\boldsymbol{N} - \underbrace{\left(\frac{1}{U\rho_0}\bar{\boldsymbol{p}}\right)\left([\![\boldsymbol{H}_0]\!] \times \boldsymbol{N}\right) \cdot \boldsymbol{N}}_{=\boldsymbol{0}} \\
&= [\![\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{D}_0]\!]\boldsymbol{N}.
\end{aligned} \tag{44}$$

Similarly, the first term on the right hand side of above equation (43) can

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be alternatively written as

$$\begin{aligned}
\bar{\boldsymbol{F}}\left(\left[\!\left[\boldsymbol{H}_0\right]\!\right]\times\boldsymbol{N}\right) &= -\left(\bar{\boldsymbol{F}}\boldsymbol{\times}\boldsymbol{N}\right)\left[\!\left[\boldsymbol{H}_0\right]\!\right] \\
&= -\left[\!\left[\left(\boldsymbol{F}\boldsymbol{\times}\boldsymbol{N}\right)\boldsymbol{H}_0\right]\!\right] + \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{N}\right]\!\right]\bar{\boldsymbol{H}}_0 \\
&= \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{H}_0\right]\!\right]\boldsymbol{N} - \frac{1}{U\rho_0}\left(\left(\left[\!\left[\boldsymbol{p}\right]\!\right]\otimes\boldsymbol{N}\right)\boldsymbol{\times}\boldsymbol{N}\right)\bar{\boldsymbol{H}}_0 \\
&= \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{H}_0\right]\!\right]\boldsymbol{N} - \underbrace{\frac{1}{U\rho_0}\left(\left[\!\left[\boldsymbol{p}\right]\!\right]\otimes\left(\boldsymbol{N}\times\boldsymbol{N}\right)\right)\bar{\boldsymbol{H}}_0}_{=\boldsymbol{0}} \\
&= \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{H}_0\right]\!\right]\boldsymbol{N}.
\end{aligned} \tag{45}$$

Introduction of both equations (44) and (45) into equation (43) finally yields the jump condition for $\boldsymbol{d}$

$$U\left[\!\left[\boldsymbol{d}\right]\!\right] = -\left(\left[\!\left[\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{D}_0\right]\!\right] - \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{H}_0\right]\!\right]\right)\boldsymbol{N}. \tag{46}$$

Combination of the local conservation law for $\boldsymbol{d}$ in (40) and the jump conditions in (46) finally gives the global conservation law for this variable as

$$\frac{d}{dt}\int_V \boldsymbol{d}\,dV - \int_V\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{D}_0\right)dV + \int_{\partial V}\boldsymbol{F}\boldsymbol{\times}\boldsymbol{H}_0\,dA = -\int_V\left(\boldsymbol{F}\boldsymbol{J}_0 + \frac{\rho_0^e}{\rho_0}\boldsymbol{p}\right)dV. \tag{47}$$

3.8. Conservation of $\boldsymbol{b}$

Following a similar approach to that in the preceding Section, it is possible to derive the local form of the conservation law for the variable $\boldsymbol{b}$ $(23)_b$ and its associated jump conditions as

$$\frac{\partial\boldsymbol{b}}{\partial t} - \mathrm{DIV}\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{B}_0 + \boldsymbol{F}\boldsymbol{\times}\boldsymbol{E}_0\right) = \boldsymbol{0};\quad U\left[\!\left[\boldsymbol{b}\right]\!\right] = -\left(\left[\!\left[\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{B}_0\right]\!\right] + \left[\!\left[\boldsymbol{F}\boldsymbol{\times}\boldsymbol{E}_0\right]\!\right]\right)\boldsymbol{N}. \tag{48}$$

Combination of the local conservation law for $\boldsymbol{b}$ and its associated jump condition in above equation (48) enables to obtain the global conservation

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law for $\boldsymbol{b}$ as

$$\frac{d}{dt}\int_V \boldsymbol{b}\,dV - \int_{\partial V}\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{B}_0\right)d\boldsymbol{A} - \int_{\partial V}\left(\boldsymbol{F}\times\boldsymbol{E}_0\right)d\boldsymbol{A} = \boldsymbol{0}. \tag{49}$$

3.9. Combined equations

Combining the results of the Sections above, a full set of first order conservation laws (7) can be established with vector of variables $\boldsymbol{\mathcal{U}}$, vector of fluxes $\boldsymbol{\mathcal{F}}_I$ and vector of source terms $\boldsymbol{\mathcal{S}}$ defined as

$$\boldsymbol{\mathcal{U}} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ \boldsymbol{H} \\ J \\ \boldsymbol{D}_0 \\ \boldsymbol{B}_0 \\ \boldsymbol{d} \\ \boldsymbol{b} \end{bmatrix}; \qquad \boldsymbol{\mathcal{F}}_I = \begin{bmatrix} \boldsymbol{\mathcal{F}}_I^{\boldsymbol{p}} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{F}} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{H}} \\ \boldsymbol{\mathcal{F}}_I^{J} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{D}_0} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{B}_0} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{d}} \\ \boldsymbol{\mathcal{F}}_I^{\boldsymbol{b}} \end{bmatrix}; \qquad \boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{f}_0 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ 0 \\ -\boldsymbol{J}_0 \\ \boldsymbol{0} \\ -\boldsymbol{F}\boldsymbol{J}_0 - \frac{\rho_0^e}{\rho_0}\boldsymbol{p} \\ \boldsymbol{0} \end{bmatrix}, \tag{50}$$

where the components of the flux vector $\boldsymbol{\mathcal{F}}_I$ are defined as

$$\begin{aligned}
\boldsymbol{\mathcal{F}}_I^{\boldsymbol{p}} &= -\boldsymbol{P}\boldsymbol{E}_I; & \boldsymbol{\mathcal{F}}_I^{\boldsymbol{F}} &= -\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{E}_I;\\
\boldsymbol{\mathcal{F}}_I^{\boldsymbol{H}} &= -\boldsymbol{F}\times\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{E}_I\right); & \boldsymbol{\mathcal{F}}_I^{J} &= -\boldsymbol{H}:\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{E}_I\right);\\
\boldsymbol{\mathcal{F}}_I^{\boldsymbol{D}_0} &= \boldsymbol{H}_0\times\boldsymbol{E}_I; & \boldsymbol{\mathcal{F}}_I^{\boldsymbol{B}_0} &= -\boldsymbol{E}_0\times\boldsymbol{E}_I;\\
\boldsymbol{\mathcal{F}}_I^{\boldsymbol{d}} &= -\frac{1}{\rho_0}\boldsymbol{p}(\boldsymbol{D}_0\cdot\boldsymbol{E}_I) + (\boldsymbol{F}\times\boldsymbol{H}_0)\,\boldsymbol{E}_I; & \boldsymbol{\mathcal{F}}_I^{\boldsymbol{b}} &= -\frac{1}{\rho_0}\boldsymbol{p}(\boldsymbol{B}_0\cdot\boldsymbol{E}_I) - (\boldsymbol{F}\times\boldsymbol{E}_0)\,\boldsymbol{E}_I,
\end{aligned} \tag{51}$$

where $\boldsymbol{P}$, $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are defined in terms of the extended sets $\mathcal{V}$ and $\Sigma_{\mathcal{V}}$ as in equations (26a), (26b) and (26c), respectively. Let us recall the quasilinear form for the system of hyperbolic equations with vector of variables $\boldsymbol{\mathcal{U}}$, vector of fluxes $\boldsymbol{\mathcal{F}}_I$ and vector of source terms $\boldsymbol{\mathcal{S}}$ in (50), that is

$$\frac{\partial\boldsymbol{\mathcal{U}}}{\partial t} + \underbrace{\boldsymbol{\mathcal{A}}_I\frac{\partial\boldsymbol{\mathcal{U}}}{\partial X_I}}_{\alpha^\star} = \boldsymbol{\mathcal{S}}, \tag{52}$$

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where the 'non-trivial' second term on the left-hand side of above equation (52), namely $\alpha^\star$ can be expressed as

$$\alpha^\star = -\underbrace{\left[\begin{array}{c:c} \mathbf{0}_{3\times3} & \\ \frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{F}}}{\partial \boldsymbol{p}} & \\ \frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{H}}}{\partial \boldsymbol{p}} & \widetilde{\boldsymbol{W}_I}^{\star\star} \\ \frac{\partial \boldsymbol{\mathcal{F}}_I^{J}}{\partial \boldsymbol{p}} & \\ \hdashline \mathbf{0}_{3\times3} & \\ \mathbf{0}_{3\times3} & \\ \frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{d}}}{\partial \boldsymbol{p}} & \widetilde{\boldsymbol{W}_I}^{\star} \\ \frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{b}}}{\partial \boldsymbol{p}} & \end{array}\right]}_{\boldsymbol{\mathcal{A}}_I} \frac{\partial}{\partial X_I}\begin{bmatrix} \cdot\boldsymbol{p} \\ :\boldsymbol{F} \\ :\boldsymbol{H} \\ J \\ \cdot\boldsymbol{D}_0 \\ \cdot\boldsymbol{B}_0 \\ \cdot\boldsymbol{d} \\ \cdot\boldsymbol{b} \end{bmatrix}, \tag{53}$$

where the entries in the first column of above matrix $\boldsymbol{\mathcal{A}}_I$ can be written in indicial notation as

$$\left[\frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{F}}}{\partial \boldsymbol{p}}\right]_{iKk} = \frac{1}{\rho_0}\delta_{ik}\delta_{IK}; \qquad \left[\frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{H}}}{\partial \boldsymbol{p}}\right]_{iKk} = \frac{1}{\rho_0}\mathcal{E}_{imk}\mathcal{E}_{KMI}F_{mM}; \qquad \left[\frac{\partial \boldsymbol{\mathcal{F}}_I^{J}}{\partial \boldsymbol{p}}\right]_{i} = \frac{1}{\rho_0}H_{iI};$$
$$\left[\frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{d}}}{\partial \boldsymbol{p}}\right]_{ik} = \frac{1}{\rho_0}\delta_{ik}\left(\boldsymbol{D}_0\cdot\boldsymbol{E}_I\right); \qquad \left[\frac{\partial \boldsymbol{\mathcal{F}}_I^{\boldsymbol{b}}}{\partial \boldsymbol{p}}\right]_{ik} = \frac{1}{\rho_0}\delta_{ik}\left(\boldsymbol{B}_0\cdot\boldsymbol{E}_I\right). \tag{54}$$

In addition, the inner products $(\cdot)$ and $(:)$ appearing in the vector of conservation variables in (53) represent standard contraction of repeated indices. For simplicity, the expressions for both matrices $\widetilde{\boldsymbol{W}_I}^{\star\star}$ and $\widetilde{\boldsymbol{W}_I}^{\star}$ (53) have been carried out in Appendix B (see equations (B.1) and (B.4), respectively).

4. Eigenvalue structure of the equations in nonlinear electro-magneto-mechanics

The objective of this Section is to analyse the eigenvalue structure of the combined set of conservation laws for electro-magneto-mechanics presented in equation (50) and demonstrate its hyperbolicity. Two scenarios will be considered: firstly, that defined by the system of conservation laws (50) and, second, a simplified case for low frequency simulations.

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4.1. Eigenvalue structure

The eigenvalues or wave speeds and the corresponding eigenvectors of the system of conservation laws in equation (50) can be determined by identifying possible plane wave solutions (in the absence of source terms) of the type

$$\boldsymbol{\mathcal{U}} = \phi(\boldsymbol{X} \cdot \boldsymbol{N} - c_\alpha t)\bar{\boldsymbol{\mathcal{U}}}_\alpha = \phi(\boldsymbol{X} \cdot \boldsymbol{N} - c_\alpha t) \begin{bmatrix} \bar{\boldsymbol{p}}_\alpha \\ \bar{\boldsymbol{F}}_\alpha \\ \bar{\boldsymbol{H}}_\alpha \\ \bar{J}_\alpha \\ \bar{\boldsymbol{D}}_{0\alpha} \\ \bar{\boldsymbol{B}}_{0\alpha} \\ \bar{\boldsymbol{d}}_\alpha \\ \bar{\boldsymbol{b}}_\alpha \end{bmatrix}, \qquad (55)$$

where c_α are the wave speeds corresponding to the eigenmode $\bar{\boldsymbol{\mathcal{U}}}_\alpha$ and $\boldsymbol{N}$, the normalised direction of propagation. The above expression in (55) for the set of conservation variables $\boldsymbol{\mathcal{U}}$ leads to an eigenvalue problem (refer to equation (9)) given by

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{N}}\bar{\boldsymbol{\mathcal{U}}}_\alpha = c_\alpha\bar{\boldsymbol{\mathcal{U}}}_\alpha; \qquad \boldsymbol{\mathcal{A}}_{\boldsymbol{N}} = \boldsymbol{\mathcal{A}}_I N_I, \qquad (56)$$

with the flux Jacobian matrix $\boldsymbol{\mathcal{A}}_I$ defined in equation (53). Particularisation of the involution equations for $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$ in equations (17) (for the case in which $\rho_0^e = 0$) to plane wave solutions of the type described in equation (55) yields

$$(\bar{\boldsymbol{D}}_{0\alpha} \cdot \boldsymbol{N})\phi' = 0; \qquad (\bar{\boldsymbol{B}}_{0\alpha} \cdot \boldsymbol{N})\phi' = 0. \qquad (57)$$

For non trivial solutions $\phi' \neq 0$, equation (57) leads to (refer to Figure 2)

$$(\bar{\boldsymbol{D}}_{0\alpha} \cdot \boldsymbol{N}) = 0; \qquad (\bar{\boldsymbol{B}}_{0\alpha} \cdot \boldsymbol{N}) = 0, \qquad (58)$$

which enables to re-define $\bar{\boldsymbol{\mathcal{U}}}_\alpha$ (55) as

$$\bar{\boldsymbol{\mathcal{U}}}_\alpha = \begin{bmatrix} \bar{\boldsymbol{p}}_\alpha \\ \bar{\boldsymbol{F}}_\alpha \\ \bar{\boldsymbol{H}}_\alpha \\ \bar{J}_\alpha \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{D}}_{0\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{B}}_{0\alpha} \\ \bar{\boldsymbol{d}}_\alpha \\ \bar{\boldsymbol{b}}_\alpha \end{bmatrix}, \qquad (59)$$

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with the projection operator $\boldsymbol{I}_{\boldsymbol{NN}}$ defined as

$$\boldsymbol{I}_{\boldsymbol{NN}} = \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}. \tag{60}$$

Particularisation of the local conservation equation for the deformation gradient tensor $\boldsymbol{F}$ (35a) to plane wave solutions of the type described in equation (55) yields (refer to Figure 2)

$$-c_\alpha \bar{\boldsymbol{F}}_\alpha \phi' - \frac{1}{\rho_0} \bar{\boldsymbol{p}}_\alpha \otimes \boldsymbol{N} \phi' = \boldsymbol{0} \Rightarrow \bar{\boldsymbol{F}}_\alpha = \bar{\boldsymbol{f}}_\alpha \otimes \boldsymbol{N}, \tag{61}$$

where $\bar{\boldsymbol{f}}_\alpha = -\frac{1}{c_\alpha \rho_0} \bar{\boldsymbol{p}}_\alpha$. Moreover, substitution of the above plane wave-like solutions (55) in conjunction with (61) into equations (35b), (35c) and (40), (48) yields

$$\begin{aligned} \bar{\boldsymbol{H}}_\alpha &= \boldsymbol{F} \times \left(\bar{\boldsymbol{f}}_\alpha \otimes \boldsymbol{N}\right); & \bar{J}_\alpha &= \boldsymbol{H} : \left(\bar{\boldsymbol{f}}_\alpha \otimes \boldsymbol{N}\right); \\ \bar{\boldsymbol{d}}_\alpha &= \bar{\boldsymbol{f}}_\alpha \left(\boldsymbol{D}_0 \cdot \boldsymbol{N}\right) + \boldsymbol{F}\left(\boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{D}}_{0_\alpha}\right); & \bar{\boldsymbol{b}}_\alpha &= \bar{\boldsymbol{f}}_\alpha \left(\boldsymbol{B}_0 \cdot \boldsymbol{N}\right) + \boldsymbol{F}\left(\boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{B}}_{0_\alpha}\right). \end{aligned} \tag{62}$$

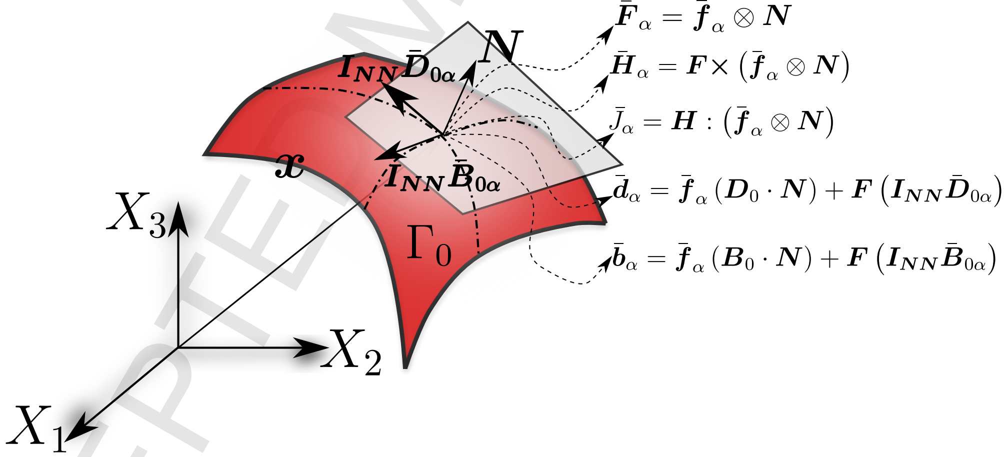

Figure 2: Eigenmodes $\bar{\boldsymbol{F}}_\alpha$, $\bar{\boldsymbol{H}}_\alpha$, $\bar{J}_\alpha$, $\boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{D}}_{0_\alpha}$, $\boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{B}}_{0_\alpha}$, $\bar{\boldsymbol{d}}_\alpha$ and $\bar{\boldsymbol{b}}_\alpha$ at a discontinuity surface Γ_0. The eigenmodes $\bar{\boldsymbol{D}}_{0_\alpha}$, $\bar{\boldsymbol{B}}_{0_\alpha}$ are perpendicular to the vector of propagation $\boldsymbol{N}$, thus complying with the involution equations (58).

The expressions in above equations (61)-(62), where the eigenmodes $\bar{\boldsymbol{F}}_\alpha$, $\bar{\boldsymbol{H}}_\alpha$, $\bar{J}_\alpha$, $\bar{\boldsymbol{d}}_\alpha$ and $\bar{\boldsymbol{b}}_\alpha$ have been expressed in terms of a reduced set of eigenmodes $\{\bar{\boldsymbol{f}}_\alpha, \boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{D}}_{0_\alpha}, \boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{B}}_{0_\alpha}\}$, enable to re-write and reduce the eigenvalue

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problem in (56) (with $\boldsymbol{\mathcal{A}}_{\boldsymbol{I}}$ defined as in equation (53)) to

$$c_\alpha \bar{\boldsymbol{\mathcal{U}}}^\star_\alpha = \boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}} \bar{\boldsymbol{\mathcal{U}}}^\star_\alpha, \tag{63}$$

in terms of the reduced set of eigenvectors $\bar{\boldsymbol{\mathcal{U}}}^\star_\alpha$ and of the reduced Jacobian matrix $\boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}}$ defined as

$$\bar{\boldsymbol{\mathcal{U}}}^\star_\alpha = \begin{bmatrix} \bar{\boldsymbol{p}}_\alpha \\ \bar{\boldsymbol{f}}_\alpha \\ \boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{D}}_{0\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}} \bar{\boldsymbol{B}}_{0\alpha} \end{bmatrix}; \quad \boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{\mathcal{C}}_{\boldsymbol{NN}} & \boldsymbol{\mathcal{Q}}_{\boldsymbol{N}}{}^T & \boldsymbol{\mathcal{T}}_{\boldsymbol{N}}{}^T \\ \frac{1}{\rho_0}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{W}\boldsymbol{\mathcal{T}}_{\boldsymbol{N}} & -\boldsymbol{W}\boldsymbol{R}^T & -\boldsymbol{W}\boldsymbol{\vartheta} \\ \boldsymbol{0} & \boldsymbol{W}\boldsymbol{\mathcal{Q}}_{\boldsymbol{N}} & \boldsymbol{W}\boldsymbol{\theta} & \boldsymbol{W}\boldsymbol{R} \end{bmatrix}, \tag{64}$$

where the second order tensor $\boldsymbol{W}$ in above equation (64) is defined as $W_{IJ} = \mathcal{E}_{IJK} N_K$. Moreover, the second order tensors $\boldsymbol{\mathcal{C}}_{\boldsymbol{NN}}$, $\boldsymbol{\mathcal{Q}}_{\boldsymbol{N}}$ and $\boldsymbol{\mathcal{T}}_{\boldsymbol{N}}$ in (64) are defined as

$$(\mathcal{C}_{NN})_{ij} = \mathcal{C}_{iIjJ} N_I N_J; \qquad (\mathcal{Q}_N)_{Ij} = \mathcal{Q}_{IjJ} N_J; \qquad (\mathcal{T}_N)_{Ij} = \mathcal{T}_{IjJ} N_J, \tag{65}$$

and with the different constitutive tensors $\boldsymbol{\mathcal{C}}$ (fourth order elasticity tensor), $\boldsymbol{\mathcal{Q}}$ (third order piezoelectric tensor), $\boldsymbol{\mathcal{T}}$ (third order piezomagnetic tensor), $\boldsymbol{\theta}$ (second order dielectric tensor), $\boldsymbol{R}$ (second order magnetoelectric tensor) and $\boldsymbol{\vartheta}$ (second order permeability tensor) featuring in equations (64) and (65) defined in equations (C.2), (C.5), (C.8),(C.11) , (C.15) and (C.13), respectively, in terms of the components of the Hessian operator $[\mathbb{H}_W]$ (27).

Careful analysis of the matrix $\boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}}$ in (64) enables to identify the following multiplicative decomposition of $\boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}}$ as

$$\boldsymbol{\mathcal{A}}^\star_{\boldsymbol{N}} = \boldsymbol{S}\boldsymbol{U}, \tag{66}$$

where the second order tensors $\boldsymbol{S}$ and $\boldsymbol{U}$ are defined as

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W} & \boldsymbol{0} \end{bmatrix}; \qquad \boldsymbol{U} = \begin{bmatrix} \frac{1}{\rho_0}\boldsymbol{I} & \boldsymbol{0}_{9\times 9} \\ \boldsymbol{0}_{9\times 9} & \boldsymbol{Q} \end{bmatrix}, \tag{67}$$

with $\boldsymbol{Q}$ the so-called electro-magneto-mechanical acoustic tensor, defined as

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{\mathcal{C}}_{\boldsymbol{NN}} & \boldsymbol{\mathcal{Q}}_{\boldsymbol{N}}{}^T & \boldsymbol{\mathcal{T}}_{\boldsymbol{N}}{}^T \\ \boldsymbol{\mathcal{Q}}_{\boldsymbol{N}} & \boldsymbol{\theta} & \boldsymbol{R} \\ \boldsymbol{\mathcal{T}}_{\boldsymbol{N}} & \boldsymbol{R}^T & \boldsymbol{\vartheta} \end{bmatrix}. \tag{68}$$

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Following a fundamental theorem in the theory of hyperbolic equations [55], summarised in Appendix D for completeness, given the symmetric nature of $\boldsymbol{S}$ $(67)_a$, a sufficient condition for the existence of real eigenvalues for the above Jacobian matrix $\boldsymbol{\mathcal{A}}^{\star}_{\boldsymbol{N}}$ (refer to (64) and (66)) would be the symmetric positive definiteness of $\boldsymbol{U}$ $(67)_b$ and hence, of $\boldsymbol{Q}$ (68). Moreover, positive definiteness of the electro-magneto-mechanical acoustic tensor $\boldsymbol{Q}$ can be shown by noticing the following relationship between $\boldsymbol{Q}$ and the Hessian operator $[\mathbb{H}_W]$, namely

$$\begin{bmatrix} \bar{\boldsymbol{f}}_{\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{D}}_{0_\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{B}}_{0_\alpha} \end{bmatrix}^T \boldsymbol{Q} \begin{bmatrix} \bar{\boldsymbol{f}}_{\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{D}}_{0_\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{B}}_{0_\alpha} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{F}}_{\alpha} \\ \bar{\boldsymbol{H}}_{\alpha} \\ \bar{J}_{\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{D}}_{0\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{B}}_{0\alpha} \\ \bar{\boldsymbol{d}}_{\alpha} \\ \bar{\boldsymbol{b}}_{\alpha} \end{bmatrix}^T [\mathbb{H}_W] \begin{bmatrix} \bar{\boldsymbol{F}}_{\alpha} \\ \bar{\boldsymbol{H}}_{\alpha} \\ \bar{J}_{\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{D}}_{0\alpha} \\ \boldsymbol{I}_{\boldsymbol{NN}}\bar{\boldsymbol{B}}_{0\alpha} \\ \bar{\boldsymbol{d}}_{\alpha} \\ \bar{\boldsymbol{b}}_{\alpha} \end{bmatrix} > 0. \tag{69}$$

As can be seen, the positive definiteness of the electro-magneto-mechanical acoustic tensor $\boldsymbol{Q}$ is implied by the positive definiteness of the extended Hessian operator $[\mathbb{H}_W]$, always satisfied by convex multi-variable energy functionals (21). Results in this Section enable the extension of the multi-variable convexity concepts presented in [1] to more general 'dynamic' and 'electro-magneto-mechanics' scenarios.

Remark 1. Equation (69) makes it possible to highlight the relationship between multi-variable convexity, ellipticity (rank-one convexity) and the Legendre-Hadamard condition (existence of real wave speeds). Notice first that it is possible to re-write the first term on the left hand side of (69) as

$$\begin{bmatrix} \bar{\boldsymbol{f}}_{\alpha} \otimes \boldsymbol{N} : & \boldsymbol{N}_{\perp,1}\cdot & \boldsymbol{N}_{\perp,2}\cdot \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{Q}}^T & \boldsymbol{\mathcal{T}}^T \\ \boldsymbol{\mathcal{Q}} & \boldsymbol{\theta} & \boldsymbol{R} \\ \boldsymbol{\mathcal{T}} & \boldsymbol{R}^T & \boldsymbol{\vartheta} \end{bmatrix} \begin{bmatrix} :\bar{\boldsymbol{f}}_{\alpha} \otimes \boldsymbol{N} \\ \boldsymbol{N}_{\perp,1} \\ \boldsymbol{N}_{\perp,2} \end{bmatrix} > 0, \tag{70}$$

where $\boldsymbol{N}_{\perp,1}$ and $\boldsymbol{N}_{\perp,2}$ represent two arbitrary vectors orthogonal to $\boldsymbol{N}$. Above expression (70) is a generalisation of the concept of ellipticity [24] to the case of electro-magneto-mechanics. Moreover, this equation, in conjunction with (69) links this property to the satisfaction of multi-variable convexity and, as a result of Appendix D, to the Legendre-Hadamard condition.

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Finally, notice that expression (70) is nothing more than the generalisation to electro-magneto-mechanical scenarios of the simpler electro-mechanical condition

$$\left[\bar{\boldsymbol{f}}_\alpha \otimes \boldsymbol{N} : \quad \boldsymbol{N}_{\perp,1}\cdot\right] \begin{bmatrix} \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{Q}}^T \\ \boldsymbol{\mathcal{Q}} & \boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} : \bar{\boldsymbol{f}}_\alpha \otimes \boldsymbol{N} \\ \boldsymbol{N}_{\perp,1} \end{bmatrix} > 0. \tag{71}$$

Above equation (71) is identical to Remark 4 in Reference [1], which establishes the relationship between multi-variable convexity and ellipticity.

4.2. Low frequency scenarios. Constrained case

In this Section, a particular scenario, interesting from the point of view of possible engineering applications characterised by the absence of current and electric charge densities ($\boldsymbol{J}_0 = \boldsymbol{0}$ ad $\rho_0^e = 0$) and by low frequency electromagnetic waves is analysed. This scenario enables the analysis of the system of conservation laws presented in equation (50) to be dramatically simplified. Specifically, the Maxwell equations (refer to both (16a) and (16b)) are now treated as constraints[3], as shown in Reference [67]. The resulting system of hyperbolic equations is then

$$\boldsymbol{\mathcal{U}} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ \boldsymbol{H} \\ J \\ \boldsymbol{d} \\ \boldsymbol{b} \end{bmatrix}; \qquad \boldsymbol{\mathcal{F}}_I = -\begin{bmatrix} \boldsymbol{P}\boldsymbol{E}_I \\ \frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{E}_I \\ \boldsymbol{F} \times \left(\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{E}_I\right) \\ \boldsymbol{H} : \left(\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{E}_I\right) \\ \frac{1}{\rho_0}\boldsymbol{p}(\boldsymbol{D}_0 \cdot \boldsymbol{E}_I) \\ \frac{1}{\rho_0}\boldsymbol{p}(\boldsymbol{B}_0 \cdot \boldsymbol{E}_I) \end{bmatrix}; \qquad \boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{f}_0 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ 0 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \tag{72}$$

subjected to the following constraints (different to the concept of involutions [53])

$$\mathrm{CURL}\boldsymbol{E}_0 = \boldsymbol{0}; \qquad \mathrm{CURL}\boldsymbol{H}_0 = \boldsymbol{0}. \tag{73}$$

For the constrained set of first hyperbolic equations in equation (72), the associated eigenvalue problem can be obtained following a similar procedure to that presented in Section (4.1) for the unconstrained case, yielding a similar expression for the eigenvalue problem as that presented in equation (64),

[3]For low frequency scenarios, it is customary to adopt the simplification $\frac{\partial \boldsymbol{D}_0}{\partial t} \approx \boldsymbol{0}$ and $\frac{\partial \boldsymbol{B}_0}{\partial t} \approx \boldsymbol{0}$.

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now subjected to the constraints in equation (73), that is

$$c_\alpha \begin{bmatrix} \bar{\boldsymbol{p}}_\alpha \\ \bar{\boldsymbol{f}}_\alpha \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{\mathcal{C}_{NN}} & \boldsymbol{\mathcal{Q}_N}^T & \boldsymbol{\mathcal{T}_N}^T \\ \frac{1}{\rho_0}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{W}\boldsymbol{\mathcal{T}_N} & -\boldsymbol{W}\boldsymbol{R}^T & -\boldsymbol{W}\boldsymbol{\vartheta} \\ \boldsymbol{0} & \boldsymbol{W}\boldsymbol{\mathcal{Q}_N} & \boldsymbol{W}\boldsymbol{\theta} & \boldsymbol{W}\boldsymbol{R} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{p}}_\alpha \\ \bar{\boldsymbol{f}}_\alpha \\ \bar{\boldsymbol{D}}_{0\alpha} \\ \bar{\boldsymbol{B}}_{0\alpha} \end{bmatrix} = \boldsymbol{0}. \tag{74}$$

Notice that the last two rows in above equation (74) are in fact constraints imposed on the eigenmodes $\bar{\boldsymbol{D}}_{0\alpha}$ and $\bar{\boldsymbol{B}}_{0\alpha}$. These two rows can alternatively be written in terms of the auxiliary scalar parameters $\lambda_{\boldsymbol{D}_0}$ and $\lambda_{\boldsymbol{B}_0}$ as

$$\begin{aligned} \boldsymbol{\mathcal{T}_N}\bar{\boldsymbol{f}}_\alpha + \boldsymbol{R}^T\bar{\boldsymbol{D}}_{0\alpha} + \boldsymbol{\vartheta}\bar{\boldsymbol{B}}_{0\alpha} &= \lambda_{\boldsymbol{B}_0}\boldsymbol{N}; \\ \boldsymbol{\mathcal{Q}_N}\bar{\boldsymbol{f}}_\alpha + \boldsymbol{\theta}\bar{\boldsymbol{D}}_{0\alpha} + \boldsymbol{R}\bar{\boldsymbol{B}}_{0\alpha} &= \lambda_{\boldsymbol{D}_0}\boldsymbol{N}. \end{aligned} \tag{75}$$

Notice that the scalar parameters $\lambda_{\boldsymbol{D}_0}$ and $\lambda_{\boldsymbol{B}_0}$ in above equation (75) are used to indicate the collinear nature between the vector $\boldsymbol{N}$ and the vectors $\boldsymbol{\mathcal{Q}_N}\bar{\boldsymbol{f}}_\alpha + \boldsymbol{\theta}\bar{\boldsymbol{D}}_{0\alpha} + \boldsymbol{R}\bar{\boldsymbol{B}}_{0\alpha}$ and $\boldsymbol{\mathcal{T}_N}\bar{\boldsymbol{f}}_\alpha + \boldsymbol{R}^T\bar{\boldsymbol{D}}_{0\alpha} + \boldsymbol{\vartheta}\bar{\boldsymbol{B}}_{0\alpha}$, respectively. From equation (75), it is possible to obtain both $\bar{\boldsymbol{B}}_{0\alpha}$ and $\bar{\boldsymbol{D}}_{0\alpha}$ as

$$\begin{aligned} \bar{\boldsymbol{B}}_{0\alpha} &= \boldsymbol{\vartheta}^{-1}\left(\lambda_{\boldsymbol{B}_0}\boldsymbol{N} - \boldsymbol{\mathcal{T}_N}\bar{\boldsymbol{f}}_\alpha - \boldsymbol{R}^T\bar{\boldsymbol{D}}_{0\alpha}\right); \\ \bar{\boldsymbol{D}}_{0\alpha} &= \boldsymbol{\theta}^{-1}\left(\lambda_{\boldsymbol{D}_0}\boldsymbol{N} - \boldsymbol{\mathcal{Q}_N}\bar{\boldsymbol{f}}_\alpha - \boldsymbol{R}\bar{\boldsymbol{B}}_{0\alpha}\right). \end{aligned} \tag{76}$$

Consideration of the involutions in equation (58) (namely $\bar{\boldsymbol{D}}_{0_\alpha} \cdot \boldsymbol{N} = \bar{\boldsymbol{B}}_{0_\alpha} \cdot \boldsymbol{N} = 0$) yields an expression for the two scalar parameters $\lambda_{\boldsymbol{B}_0}$ and $\lambda_{\boldsymbol{D}_0}$ as

$$\lambda_{\boldsymbol{B}_0} = \frac{\boldsymbol{N}\cdot\boldsymbol{\vartheta}^{-1}\left(\boldsymbol{\mathcal{T}_N}\bar{\boldsymbol{f}}_{\boldsymbol{\alpha}} + \boldsymbol{R}^T\bar{\boldsymbol{D}}_{0\alpha}\right)}{\boldsymbol{N}\cdot\boldsymbol{\vartheta}^{-1}\boldsymbol{N}}; \qquad \lambda_{\boldsymbol{D}_0} = \frac{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\left(\boldsymbol{\mathcal{Q}_N}\bar{\boldsymbol{f}}_{\boldsymbol{\alpha}} + \boldsymbol{R}\bar{\boldsymbol{B}}_{0\alpha}\right)}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}. \tag{77}$$

Substitution of equation (77) into equation (76) enables both eigenmodes $\bar{\boldsymbol{B}}_{0\alpha}$ and $\bar{\boldsymbol{D}}_{0\alpha}$ to be expressed in terms of the eigenmodes $\bar{\boldsymbol{f}}_\alpha$ as

$$\begin{aligned} \bar{\boldsymbol{B}}_{0\alpha} &= -\left[\boldsymbol{\vartheta} - \boldsymbol{B}\boldsymbol{R}^T\boldsymbol{\theta}^{-1}\boldsymbol{C}\boldsymbol{R}\right]^{-1}\left(\boldsymbol{\mathcal{T}_N}^\star - \boldsymbol{B}\boldsymbol{R}^T\boldsymbol{\theta}^{-1}\boldsymbol{\mathcal{Q}_N}\right)\bar{\boldsymbol{f}}_\alpha; \\ \bar{\boldsymbol{D}}_{0\alpha} &= -\left[\boldsymbol{\theta} - \boldsymbol{C}\boldsymbol{R}\boldsymbol{\vartheta}^{-1}\boldsymbol{B}\boldsymbol{R}^T\right]^{-1}\left(\boldsymbol{\mathcal{Q}_N}^\star - \boldsymbol{C}\boldsymbol{R}\boldsymbol{\vartheta}^{-1}\boldsymbol{\mathcal{T}_N}\right)\bar{\boldsymbol{f}}_\alpha, \end{aligned} \tag{78}$$

where the modified (coupled) matrices $\boldsymbol{\mathcal{Q}_N}^\star$, $\boldsymbol{\mathcal{T}_N}^\star$, $\boldsymbol{R}^\star_{\boldsymbol{\theta}}$ and $\boldsymbol{R}^\star_{\boldsymbol{\vartheta}}$ are defined

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as

$$\boldsymbol{\mathcal{Q}_N}^\star = \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\right]\boldsymbol{\mathcal{Q}_N}; \qquad \boldsymbol{\mathcal{T}_N}^\star = \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N}\otimes\boldsymbol{\vartheta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N}\cdot\boldsymbol{\vartheta}^{-1}\boldsymbol{N}}\right]\boldsymbol{\mathcal{T}_N};$$
$$\boldsymbol{B} = \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N}\otimes\boldsymbol{\vartheta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N}\cdot\boldsymbol{\vartheta}^{-1}\boldsymbol{N}}\right]; \qquad \boldsymbol{C} = \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\right]; \quad . \tag{79}$$

Finally, the first two equations (rows) in above equation (74) can be written as

$$c_\alpha \begin{bmatrix}\bar{\boldsymbol{p}}_\alpha\\ \bar{\boldsymbol{f}}_\alpha\end{bmatrix} + \begin{bmatrix}\boldsymbol{0} & \boldsymbol{\mathcal{C}_{NN}} & \boldsymbol{\mathcal{Q}_N}^T & \boldsymbol{\mathcal{T}_N}^T\\ \frac{1}{\rho_0}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{p}}_\alpha\\ \bar{\boldsymbol{f}}_\alpha\\ \bar{\boldsymbol{D}}_{0\alpha}\\ \bar{\boldsymbol{B}}_{0\alpha}\end{bmatrix} = \boldsymbol{0}, \tag{80}$$

with $\bar{\boldsymbol{D}}_{0\alpha}$ and $\bar{\boldsymbol{B}}_{0\alpha}$ defined in terms of $\bar{\boldsymbol{f}}_\alpha$ in equation (78). Finally, substitution of the second row into the first row in above equation (80) and consideration of the expressions for both $\bar{\boldsymbol{D}}_{0\alpha}$ and $\bar{\boldsymbol{B}}_{0\alpha}$ in terms of $\bar{\boldsymbol{f}}_\alpha$ (refer to equation (78)) yields the following eigenvalue problem

$$\left(\rho_0 c_\alpha^2 \boldsymbol{I} - \boldsymbol{Q}^\star\right)\bar{\boldsymbol{p}}_\alpha = \boldsymbol{0}, \tag{81}$$

where the generalised electro-magneto-mechanical acoustic tensor $\boldsymbol{Q}^\star$ in above equation (81) is defined as

$$\begin{aligned}\boldsymbol{Q}^\star = \underbrace{\boldsymbol{\mathcal{C}_{NN}}}_{\text{Classical acoustic tensor}} & \underbrace{-\boldsymbol{\mathcal{Q}_N}^T\left[\boldsymbol{\theta} - \boldsymbol{C}\boldsymbol{R}\boldsymbol{\vartheta}^{-1}\boldsymbol{B}\boldsymbol{R}^T\right]^{-1}\left(\boldsymbol{\mathcal{Q}_N}^\star - \boldsymbol{C}\boldsymbol{R}\boldsymbol{\vartheta}^{-1}\boldsymbol{\mathcal{T}_N}\right)}_{\text{Electric contribution to the acoustic tensor}}\\ & \underbrace{-\boldsymbol{\mathcal{T}_N}^T\left[\boldsymbol{\vartheta} - \boldsymbol{B}\boldsymbol{R}^T\boldsymbol{\theta}^{-1}\boldsymbol{C}\boldsymbol{R}\right]^{-1}\left(\boldsymbol{\mathcal{T}_N}^\star - \boldsymbol{B}\boldsymbol{R}^T\boldsymbol{\theta}^{-1}\boldsymbol{\mathcal{Q}_N}\right)}_{\text{Magnetic contribution to the acoustic tensor}}.\end{aligned} \tag{82}$$

Remark 2. Particularisation of the expression for the generalised electro-magneto-mechanical acoustic tensor $\boldsymbol{Q}^\star$ in equation (82) to the field of electromechanics yields the following expression for the generalised electromechanical acoustic tensor, also derived in Reference [25], as

$$\boldsymbol{Q}^\star = \boldsymbol{\mathcal{C}_{NN}} - \boldsymbol{\mathcal{Q}_N}^T\boldsymbol{\theta}^{-1}\left[\boldsymbol{I} - \frac{\left(\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\right]\boldsymbol{\mathcal{Q}_N}. \tag{83}$$

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Similarly, the expression for the generalised magnetomechanical acoustic tensor can be obtained as

$$\boldsymbol{Q}^{\star} = \boldsymbol{\mathcal{C}}_{\boldsymbol{N}\boldsymbol{N}} - \boldsymbol{\mathcal{T}_N}^T \boldsymbol{\vartheta}^{-1} \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N} \otimes \boldsymbol{\vartheta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N} \cdot \boldsymbol{\vartheta}^{-1}\boldsymbol{N}} \right] \boldsymbol{\mathcal{T}_N}. \tag{84}$$

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5. Symmetrisation of the equations in nonlinear electro-magneto-mechanics

For completeness, the objective of this Section is to show that multi-variable convexity of the internal energy enables the introduction of a generalised convex entropy function which, in turn, guarantees symmetrisation of the system of conservation laws [55, 68] presented in equation (50). This symmetrisation confirms (see Section 4.1) existence of travelling waves in the material, as shown by Hughes *et al.* [55] in the context of Computational Fluid Dynamics.

5.1. Entropy

Quasi-linear first order hyperbolic systems can be symmetrised with an appropriate change of variables [69], if there exists a generalised convex entropy function $S(\boldsymbol{\mathcal{U}})$, a corresponding entropy flux $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{U}})$ and a source term $W_{\boldsymbol{\mathcal{S}}}$ such that for all admissible solutions

$$\frac{\partial S}{\partial t} + \text{DIV}\boldsymbol{\Lambda} - W_{\boldsymbol{\mathcal{S}}} \geq 0. \tag{85}$$

The above inequality becomes an equality for smooth solutions in the absence of dissipative effects such as viscosity or heat flow (including Joule effect). It is now possible to define a new set of conjugate entropy variables $\boldsymbol{\mathcal{V}} = \frac{\partial S}{\partial \boldsymbol{\mathcal{U}}}$ for which the conservation laws (9) become

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{\mathcal{V}}}{\partial t} + \tilde{\boldsymbol{\mathcal{A}}}_I \frac{\partial \boldsymbol{\mathcal{V}}}{\partial X_I} = \boldsymbol{\mathcal{S}}, \tag{86}$$

where the symmetric system matrices $\boldsymbol{\mathcal{A}}_0$ and $\tilde{\boldsymbol{\mathcal{A}}}_I$ are

$$\boldsymbol{\mathcal{A}}_0 = \frac{\partial \boldsymbol{\mathcal{U}}}{\partial \boldsymbol{\mathcal{V}}} = \left[\frac{\partial^2 S}{\partial \boldsymbol{\mathcal{U}} \partial \boldsymbol{\mathcal{U}}} \right]^{-1}; \qquad \tilde{\boldsymbol{\mathcal{A}}}_I = \boldsymbol{\mathcal{A}}_I \boldsymbol{\mathcal{A}}_0 = \tilde{\boldsymbol{\mathcal{A}}}_I^T, \tag{87}$$

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and the entropy flux Λ_I satisfies

$$\frac{\partial \Lambda_I}{\partial \boldsymbol{\mathcal{U}}} = \boldsymbol{\mathcal{V}}^T \boldsymbol{\mathcal{A}}_I. \tag{88}$$

Consideration of the symmetry of the matrix $\tilde{\boldsymbol{\mathcal{A}}}_I$ enables to re-write the above conjugate set of equations for the entropy variables as

$$\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t} + \boldsymbol{\mathcal{A}}_I^T \frac{\partial \boldsymbol{\mathcal{V}}}{\partial X_I} = \boldsymbol{\mathcal{A}}_0^{-1} \boldsymbol{\mathcal{S}}. \tag{89}$$

5.2. Conservation of energy and generalised convex entropy

In order to derive the suitable generalised entropy and entropy flux functions for electro-magneto-mechanics, let us consider the following convex entropy function defined as

$$S(\boldsymbol{p}, \boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}) = \frac{1}{2\rho_0} \boldsymbol{p} \cdot \boldsymbol{p} + W(\boldsymbol{F}, \boldsymbol{H}, J, \boldsymbol{D}_0, \boldsymbol{B}_0, \boldsymbol{d}, \boldsymbol{b}), \tag{90}$$

which clearly represents the kinetic and internal energy per unit undeformed volume. The corresponding flux vector is defined as

$$\boldsymbol{\Lambda} = -\frac{1}{\rho_0} \boldsymbol{P}^T \boldsymbol{p} + \boldsymbol{S}_0; \qquad \boldsymbol{S}_0 = \boldsymbol{E}_0 \times \boldsymbol{H}_0, \tag{91}$$

where $\boldsymbol{S}_0$ in above equation (91) is the Poynting vector in the Lagrangian or material configuration. In addition, the source term $W_{\boldsymbol{\mathcal{S}}}$ is defined as

$$W_{\boldsymbol{\mathcal{S}}} = \boldsymbol{v} \cdot \boldsymbol{f}_0 - \boldsymbol{E}_0 \cdot \boldsymbol{J}_0. \tag{92}$$

Having defined the entropy S of the system, note first that the conjugate entropy variables are given by the derivatives of S as,

$$\boldsymbol{\mathcal{V}} = \frac{\partial S}{\partial \boldsymbol{\mathcal{U}}} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\Sigma}_{\boldsymbol{F}} \\ \boldsymbol{\Sigma}_{\boldsymbol{H}} \\ \Sigma_J \\ \boldsymbol{\Sigma}_{\boldsymbol{D}_0} \\ \boldsymbol{\Sigma}_{\boldsymbol{B}_0} \\ \boldsymbol{\Sigma}_{\boldsymbol{d}} \\ \boldsymbol{\Sigma}_{\boldsymbol{b}} \end{bmatrix}. \tag{93}$$

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Multiplication of each of the conservation laws in equation (50) by the corresponding conjugate variables (93), with the use of the involution equations (4) for the deformation gradient and its Co-factor and (17) for $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$ yields

$$\begin{aligned}\dot{S} &= \boldsymbol{v}\cdot\dot{\boldsymbol{p}} + \boldsymbol{\Sigma}_{\boldsymbol{F}} : \dot{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} : \dot{\boldsymbol{H}} + \Sigma_J \dot{J} \\ &\quad + \boldsymbol{\Sigma}_{\boldsymbol{D}_0}\cdot\dot{\boldsymbol{D}}_0 + \boldsymbol{\Sigma}_{\boldsymbol{B}_0}\cdot\dot{\boldsymbol{B}}_0 + \boldsymbol{\Sigma}_{\boldsymbol{d}}\cdot\dot{\boldsymbol{d}} + \boldsymbol{\Sigma}_{\boldsymbol{b}}\cdot\dot{\boldsymbol{b}} \\ &= \boldsymbol{v}\cdot\mathrm{DIV}\boldsymbol{P} + \boldsymbol{v}\cdot\boldsymbol{f}_0 + \boldsymbol{\Sigma}_{\boldsymbol{F}} : \boldsymbol{\nabla}_0\boldsymbol{v} + \boldsymbol{\Sigma}_{\boldsymbol{H}} : \mathrm{CURL}\left(\boldsymbol{v}\boldsymbol{\times}\boldsymbol{F}\right) + \Sigma_J\mathrm{DIV}\left(\boldsymbol{H}^T\boldsymbol{v}\right) \\ &\quad + \boldsymbol{\Sigma}_{\boldsymbol{D}_0}\cdot\dot{\boldsymbol{D}}_0 + \boldsymbol{\Sigma}_{\boldsymbol{B}_0}\cdot\dot{\boldsymbol{B}}_0 + \boldsymbol{\Sigma}_{\boldsymbol{d}}\cdot(\boldsymbol{\nabla}_0\boldsymbol{v}\boldsymbol{D}_0 + \boldsymbol{F}\dot{\boldsymbol{D}}_0) + \boldsymbol{\Sigma}_{\boldsymbol{b}}\cdot(\boldsymbol{\nabla}_0\boldsymbol{v}\boldsymbol{B}_0 + \boldsymbol{F}\dot{\boldsymbol{B}}_0).\end{aligned} \tag{94}$$

Introduction of the expressions for $\boldsymbol{P}$, $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ in equations (26a), (26b) and (26c) into above equation (94) yields

$$\dot{S} = \boldsymbol{v}\cdot\mathrm{DIV}\boldsymbol{P} + \boldsymbol{v}\cdot\boldsymbol{f}_0 + \boldsymbol{P} : \boldsymbol{\nabla}_0\boldsymbol{v} + \boldsymbol{E}_0\cdot\dot{\boldsymbol{D}}_0 + \boldsymbol{H}_0\cdot\dot{\boldsymbol{B}}_0. \tag{95}$$

Finally, use of equations (16a)-(16b) enable to re-write above equation (95) as

$$\dot{S} = \mathrm{DIV}\left(\boldsymbol{P}^T\boldsymbol{v} - \boldsymbol{S}_0\right) + \boldsymbol{v}\cdot\boldsymbol{f}_0 - \boldsymbol{E}_0\cdot\boldsymbol{J}_0, \tag{96}$$

with $\boldsymbol{S}_0$ defined in (91). Notice that use of the property

$$\boldsymbol{E}_0\cdot\mathrm{CURL}\boldsymbol{H}_0 - \boldsymbol{H}_0\cdot\mathrm{CURL}\boldsymbol{E}_0 = -\mathrm{DIV}\left(\boldsymbol{E}_0\times\boldsymbol{H}_0\right) \tag{97}$$

has been made in above equation (96). The statement in equation (94) is in fact a simplified version of the energy conservation law, or first law of thermodynamics, which, in the absence of the heat sources and heat flow is globally stated as

$$\frac{d}{dt}\int_V E\,dV = \int_{\partial V}\boldsymbol{t}_0\cdot\boldsymbol{v}\,dA + \int_V \boldsymbol{f}_0\cdot\boldsymbol{v}\,dV - \int_{\partial V}\boldsymbol{S}_0\cdot\,d\boldsymbol{A} - \int_V \boldsymbol{E}_0\cdot\boldsymbol{J}_0\,dV, \tag{98}$$

where E denotes the total energy per unit undeformed volume. The local version of this equation gives,

$$\frac{\partial E}{\partial t} - \mathrm{DIV}\left(\boldsymbol{P}^T\boldsymbol{v} - \boldsymbol{S}_0\right) = \boldsymbol{f}_0\cdot\boldsymbol{v} - \boldsymbol{E}_0\cdot\boldsymbol{J}_0. \tag{99}$$

From the comparison of both equations (96) and (99), it is therefore clear that for the reversible scenarios and under consideration of smooth solutions (i.e. in the absence of physical shocks), the generalised entropy can be simply identified as the total energy per unit undeformed volume, that is, $S = E$, coinciding with the notion of Hamiltonian per unit of undeformed volume.

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5.3. Symmetric hyperbolic equations for elastodynamics

With the definition of the conjugate entropy variables given above (93), it is now possible to derive a symmetric quasi-linear system for these variables. Consideration of the involution equations for $\boldsymbol{F}$ and $\boldsymbol{H}$ in (4) and those for $\boldsymbol{D}_0$ and $\boldsymbol{B}_0$ in (17), enables us to re-write the local form of the conservation law for linear momentum $\boldsymbol{p}$ (refer to equation (13) where first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is substituted by its equivalent representation in (26a)), the deformation gradient tensor $\boldsymbol{F}$ (refer to equation (35a)), the Co-factor $\boldsymbol{H}$ (refer to (35b)), the Jacobian J (refer to (35c)), $\boldsymbol{D}_0$ (refer to (16b) where $\boldsymbol{H}_0$ is substituted by its equivalent representation (26c)), $\boldsymbol{B}_0$ (refer to (16a) where $\boldsymbol{E}_0$ is substituted by its equivalent representation (26b)), $\boldsymbol{d}$ (refer to (40) where $\boldsymbol{H}_0$ is substituted by its equivalent representation (26c)) and $\boldsymbol{b}$ (refer to (48) where $\boldsymbol{E}_0$ is substituted by its equivalent representation (26b)) as

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} - \text{DIV}\boldsymbol{\Sigma}_{\boldsymbol{F}} + \boldsymbol{F}_{\boldsymbol{\Sigma}} \times \text{CURL}\boldsymbol{\Sigma}_{\boldsymbol{H}} - \boldsymbol{H}_{\boldsymbol{\Sigma}} \boldsymbol{\nabla}_0 \Sigma_J - \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{d}} \boldsymbol{D}_{0\boldsymbol{\Sigma}} - \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{b}} \boldsymbol{B}_{0\boldsymbol{\Sigma}} = \boldsymbol{f}_0 - \rho_0^e \boldsymbol{\Sigma}_{\boldsymbol{d}}, \tag{100}$$

which represents the conservation of linear momentum and

$$[\mathbb{H}_W]^{-1} \frac{\partial}{\partial t} \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{F}} \\ \boldsymbol{\Sigma}_{\boldsymbol{H}} \\ \Sigma_J \\ \boldsymbol{\Sigma}_{\boldsymbol{D}_0} \\ \boldsymbol{\Sigma}_{\boldsymbol{B}_0} \\ \boldsymbol{\Sigma}_{\boldsymbol{d}} \\ \boldsymbol{\Sigma}_{\boldsymbol{b}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\nabla}_0 \boldsymbol{v} \\ \boldsymbol{F}_{\Sigma} \boldsymbol{\times} \boldsymbol{\nabla}_0 \boldsymbol{v} \\ \boldsymbol{H}_{\boldsymbol{\Sigma}} : \boldsymbol{\nabla}_0 \boldsymbol{v} \\ \text{CURL}\boldsymbol{\Sigma}_{\boldsymbol{B}_0} + \boldsymbol{F}_{\boldsymbol{\Sigma}}{}^T \times \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{b}}{}^T \\ -\text{CURL}\boldsymbol{\Sigma}_{\boldsymbol{D}_0} - \boldsymbol{F}_{\boldsymbol{\Sigma}}{}^T \times \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{d}}{}^T \\ \boldsymbol{\nabla}_0 \boldsymbol{v} \otimes \boldsymbol{D}_0 - \boldsymbol{F}_{\boldsymbol{\Sigma}} \left(\text{CURL}\boldsymbol{\Sigma}_{\boldsymbol{B}_0} + \boldsymbol{F}_{\boldsymbol{\Sigma}}{}^T \times \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{b}}{}^T \right) \\ \boldsymbol{\nabla}_0 \boldsymbol{v} \otimes \boldsymbol{B}_0 + \boldsymbol{F}_{\boldsymbol{\Sigma}} \left(\text{CURL}\boldsymbol{\Sigma}_{\boldsymbol{D}_0} + \boldsymbol{F}_{\boldsymbol{\Sigma}}{}^T \times \boldsymbol{\nabla}_0 \boldsymbol{\Sigma}_{\boldsymbol{d}}{}^T \right) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ 0 \\ \boldsymbol{J}_0 \\ \boldsymbol{0} \\ \boldsymbol{F}_{\Sigma} \boldsymbol{J}_0 \\ \boldsymbol{0} \end{bmatrix}, \tag{101}$$

which represents the remaining set of conservation laws. Notice in above equation (101) that the notations $\boldsymbol{F}_{\boldsymbol{\Sigma}}$, $\boldsymbol{H}_{\boldsymbol{\Sigma}}$, $\boldsymbol{D}_{0\boldsymbol{\Sigma}}$ and $\boldsymbol{B}_{0\boldsymbol{\Sigma}}$ have been used here to explicitly indicate that these variables are now being evaluated from the conjugate variables in (93) using the reverse constitutive relationships, namely

$$\begin{aligned} \boldsymbol{F}_{\boldsymbol{\Sigma}} &= \boldsymbol{F}(\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J, \boldsymbol{\Sigma}_{\boldsymbol{D}_0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_0}, \boldsymbol{\Sigma}_{\boldsymbol{d}}, \boldsymbol{\Sigma}_{\boldsymbol{b}}); \\ \boldsymbol{H}_{\boldsymbol{\Sigma}} &= \boldsymbol{H}(\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J, \boldsymbol{\Sigma}_{\boldsymbol{D}_0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_0}, \boldsymbol{\Sigma}_{\boldsymbol{d}}, \boldsymbol{\Sigma}_{\boldsymbol{b}}); \\ \boldsymbol{D}_{0\boldsymbol{\Sigma}} &= \boldsymbol{D}_0(\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J, \boldsymbol{\Sigma}_{\boldsymbol{D}_0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_0}, \boldsymbol{\Sigma}_{\boldsymbol{d}}, \boldsymbol{\Sigma}_{\boldsymbol{b}}); \\ \boldsymbol{B}_{0\boldsymbol{\Sigma}} &= \boldsymbol{B}_0(\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J, \boldsymbol{\Sigma}_{\boldsymbol{D}_0}, \boldsymbol{\Sigma}_{\boldsymbol{B}_0}, \boldsymbol{\Sigma}_{\boldsymbol{d}}, \boldsymbol{\Sigma}_{\boldsymbol{b}}). \end{aligned} \tag{102}$$

The reader is referred to Appendix A for a better understanding of the tensor cross product operation $\boldsymbol{\times}$ and of the different cross products $\times$

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between second order tensors featuring in equations (100) and (101) (see [41, 49, 58] for further details). The symmetric nature of this system is more easily appreciated combining the above set of equations (100) and (101) to give

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{\mathcal{V}}}{\partial t} + \underbrace{\tilde{\boldsymbol{\mathcal{A}}}_I \frac{\partial \boldsymbol{\mathcal{V}}}{\partial X_I}}_{\beta^\star} = \boldsymbol{\mathcal{S}}, \tag{103}$$

where matrix $\boldsymbol{\mathcal{A}}_0$ in above equation (103) is defined as

$$\boldsymbol{\mathcal{A}}_0 = \begin{bmatrix} \rho_0 \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & [\mathbb{H}_W]^{-1} \end{bmatrix}, \tag{104}$$

with $\mathbb{H}_W$ defined in equation (27). It is therefore certain that the matrix $\boldsymbol{\mathcal{A}}_0$ is symmetric positive definite provided that the internal energy e (18) is convex multi-variable in the sense described in equation (21). Finally, the second term on the left-hand side of (103), namely $\beta^\star$ can be expanded in indicial notation as

$$\beta^\star = -\underbrace{\begin{bmatrix} \boldsymbol{0} & \delta_{ik}\delta_{KI} & \mathcal{E}_{ijk}\mathcal{E}_{IJK}F_{jJ} & H_{iI} & \boldsymbol{0} & \boldsymbol{0} & \delta_{ik}D_{0I} & \delta_{ik}B_{0I} \\ & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ & & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ & & & 0 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ & & & & \boldsymbol{0} & \mathcal{E}_{JIK} & \boldsymbol{0} & \mathcal{E}_{JIK}F_{kK} \\ & & & & & \boldsymbol{0} & \mathcal{E}_{JKI}F_{kK} & \boldsymbol{0} \\ & & \text{sym} & & & & \boldsymbol{0} & \mathcal{E}_{JIK}F_{iJ}F_{kK} \\ & & & & & & & \boldsymbol{0} \end{bmatrix}}_{\tilde{\boldsymbol{\mathcal{A}}}_I} \frac{\partial}{\partial X_I} \begin{bmatrix} v_k \\ [\boldsymbol{\Sigma_F}]_{kK} \\ [\boldsymbol{\Sigma_H}]_{kK} \\ \Sigma_J \\ \Sigma_{D_0 K} \\ \Sigma_{B_0 K} \\ \Sigma_{dk} \\ \Sigma_{bk} \end{bmatrix}. \tag{105}$$

The symmetrisation displayed in (103) confirms once again the existence of real waves (e.g. legendre-Hadamard condition) and opens up very interesting possibilities to use stabilised based formulations, as in [55] in the context of Computational Fluid Dynamics and in line with those published in References [36, 42, 48, 49, 52, 56] in the context of nonlinear elasticity.

6. Numerical examples

The objective of this Section is to analyse the behaviour of convex multi-variable constitutive laws in different scenarios. First, the convex multi-variable electro-magneto-mechanical constitutive law in equation (28) will be

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analysed. In particular, the speed of propagation of acoustic and electromechanical waves for this model will be studied for a particular experimental set-up. Finally, an interesting analysis of convex multi-variable constitutive models for the particular case of electromechanics will also be carried out in this Section.

6.1. Analysis of eigenvalues (wave speeds) for convex multi-variable electro-magneto-mechanical materials

Let us consider the constitutive model presented in equation (28), suitable for the description of the behaviour of electro-magnetrostrictive materials in the large deformation and large electric and magnetic fields scenarios. For this specific simple convex multi-variable constitutive law, the non-zero components of the matrix $\boldsymbol{\mathcal{A}}^{\star}_{\boldsymbol{N}}$ in equation (64) can be written as

$$\begin{aligned}
\boldsymbol{\mathcal{C}}_{\boldsymbol{N}\boldsymbol{N}} &= 2\mu_1\boldsymbol{I} + 2\mu_2\left[(\boldsymbol{F}\boldsymbol{N}\otimes\boldsymbol{F}\boldsymbol{N}) - (\boldsymbol{F}\boldsymbol{N}\cdot\boldsymbol{F}\boldsymbol{N})\boldsymbol{I} - \boldsymbol{F}\boldsymbol{F}^T + (\boldsymbol{F}:\boldsymbol{F})\boldsymbol{I}\right]\\
&\quad + f''(J)(\boldsymbol{H}\boldsymbol{N}\otimes\boldsymbol{H}\boldsymbol{N}) + \frac{1}{\varepsilon_2}\boldsymbol{D}^2_{0\boldsymbol{N}}\boldsymbol{I} + \frac{1}{\hat{\mu}_2}\boldsymbol{B}^2_{0\boldsymbol{N}}\boldsymbol{I};\\
\boldsymbol{\mathcal{Q}}_{\boldsymbol{N}}{}^T &= \frac{1}{\varepsilon_2}\left(\boldsymbol{d}\otimes\boldsymbol{N} + \boldsymbol{F}\left(\boldsymbol{D}_0\cdot\boldsymbol{N}\right)\right);\\
\boldsymbol{\mathcal{T}}_{\boldsymbol{N}}{}^T &= \frac{1}{\hat{\mu}_2}\left(\boldsymbol{b}\otimes\boldsymbol{N} + \boldsymbol{F}\left(\boldsymbol{B}_0\cdot\boldsymbol{N}\right)\right);\\
\boldsymbol{\theta} &= \frac{1}{\varepsilon_1}\boldsymbol{I} + \frac{1}{\varepsilon_2}\boldsymbol{F}^T\boldsymbol{F};\\
\boldsymbol{\vartheta} &= \frac{1}{\hat{\mu}_1}\boldsymbol{I} + \frac{1}{\hat{\mu}_2}\boldsymbol{F}^T\boldsymbol{F}.
\end{aligned} \tag{106}$$

In the reference configuration, $\boldsymbol{D}_0 = \boldsymbol{B}_0 = \boldsymbol{d} = \boldsymbol{b} = \boldsymbol{0}$, $\boldsymbol{F} = \boldsymbol{H} = \boldsymbol{I}$ and $J = 1$, an explicit expression for the the eigenvalues of the matrix $\boldsymbol{\mathcal{A}}^{\star}_{\boldsymbol{N}}$ can be obtained (e.g. by using a standard symbolic algebra computer package)

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as

$$
\begin{aligned}
&c_1 = c_2 = 0; \\
&c_l = c_3 = c_4 = -c_5 = -c_6 = \sqrt{\frac{(\hat{\mu}_2 + \hat{\mu}_1)(\varepsilon_2 + \varepsilon_1)}{\varepsilon_2 \varepsilon_1 \hat{\mu}_2 \hat{\mu}_2}} = c_0 c_r; \\
&c_s = c_7 = c_8 = -c_9 = -c_{10} = \sqrt{\frac{2(\mu_1 + \mu_2)}{\rho_0}} = \sqrt{\frac{\mu}{\rho_0}}; \\
&c_p = c_{11} = c_{12} = \sqrt{\frac{2(\mu_1 + \mu_2) + \lambda}{\rho_0}} = \sqrt{\frac{\mu + \lambda}{\rho_0}}.
\end{aligned}
\tag{107}
$$

The set of eigenvalues denoted as c_l represents the speed of light in the material. Therefore, c_0 and $c_r \leq 1$ in above equation (107) represent the speed of light in vacuum and the dimensionless relative speed of light in the material, respectively. The set of eigenvalues denoted as c_s corresponds to the speed of propagation of shear waves [48]. Finally, the set of eigenvalues c_p corresponds to the speed of propagation of pressure waves [48] in the material. For a Young's modulus $E = 10^5\, Pa$, Poisson ratio $\nu = 0.48$[4], relative permittivity $\varepsilon_r = 4$, relative permeability $\hat{\mu}_r = 4$ and a density of $720\, kg/m^3$, all of them in the reference configuration, the speed of propagation for shear, pressure and electromagnetic (speed of light) waves is

$$
c_s = 6.85\, m/s; \qquad c_p = 34.25\, m/s; \qquad c_r = 0.25. \tag{109}
$$

Let us now consider a different configuration characterised by a purely volumetric deformation gradient tensor defined in terms of the uniform stretch λ as $\boldsymbol{F} = \lambda \boldsymbol{I}$. Moreover, let the electric displacement and magnetic induction be defined in terms of their dimensionless moduli, $\tilde{D}_0$ and $\tilde{B}_0$ respectively, as $\boldsymbol{D}_0 = \sqrt{\mu\varepsilon}\tilde{D}_0 \boldsymbol{E}_2$ and $\boldsymbol{B}_0 = \sqrt{\mu\hat{\mu}}\tilde{B}_0 \boldsymbol{E}_3$, with the unit vectors $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ defined in equation (10). For this specific configuration, Figure 3 shows the speed of propagation of the pressure waves for different values of $\{\lambda, \tilde{D}_0, \tilde{B}_0\}$

[4] Given the Young's modulus E and the Poisson ratio ν, the shear modulus μ and the first Lamé parameter λ are obtained as

$$
\mu = \frac{E}{2(1+\nu)}; \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{108}
$$

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and for different orientations of the propagation vector $\boldsymbol{N}$, which can be spherically characterised in terms of the angles $0 \leq \alpha \leq 2\pi$ and $0 \leq \beta \leq \pi$ as

$$\boldsymbol{N} = \begin{bmatrix}\sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta\end{bmatrix}^T . \tag{110}$$

Similarly, Figure 4 displays the evolution of the speed of propagation of shear waves for different values of $\{\lambda, \tilde{D}_0, \tilde{B}_0\}$ and for different orientations of the propagation vector $\boldsymbol{N}$. The speed of light in the material c_l remains practically unaltered throughout the entire range of deformations. This confirms that the speed of propagation of acoustic waves are dramatically changed in the presence of an electric field and/or magnetic field. However, the speed of propagation of electromagnetic waves is almost unaffected by the deformation in the material and hence, not displayed.

6.2. Analysis of electro-mechanical constitutive models

In this Section, an analysis of the behaviour of electro-active materials described via convex and non-convex multi-variable constitutive models is carried out.

6.2.1. Analytical derivation of wave speeds in simple convex multi-variable constitutive models for dielectric elastomers

The aim of this Section is to obtain the explicit representation of the speed of propagation of the acoustic waves for the particular scenario of electromechanics, for a simple convex multi-variable constitutive model. For the constitutive model in equation (28) (and neglecting magnetic contributions), the expression for the generalised electromechanical acoustic tensor $\boldsymbol{Q}^\star$ (83) is

$$\begin{aligned}\boldsymbol{Q}^\star &= 2\mu_1\boldsymbol{I} + 2\mu_2\left[(\boldsymbol{FN}\otimes\boldsymbol{FN}) - (\boldsymbol{FN}\cdot\boldsymbol{FN})\boldsymbol{I} - \boldsymbol{FF}^T + (\boldsymbol{F}:\boldsymbol{F})\boldsymbol{I}\right] + f''(J)\,(\boldsymbol{HN}\otimes\boldsymbol{HN})\\ &\quad + \varepsilon_2\left(\frac{\boldsymbol{D}_0\cdot\boldsymbol{N}}{\varepsilon_2}\right)^2 - \left(\frac{\boldsymbol{D}_0\cdot\boldsymbol{N}}{\varepsilon_2}\right)^2 \underbrace{\boldsymbol{F}\boldsymbol{I}_{\boldsymbol{NN}}\boldsymbol{\theta}^{-1}\left[\boldsymbol{I} - \frac{(\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N})}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\right]\boldsymbol{F}^T}_{\boldsymbol{\mathcal{C}_D}} .\end{aligned} \tag{111}$$

In this Section, we restrict our analysis to the case where the propagation vector $\boldsymbol{N}$ is a principal direction of the deformation. In this case, let the eigenvalues of the deformation $\{\lambda_1, \lambda_2, \lambda_3\}$ be associated to the (unitary) principal directions in the deformed and undeformed configurations, denoted

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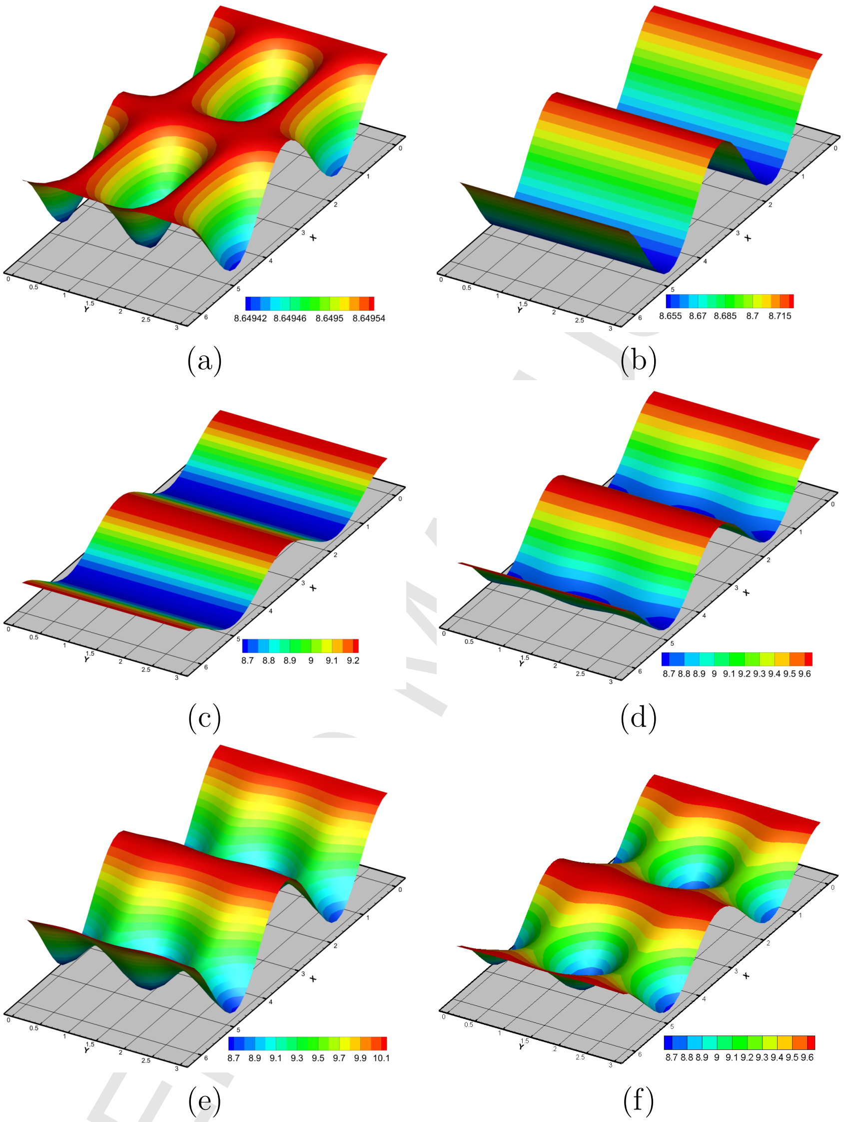

Figure 3: Representation of the pressure waves for the convex multi-variable constitutive model in equation (28) for different orientations of the propagation vector $\boldsymbol{N}$, spherically parametrised as in equation (110). The values for the set $\{\lambda, \tilde{D}_0, \tilde{B}_0\}$ are a) $\{0.54, 0.01, 1\}$; b) $\{0.58, 0.1, 1\}$; c) $\{0.62, 0.2, 1\}$; d) $\{1, 0.2, 1\}$; e) $\{1.4, 0.2, 1\}$ and f) $\{1.14, 0.3, 1\}$.

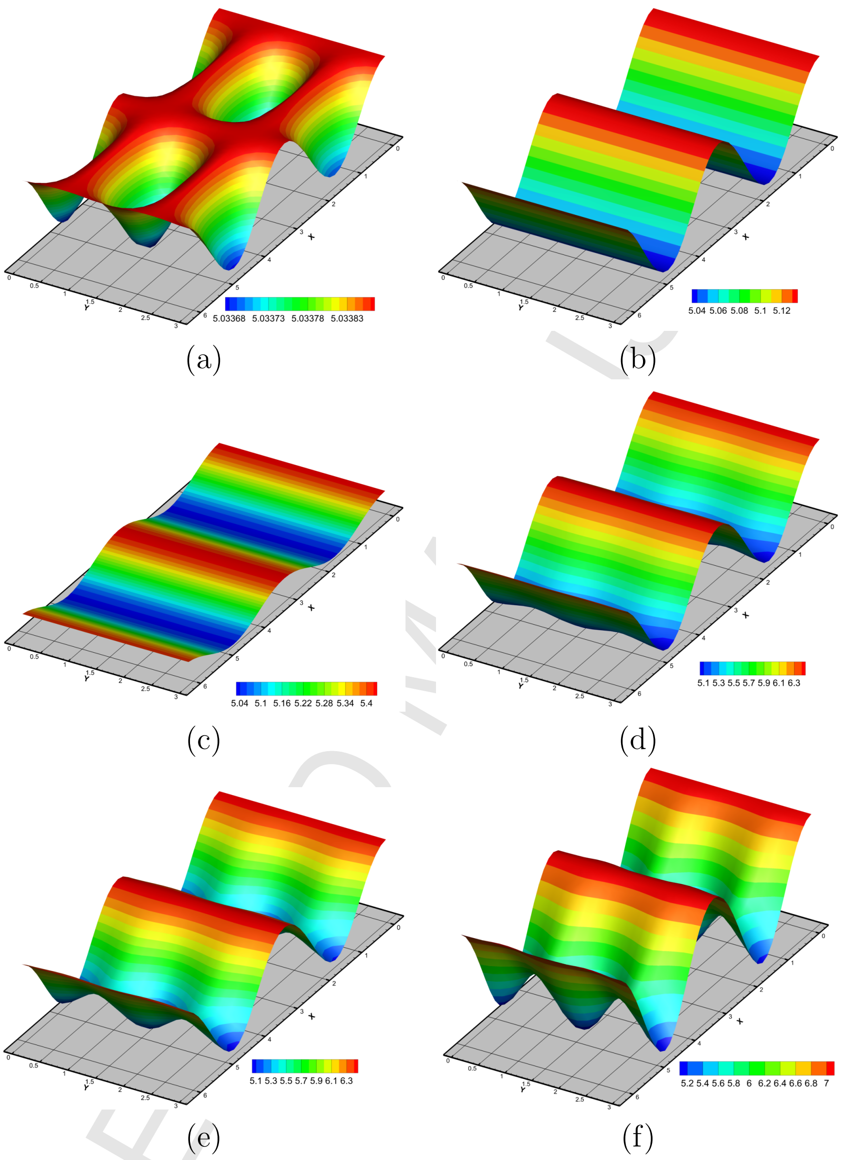

Figure 4: Representation of the shear waves for the convex multi-variable constitutive model in equation (28) for different orientations of the propagation vector $\boldsymbol{N}$, spherically parametrised as in equation (110). The values for the set $\{\lambda, \tilde{D}_0, \tilde{B}_0\}$ are a) $\{0.54, 0.01, 1\}$; b) $\{0.58, 0.1, 1\}$; c) $\{0.62, 0.2, 1\}$; d) $\{1, 0.2, 1\}$; e) $\{1.4, 0.2, 1\}$ and f) $\{1.14, 0.3, 1\}$.

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as $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{n}\}$ and $\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{N}\}$[5]. In that case, the following relations are easy to verify

$$\begin{aligned} \boldsymbol{F}\boldsymbol{T}_\alpha &= \lambda_\alpha \boldsymbol{t}_\alpha; & \boldsymbol{F}^T\boldsymbol{t}_\alpha &= \lambda_\alpha \boldsymbol{T}_\alpha; \\ \boldsymbol{F}\boldsymbol{N} &= \lambda_3 \boldsymbol{n}; & \boldsymbol{F}^T\boldsymbol{n} &= \lambda_3 \boldsymbol{N}; \\ \boldsymbol{H}\boldsymbol{T}_\alpha &= \frac{J}{\lambda_\alpha}\boldsymbol{t}_\alpha; & \boldsymbol{H}\boldsymbol{t}_\alpha &= \frac{J}{\lambda_\alpha}\boldsymbol{T}_\alpha; \\ \boldsymbol{H}\boldsymbol{N} &= \frac{J}{\lambda_3}\boldsymbol{n}; & \boldsymbol{H}\boldsymbol{n} &= \frac{J}{\lambda_3}\boldsymbol{N}. \end{aligned} \tag{112}$$

Introduction of the different identities in equation (112) into equation (111) for the specialised scenario considered yields the following expression for the generalised electromechanical acoustic tensor $\boldsymbol{Q}^*$

$$\begin{aligned} \boldsymbol{Q}^\star = {} & 2\mu_1 \boldsymbol{I} + 2\mu_2 \left(\left(\lambda_1^2 + \lambda_2^2\right) \boldsymbol{I} - \boldsymbol{\Lambda}_T \right) + f''(J) \left(\frac{J}{\lambda_3} \right)^2 \boldsymbol{n} \otimes \boldsymbol{n} \\ & + \varepsilon_2 \left(\frac{\boldsymbol{D}_0 \cdot \boldsymbol{N}}{\varepsilon_2} \right)^2 - \left(\frac{\boldsymbol{D}_0 \cdot \boldsymbol{N}}{\varepsilon_2} \right)^2 \underbrace{\boldsymbol{F}\boldsymbol{I}_{\boldsymbol{N}\boldsymbol{N}}\boldsymbol{\theta}^{-1} \left[\boldsymbol{I} - \frac{\left(\boldsymbol{N} \otimes \boldsymbol{\theta}^{-1}\boldsymbol{N}\right)}{\boldsymbol{N} \cdot \boldsymbol{\theta}^{-1}\boldsymbol{N}} \right] \boldsymbol{F}^T}_{\boldsymbol{\mathcal{C}_D}}, \end{aligned} \tag{113}$$

where the rank-two tensor $\boldsymbol{\Lambda}_T$ in above equation (113) is defined as

$$\boldsymbol{\Lambda}_T = \lambda_1^2 \boldsymbol{t}_1 \otimes \boldsymbol{t}_1 + \lambda_2^2 \boldsymbol{t} \otimes \boldsymbol{t}_2. \tag{114}$$

The wave speeds then can be obtained by solving the following equation (refer to equation (81))

$$\rho_0 c_\alpha^2 = \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{Q}^\star \bar{\boldsymbol{p}}_\alpha. \tag{115}$$

Combination of equation (115) and the expression for $\boldsymbol{Q}^\star$ in (113) finally

[5] The deformation gradient tensor is decomposed as $\boldsymbol{F} = \lambda_1 \boldsymbol{t}_1 \otimes \boldsymbol{T}_1 + \lambda_2 \boldsymbol{t}_2 \otimes \boldsymbol{T}_2 + \lambda_3 \boldsymbol{n} \otimes \boldsymbol{N}$ [59].

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yields

$$\rho_0 c_\alpha^2 = 2\mu_1 + 2\mu_2 \left(\left(\lambda_1^2 + \lambda_2^2\right) - \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\Lambda}_T \bar{\boldsymbol{p}}_\alpha \right) + f''(J) \left(\frac{J}{\lambda_3} \right)^2 \left(\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{n} \right)^2$$
$$+ \left(\frac{D_{0N}}{\varepsilon_2} \right)^2 \left(1 - \underbrace{\left(\sum_{\alpha=1}^{2} \lambda_\alpha \boldsymbol{t}_\alpha \cdot \bar{\boldsymbol{p}}_\alpha \right) \left(\left(\boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha \right) - \frac{\left(\boldsymbol{T}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N} \right) \left(\boldsymbol{\theta}^{-1} \boldsymbol{N} \cdot \boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha \right)}{\boldsymbol{N} \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N}} \right)}_{\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}_D} \bar{\boldsymbol{p}}_\alpha} \right), \tag{116}$$

where $D_{0N} = \boldsymbol{D}_0 \cdot \boldsymbol{N}$ and where the expression for $\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}_D} \bar{\boldsymbol{p}}_\alpha$ in above equation (116) has been obtained in equation (E.11). The first set of eigenvalues corresponding to $p-$waves is obtained by taking $\bar{\boldsymbol{p}}_\alpha = \boldsymbol{n}$ to give

$$c_{1,2} = \pm \sqrt{\frac{\mu_1 + \mu_2 \left(\lambda_1^2 + \lambda_2^2\right) + f''(J) \left(\frac{J}{\lambda_3} \right)^2 + \frac{D_{0N}^2}{\varepsilon_2}}{\rho_0}}. \tag{117}$$

The next four eigenvalues correspond to shear waves where the vibration takes place on the propagation plane. The corresponding velocity vectors are orthogonal to $\boldsymbol{n}$ and in the directions of the unit eigenvectors $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ of the rank-two tensor $\boldsymbol{\Lambda}_T$. Particularisation of the expression for $\boldsymbol{\mathcal{C}_D}$ in equation (E.11) to the case of shear waves gives

$$c_{3,4} = \pm \sqrt{\frac{\mu_1 + \mu_2 \lambda_2^2 + f''(J) \left(\frac{J}{\lambda_3} \right)^2 + \left(\frac{D_{0N}}{\varepsilon_2} \right)^2 \left(\varepsilon_2 - \lambda_1 \left(\alpha_1 + \alpha_2 \lambda_1^2 + \alpha_3 \left(\frac{J}{\lambda_1} \right)^2 \right) \right)}{\rho_0}};$$
$$c_{5,6} = \pm \sqrt{\frac{\mu_1 + \mu_2 \lambda_1^2 + f''(J) \left(\frac{J}{\lambda_3} \right)^2 + \left(\frac{D_{0N}}{\varepsilon_2} \right)^2 \left(\varepsilon_2 - \lambda_2 \left(\alpha_1 + \alpha_2 \lambda_1^2 + \alpha_3 \left(\frac{J}{\lambda_2} \right)^2 \right) \right)}{\rho_0}}, \tag{118}$$

where the expression for the coefficients $\{\alpha_1, \alpha_2, \alpha_3\}$ in above equation (118) has been determined in equation (E.7).

For more complex convex multi-variable constitutive models, the same procedure as that followed in the current Section can be followed in order to derive an explicit representation of the wave speeds for a specific direction of propagation coinciding with a principal direction of the deformation. However, the complexity of the model, associated to possible nonlinear effects

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such as electrostriction [70] or electric saturation [70] could lead to cumbersome algebraic manipulations.

6.2.2. Analysis of material stability in dielectric elastomers

The Legendre-Hadamard condition is strongly related to the material stability of the constitutive equations [26]. The objective of this Section is to analyse the material stability of some relevant convex and non-convex multi-variable constitutive models suitable for the description of isotropic electrostrictive [70] dielectric elastomers.

In particular, we focus on dielectric elastomers in the experimental set-up presented in Reference [2] and depicted in Figure 5. In this set-up, a thin film of an incompressible dielectric elastomer is subjected to an electric field applied across its thickness, i.e. in direction OX_3. Consequently, a uniform deformation in the plane of the film (perpendicular to the axis OX_3) characterised by the stretch λ, is observed. However, a slightly different scenario to that in Reference [2] is analysed in the present Section, in which the deformation in the OX_2 is completely constrained. Therefore, the deformation gradient tensor (accounting for the incompressibility constraint) can be expressed as

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}. \tag{119}$$

The behaviour of two constitutive models is studied in this Section. First, the following convex multi-variable constitutive model (slightly different to that presented in Reference [1]) is defined as

$$\begin{aligned} W_{el,1} = \mu_1 II_{\boldsymbol{F}} + \mu_2 II_{\boldsymbol{H}} + \frac{1}{2\varepsilon_1} II_{\boldsymbol{d}} + \mu_e \underbrace{\left(II_{\boldsymbol{F}}^2 + \frac{2}{\mu_e \varepsilon_e} II_{\boldsymbol{F}} II_{\boldsymbol{d}} + \frac{1}{\mu_e^2 \varepsilon_e^2} II_{\boldsymbol{d}} \right)}_{\text{Stabilised electrostrictive invariant}} \\ + \frac{1}{2\varepsilon_2} II_{\boldsymbol{D}_0} - 2(\alpha + 2\beta) \ln J + \frac{\kappa}{2} (J-1)^2 ; \end{aligned} \tag{120}$$

An alternative non-convex multi-variable constitutive model (slightly differ-

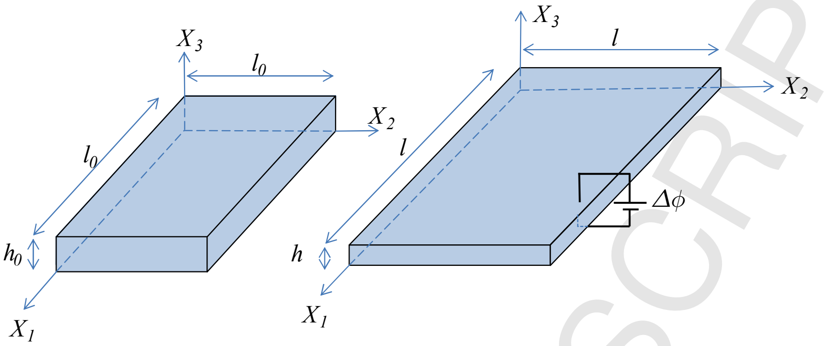

Figure 5: Experimental set-up. The application of a uniform electric potential gradient across the thickness of the incompressible dielectric elastomer film (parallel to the axis OX_3) of initial length and thickness l_0 and h_0 respectively, leads to a uniform axial expansion in the OX_1 direction and final thickness $h = 1/\lambda h_0$, with λ the stretch in the dielectric elastomer.

ent to that presented in Reference [1]) is defined as

$$W_{el,2} = \tilde{\mu}_1 II_{\boldsymbol{F}} + \tilde{\mu}_2 II_{\boldsymbol{H}} + \frac{1}{2\tilde{\varepsilon}_1} II_{\boldsymbol{d}} + \underbrace{\frac{2}{\tilde{\varepsilon}_e} II_{\boldsymbol{F}} II_{\boldsymbol{d}}}_{\text{Non-convex multi-variable invariant}} + \frac{1}{2\tilde{\varepsilon}_2} II_{\boldsymbol{D}_0} - 2\left(\alpha + 2\beta\right)\ln J + \frac{\tilde{\kappa}}{2}\left(J-1\right)^2 . \tag{121}$$

Notice that the only non-convex multi-variable invariant (it cannot be expressed as a convex combination of the elements in $\mathcal{V}$ (22)) in the constitutive model in equation (121) has been underlined. A convex regularisation (stabilisation) of that invariant has been applied on the constitutive model denoted as $W_{el,1}$ in above equation (120), which entails multi-variable convexity of the resulting stabilised invariant, as shown in Reference [1]. This non-convex multi-variable invariant and its regularised counterpart have been introduced in Reference [1] due to their ability to replicate physical electrostrictive behaviour in dielectric elastomers [70]. Both material parameters ε_e and $\tilde{\varepsilon}_e$ in (120) and (121) have been denoted as electrostrictive parameters. The elastic material parameter μ_e in above equation (120) can be understood as a mechanical stiffening parameter.

As shown in Reference [1], material characterisation in the reference configuration enables the material parameters $\{\mu_1, \mu_2, \mu_e, \kappa, \varepsilon_1, \varepsilon_2, \varepsilon_e\}$ and

$\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\kappa}, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_e\}$ to be related to the shear modulus μ, the first Lamé parameter $\hat{\lambda}$ and the electric permittivity ε in the reference configuration as

$$\mu = 2\mu_1 + 2\mu_2 + 12\mu_e; \qquad \hat{\lambda} = \kappa + 8\mu_e + 4\mu_2; \qquad \frac{1}{\varepsilon} = \frac{1}{\varepsilon_1} + \frac{12}{\varepsilon_e} + \frac{1}{\varepsilon_2};$$
$$\mu = 2\tilde{\mu}_1 + 2\tilde{\mu}_2; \qquad \hat{\lambda} = \tilde{\kappa} + 4\tilde{\mu}_2; \qquad \frac{1}{\varepsilon} = \frac{1}{\tilde{\varepsilon}_1} + \frac{12}{\tilde{\varepsilon}_e} + \frac{1}{\tilde{\varepsilon}_2}. \tag{122}$$

As shown in Reference [1], the following electristrictive and stiffening parameters, namely f_e and f_s can be defined for the convex multi-variable constitutive model as

$$12\mu_e = f_s\mu; \qquad \varepsilon_1 = f_e\varepsilon, \tag{123}$$

Analogously, an electrostrictive parameter $\hat{f}_e$ can be defined for the non-convex multi-variable constitutive model as

$$\tilde{\varepsilon}_1 = \tilde{f}_e\varepsilon, \tag{124}$$

Determination of the value of the variables $\{f_e, f_s\}$ for the constitutive model $W_{el,1}$ in (120) which minimises the error in the variables $\{\boldsymbol{P}, \lambda, \tilde{D}, \tilde{E}\}$ between both constitutive models $W_{el,1}$ in (120) and $W_{el,2}$ in (121) for the entire range of electric fields used in the proposed experimental set-up has been carried out. Figure 7 shows the qualitatively accurate agreement in the response for both convex and non-convex multi-variable constitutive models in equations (120) and (121), respectively, for a particular value of the electrostrictive parameter of $\tilde{f}_e = 1.5$ (moderate electrostriction), which corresponds (after minimisation) to a value of the parameters $\{f_e, f_s\}$ of $f_e = 1.195$ and $f_s = 0.163$[6].

Certainly, this minimisation procedure entails a qualitative similarity of the different constitutive tensors relevant in electromechanics, namely $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\theta}$ (refer to equations (C.1), (C.7) and (C.10)) for the entire experimental set-up for both convex $W_{el,1}$ and non-convex multi-variable $W_{el,2}$ constitutive models. Figure 8 shows the graphical visualisation of these tensors for both constitutive models in equation (120) and (121) for different stages

[6]For the remaining values of $\tilde{f}_e$ (124) used in Figure 6, namely $\{1.05, 10, 100\}$, the associated value of the parameters f_e and f_s (123) resulting after minimisation are $\{1.024, 1.701, 1.833\}$ and $\{0.019, 0.887, 0.999\}$, respectively.

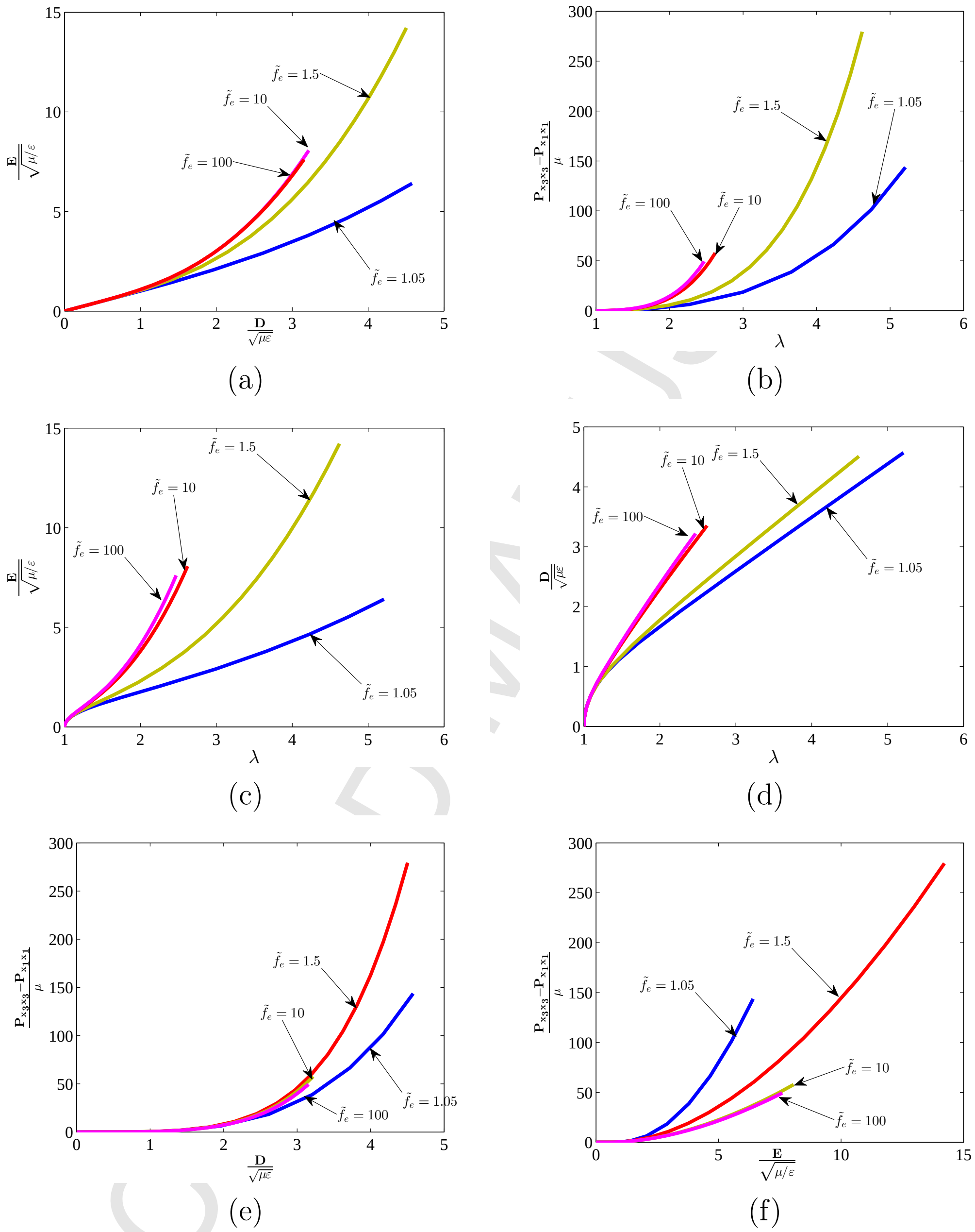

Figure 6: Numerical experiment reproducing the experimental set-up in Figure 5. Response of the non-convex multi-variable constitutive model $W_{el,2}$ in (121) for different values of the electrostrictive parameter $\tilde{f}_e$ in equation (124) for material parameters $E = 10^5\, N/m^2$, $\nu = 0.48$ and $\varepsilon_r = 4$. The following choice of material parameters was used: $\tilde{\mu}_1 = 2\tilde{\mu}_2$, $\varepsilon_2 = \infty$. Representation of the relation between (a) E vs D, (b) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs λ, (c) E vs λ, (d) D vs λ, (d) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs D and (d) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs E.

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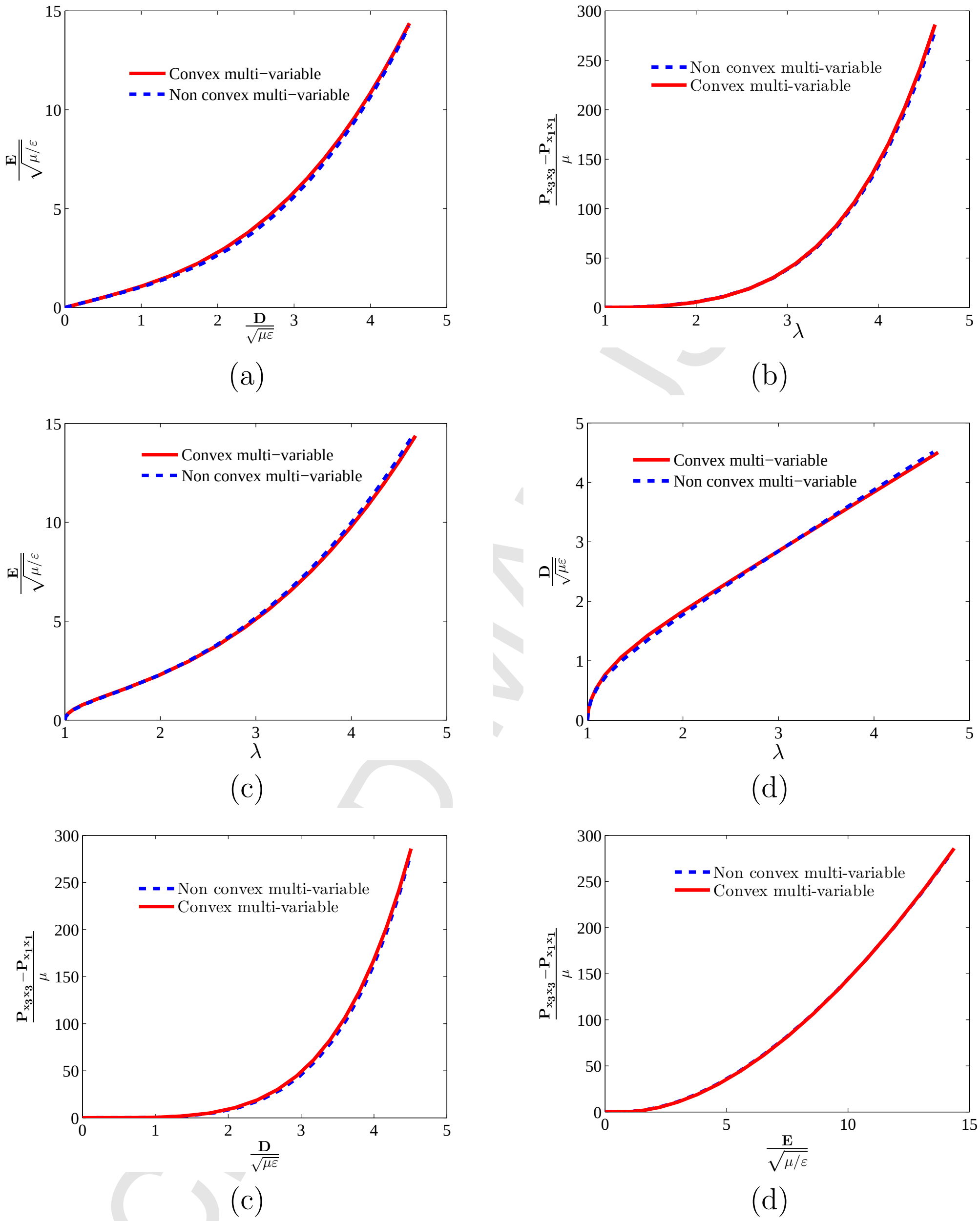

Figure 7: Numerical experiment reproducing the experimental set-up in Figure 5. Response of the non-convex multi-variable constitutive model $W_{el,2}$ in (121) for $\tilde{f}_e = 1.5$ and convex multi-variable model $W_{el,1}$ in (120) for $f_e = 1.195$ and $f_s = 0.163$. The following choice of material parameters was used: $\tilde{\mu}_1 = 2\tilde{\mu}_2$, $\varepsilon_2 = \infty$. Representation of the relation between (a) E vs D, (b) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs λ, (c) E vs λ, (d) D vs λ, (d) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs D and (d) $\boldsymbol{P}_{\boldsymbol{x}_1\boldsymbol{x}_1} - \boldsymbol{P}_{\boldsymbol{x}_3\boldsymbol{x}_3}$ vs E.

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of the experimental set-up for a value of the electrostrictive parameter for the non-convex multi-variable model $W_{el,2}$ of $\tilde{f}_e = 1.5$ $\tilde{f}_e = 1.5$ (moderate electrostriction), which corresponds (after minimisation) to a value of the parameters $\{f_e, f_s\}$ of $f_e = 1.195$ and $f_s = 0.163$ for the convex multi-variable model. Based on Reference [71] and [1], a spherical parametrisation of a vector $\boldsymbol{n}$ permits to define the moduli $\tilde{\mu}$, $\tilde{\mathcal{Q}}$ and $\tilde{\varepsilon}$ in Figure (8) as

$$\tilde{\mu} = (\boldsymbol{n} \otimes \boldsymbol{n}) : \boldsymbol{\mathcal{C}} : (\boldsymbol{n} \otimes \boldsymbol{n}) \, ; \qquad \tilde{\mathcal{Q}} = \boldsymbol{\mathcal{Q}} : (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}) \, ; \qquad \frac{1}{\tilde{\varepsilon}} = \boldsymbol{n}\boldsymbol{\theta}\boldsymbol{n}. \tag{125}$$

Figures 9 and 10 study the possible loss of material stability for the non-convex multi-variable constitutive law $W_{el,2}$ (121) and the convex multi-variable constitutive model $W_{el,1}$ (120)[7] for the current experimental set-up considered, depicted in Figure 5. The loss of material stability is associated with the loss of positive definiteness of the generalised electromechanical acoustic tensor $\boldsymbol{Q}^{\star}$ in equation (83) [26]. Therefore, a sufficient condition for material instability (refer to Reference [26]) is

$$q = \min\left(q_1, q_2, q_3\right) \leq 0; \quad q_1 = Q^{\star}_{11}; \quad q_2 = Q^{\star}_{11}Q^{\star}_{22} - Q^{\star}_{12}Q^{\star}_{21}; \quad q_3 = \det \boldsymbol{Q}^{\star}. \tag{126}$$

The evolution of the variable q in (126) for all the possible orientations of the propagation vector $\boldsymbol{N}$ for both constitutive models $W_{el,1}$ (120) and $W_{el,2}$ (121) is depicted in Figures 9 and 10, respectively. The same highly electrostrictive material as in Figure 8 is considered. It can be observed in Figure 9 that at a certain stage of the experimental set-up, the variable q becomes zero (negative values of the variable q (126) have been set to zero in order to facilitate the identification of material instability) for the non-convex multi-variable model $W_{el,2}$ (121). On the contrary, it is impossible to obtain negative values of q (126) for the convex multi-variable model $W_{el,1}$ (120) (refer to Figure 10).

7. Concluding remarks

This paper completes the series of publications [1, 2] on a new convex multi-variable variational and computational framework for the analysis of Electro Active Polymers. The approach presented in this paper allows also

[7]Convex multi-variable constitutive models in the sense described in equation (21) are always materially stable.

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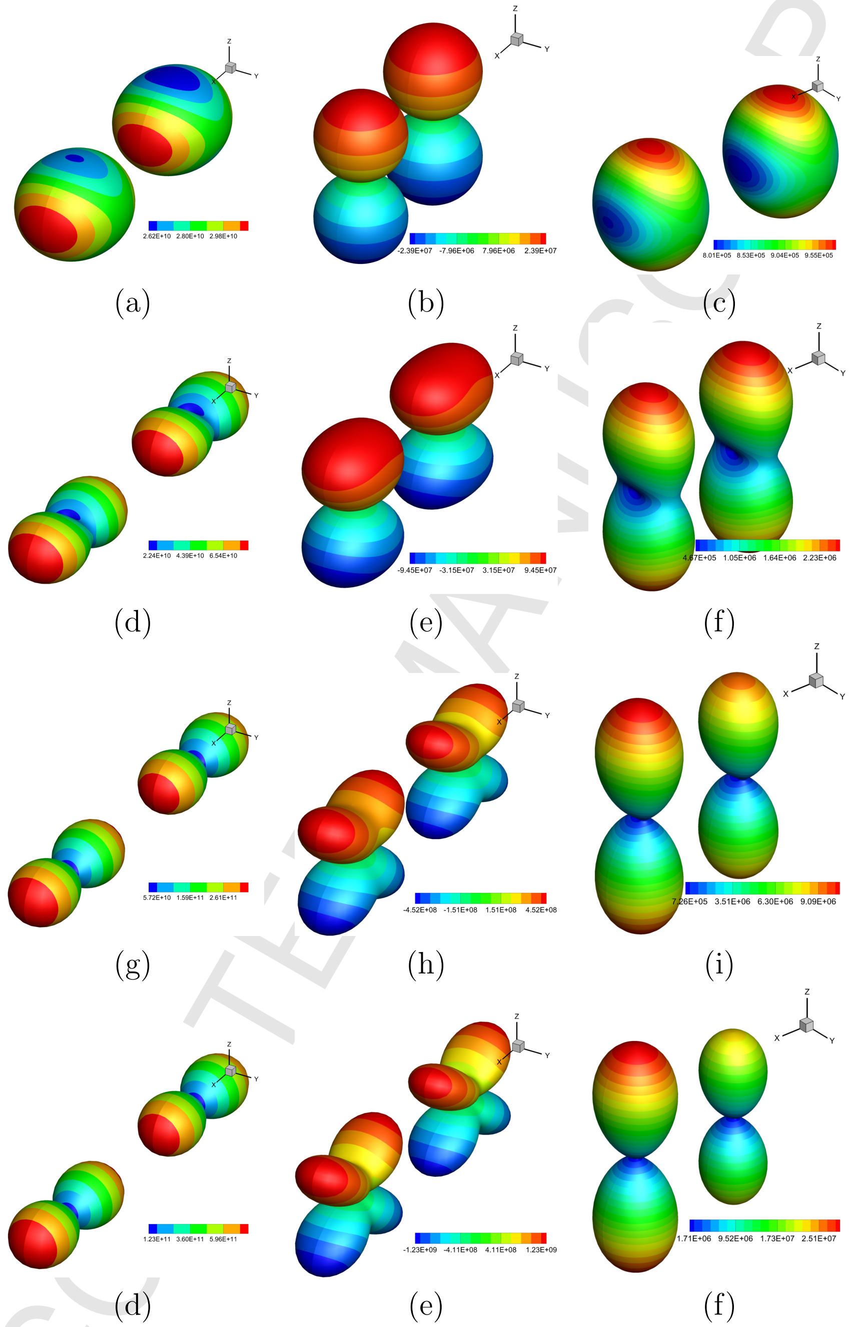

Figure 8: Numerical experiment reproducing the experimental set-up in Figure 5. Evolution of $\frac{1}{\tilde{\varepsilon}}$ (125) (left column), $\tilde{\mathcal{Q}}$ (125) (center column) and $\tilde{\mu}$ (125) (right column) for the non-convex multi-variable constitutive model $W_{el,2}$ in (121) for $\tilde{f}_e = 1.5$ and convex multi-variable model $W_{el,1}$ in (120) for $f_e = 1.195$ and $f_s = 0.163$. The following choice of material parameters was used: $\tilde{\mu}_1 = 2\tilde{\mu}_2$, $\varepsilon_2 = \infty$. Results obtained for an electrically induced strain (actuated strain) of (a)-(b)-(c) $\lambda = 1.053$, (d)-(e)-(f) $\lambda = 1.631$, (g)-(h)-(i) $\lambda = 3.081$ and (j)-(k)-l $\lambda = 3.081$

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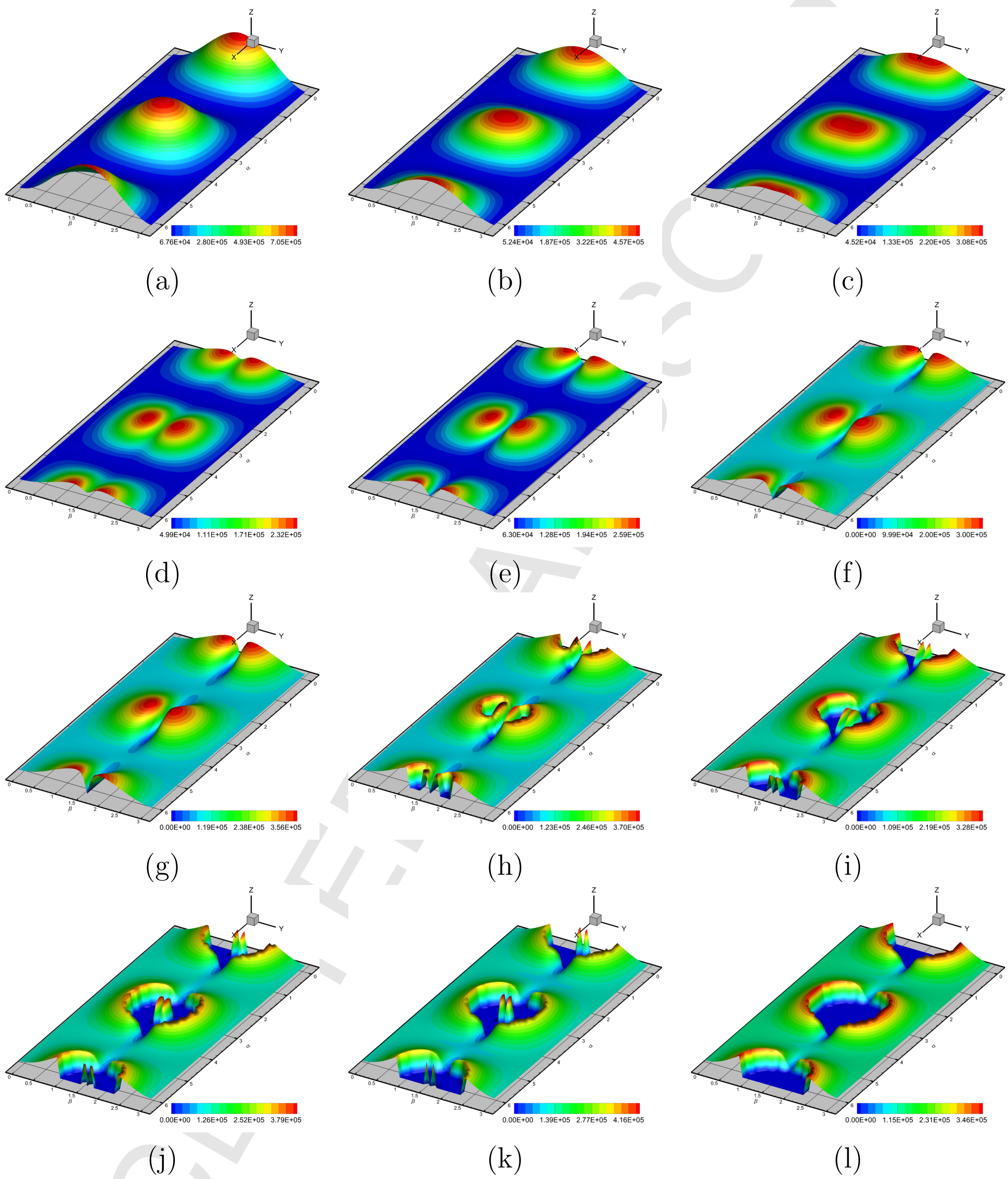

Figure 9: Numerical experiment reproducing the experimental set-up in Figure 5. Evolution of the variable q (126) (a value of zero of this variable would indicate material instability) for the non-convex multi-variable constitutive model $W_{el,2}$ in (121) for $\tilde{f}_e = 1.5$ (124). The following choice of material parameters was used: $\tilde{\mu}_1 = 2\tilde{\mu}_2$, $\varepsilon_2 = \infty$. Results obtained for an electrically induced strain (actuated strain) of (a) $\lambda = 1.053$, (b) $\lambda = 1.311$, (c) $\lambda = 1.631$, (d) $\lambda = 2.428$, (e) $\lambda = 3.081$, (f) $\lambda = 3.621$, (g) $\lambda = 4.089$, (h) $\lambda = 4.304$, (i) $\lambda = 4.508$, (j) $\lambda = 4.891$, (k) $\lambda = 5.07$ and (l) $\lambda = 5.246$.

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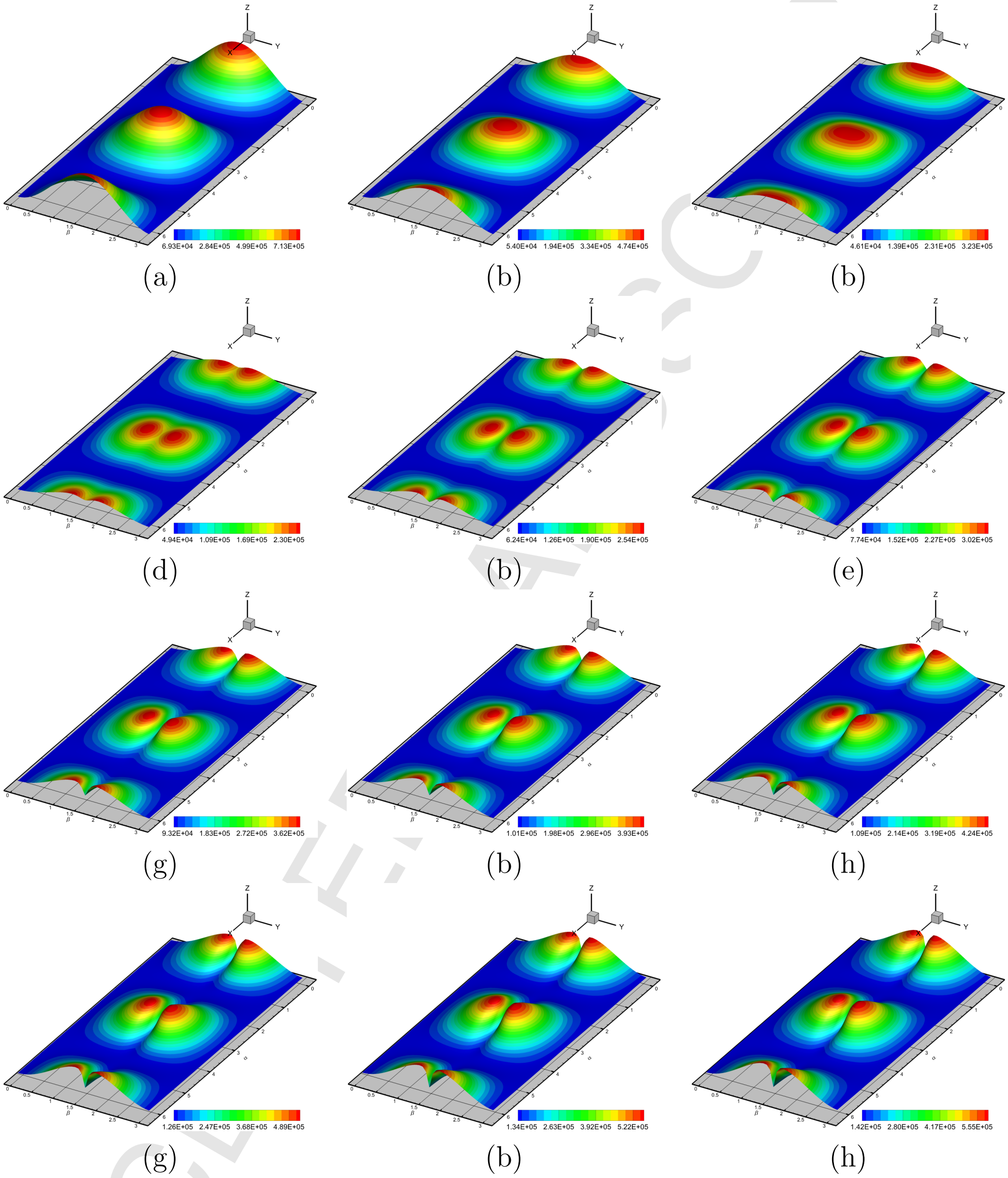

Figure 10: Numerical experiment reproducing the experimental set-up in Figure 5. Evolution of the variable q (126) (a value of zero of this variable would indicate material instability) for the convex multi-variable constitutive model $W_{el,1}$ in (120) for $f_e = 1.195$ and $f_s = 0.163$ (123). The following choice of material parameters was used: $\tilde{\mu}_1 = 2\tilde{\mu}_2$, $\varepsilon_2 = \infty$. Results obtained for an electrically induced strain (actuated strain) of (a) $\lambda = 1.053$, (b) $\lambda = 1.311$, (c) $\lambda = 1.631$, (d) $\lambda = 2.428$, (e) $\lambda = 3.081$, (f) $\lambda = 3.621$, (g) $\lambda = 4.089$, (h) $\lambda = 4.304$, (i) $\lambda = 4.508$, (j) $\lambda = 4.891$, (k) $\lambda = 5.07$ and (l) $\lambda = 5.246$.

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for a simple extension of the proposed framework to the field of Magneto Active Polymers.

One of the main contributions of this paper resides in the extension of the convex multi-variable definition of the electromechanical internal energy, postulated in [1], to the field of nonlinear electro-magneto-elasticity. This paper shows that the extended set of variables defining multi-variable convexity can be presented in the form of a system of first order conservation laws. In particular, two completely new conservation equations for the spatial vectors $\boldsymbol{d}$ and $\boldsymbol{b}$ (23) have been presented in this work. The proposed convex multi-variable definition of the internal energy in terms of conservation variables automatically leads to the fulfilment of the Legendre-Hadamard condition in the context of nonlinear electro-magneto-mechanics or, in other words, the positive definiteness nature of the electro-magneto-mechanical acoustic tensor.

The one-to-one and invertible relationship between the extended set of variables defining multi-variable convexity in electro-magneto-elasticity and its associated set of entropy variables allows to define a generalised convex entropy function and its associated flux. Following the work of [41] in the context of nonlinear Solid Dynamics, a symmetrisation of the hyperbolic equations has been carried out for the first time in the context of nonlinear electro-magneto-mechanics.

Finally, a series of challenging numerical examples have been presented in order to demonstrate the validity of the approach. Specifically, computation of wave speeds has been carried out for a series of sophisticated constitutive models and a comparison between convex and non-convex multi-variable energy functionals has also been presented.

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Appendix A. Cross product tensor product

The left cross product of a vector $\boldsymbol{v}$ and a second order tensor $\boldsymbol{A}$ to give a second order tensor denoted $\boldsymbol{v} \times \boldsymbol{A}$ is defined so that when applied to a general vector $\boldsymbol{w}$ gives:

$$(\boldsymbol{v} \times \boldsymbol{A})\, \boldsymbol{w} = \boldsymbol{v} \times (\boldsymbol{A}\boldsymbol{w})\,; \quad (\boldsymbol{v} \times \boldsymbol{A})_{ij} = \mathcal{E}_{ikl} v_k A_{lj}, \tag{A.1}$$

where $\mathcal{E}_{ikl}$ denotes the components of the standard third order alternating tensor and $\times$ is the standard vector cross product. The right cross product of a second order tensor $\boldsymbol{A}$ by a vector $\boldsymbol{v}$ to give a second order tensor denoted $\boldsymbol{A} \times \boldsymbol{v}$ is defined so that for every vector $\boldsymbol{w}$:

$$(\boldsymbol{A} \times \boldsymbol{v})\, \boldsymbol{w} = \boldsymbol{A}\,(\boldsymbol{v} \times \boldsymbol{w})\,; \quad (\boldsymbol{A} \times \boldsymbol{v})_{ij} = \mathcal{E}_{jkl} A_{ik} v_l. \tag{A.2}$$

The cross product of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ to give a new second order tensor denoted $\boldsymbol{A} \times \boldsymbol{B}$ is defined so that for any arbitrary vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ gives:

$$\boldsymbol{v} \cdot (\boldsymbol{A} \times \boldsymbol{B})\, \boldsymbol{w} = (\boldsymbol{v} \times \boldsymbol{A}) : (\boldsymbol{B} \times \boldsymbol{w})\,; \quad (\boldsymbol{A} \times \boldsymbol{B})_{ij} = \mathcal{E}_{ikl}\mathcal{E}_{jmn} A_{km} B_{ln}. \tag{A.3}$$

The cross vector product of two two-point tensors to give a spatial vector is also defined by a cross product operation with respect to the first indices and a contraction with respect to the second set of indices, so that

$$\boldsymbol{v} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{v} \cdot \boldsymbol{\mathcal{E}} : \left(\boldsymbol{A}\boldsymbol{B}^T\right); \qquad (\boldsymbol{A} \times \boldsymbol{B})_i = \mathcal{E}_{ijk} A_{jI} B_{kI}. \tag{A.4}$$

The cross vector product of the transpose of two two-point tensors to give a material vector is also defined by a cross product operation with respect to the first indices (material) and a contraction with respect to the second (spatial) set of indices, so that

$$\boldsymbol{V} \cdot \left(\boldsymbol{A}^T \times \boldsymbol{B}^T\right) = \boldsymbol{V} \cdot \boldsymbol{\mathcal{E}} : \left(\boldsymbol{A}^T\boldsymbol{B}\right); \qquad \left(\boldsymbol{A}^T \times \boldsymbol{B}^T\right)_I = \mathcal{E}_{IJK} A_{iJ} B_{iK}. \tag{A.5}$$

In the framework developed in this paper the tensor cross product will be mostly applied between two-point tensors. For this purpose the above definition can be readily particularised to second order two-point tensors as,

$$(\boldsymbol{A} \times \boldsymbol{B})_{iI} = \mathcal{E}_{ijk}\mathcal{E}_{IJK} A_{jJ} B_{kK}. \tag{A.6}$$

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When applied to a second order tensor $\boldsymbol{A}$ and a fourth order tensors $\mathbb{H}$, two possible operations are defined as:

$$(\mathbb{H} \boldsymbol{\times} \boldsymbol{A})_{pPiI} = \mathcal{E}_{ijk}\mathcal{E}_{IJK}\mathbb{H}_{pPjJ}A_{kK}; \quad (\boldsymbol{A} \boldsymbol{\times} \mathbb{H})_{iIpP} = \mathcal{E}_{ijk}\mathcal{E}_{IJK}A_{jJ}\mathbb{H}_{kKpP}. \tag{A.7}$$

Moreover, the double application of the tensor cross product between a fourth order tensor and two second order tensors is associative, namely:

$$\boldsymbol{A} \boldsymbol{\times} \mathbb{H} \boldsymbol{\times} \boldsymbol{B} = (\boldsymbol{A} \boldsymbol{\times} \mathbb{H}) \boldsymbol{\times} \boldsymbol{B} = \boldsymbol{A} \boldsymbol{\times} (\mathbb{H} \boldsymbol{\times} \boldsymbol{B}). \tag{A.8}$$

Finally, when applied to a second order tensor $\boldsymbol{A}$ and a third order tensors $\mathbb{Q}$, two possible operations are defined as:

$$(\mathbb{Q} \boldsymbol{\times} \boldsymbol{A})_{PiI} = \mathcal{E}_{ijk}\mathcal{E}_{IJK}\mathbb{Q}_{PjJ}A_{kK}; \quad (\boldsymbol{A} \boldsymbol{\times} \mathbb{Q})_{iIP} = \mathcal{E}_{ijk}\mathcal{E}_{IJK}A_{jJ}\mathbb{H}_{kKP}. \tag{A.9}$$

Some useful properties of new cross product are enumerated below. Let a be a scalar, $\boldsymbol{V}$ and $\boldsymbol{W}$ material vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$ spatial vectors, $\boldsymbol{I}$ the identity tensor with Kronecker delta components $(\boldsymbol{I})_{ij} = \delta_{ij}$ and $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ second order tensors

$$\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B} = \boldsymbol{B} \boldsymbol{\times} \boldsymbol{A}; \tag{A.10}$$

$$\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B} = \boldsymbol{A}^T \boldsymbol{\times} \boldsymbol{B}^T; \tag{A.11}$$

$$\boldsymbol{A} \boldsymbol{\times} (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \boldsymbol{\times} \boldsymbol{B} + \boldsymbol{A} \boldsymbol{\times} \boldsymbol{C}; \tag{A.12}$$

$$a\,(\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B}) = (a\boldsymbol{A}) \boldsymbol{\times} \boldsymbol{B} = \boldsymbol{A} \boldsymbol{\times} (a\boldsymbol{B}); \tag{A.13}$$

$$(\boldsymbol{v} \otimes \boldsymbol{V}) \boldsymbol{\times} (\boldsymbol{w} \otimes \boldsymbol{W}) = (\boldsymbol{v} \times \boldsymbol{w}) \otimes (\boldsymbol{V} \times \boldsymbol{W}); \tag{A.14}$$

$$\boldsymbol{v} \boldsymbol{\times} (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{V}) = (\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A}) \boldsymbol{\times} \boldsymbol{V} = \boldsymbol{v} \boldsymbol{\times} \boldsymbol{A} \boldsymbol{\times} \boldsymbol{V}; \tag{A.15}$$

$$\boldsymbol{A} \boldsymbol{\times} (\boldsymbol{v} \otimes \boldsymbol{V}) = -\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A} \boldsymbol{\times} \boldsymbol{V}; \tag{A.16}$$

$$(\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B}) : \boldsymbol{C} = (\boldsymbol{B} \boldsymbol{\times} \boldsymbol{C}) : \boldsymbol{A} = (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{C}) : \boldsymbol{B}; \tag{A.17}$$

$$(\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B})(\boldsymbol{V} \times \boldsymbol{W}) = (\boldsymbol{A}\boldsymbol{V}) \times (\boldsymbol{B}\boldsymbol{W}) + (\boldsymbol{B}\boldsymbol{V}) \times (\boldsymbol{A}\boldsymbol{W}); \tag{A.18}$$

$$\boldsymbol{A} \boldsymbol{\times} \boldsymbol{I} = (\mathrm{tr}\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^T; \tag{A.19}$$

$$\boldsymbol{I} \boldsymbol{\times} \boldsymbol{I} = 2\boldsymbol{I}; \tag{A.20}$$

$$(\boldsymbol{A} \boldsymbol{\times} \boldsymbol{A}) : \boldsymbol{A} = 6\ \det\boldsymbol{A}; \tag{A.21}$$

$$\mathrm{Cof}\boldsymbol{A} = \frac{1}{2}\boldsymbol{A} \boldsymbol{\times} \boldsymbol{A}; \tag{A.22}$$

$$(\boldsymbol{A}\boldsymbol{C}) \boldsymbol{\times} (\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B})(\mathrm{Cof}\boldsymbol{C}). \tag{A.23}$$

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Appendix B. Components of the Jacobian matrix for the full system of hyperbolic equations in convex multi-variable electro-magneto-elasticity

The objective of this Section is to present the expressions for the tensors emerging in the derivation of the Jacobian matrix $\boldsymbol{\mathcal{A}}_I$ in equation (53). This matrix is needed for the study of the eigenvalue structure of the quasilinear form of the hyperbolic equations in electro-magneto-elasticity presented in Section 4.

Matrix $\widetilde{\boldsymbol{W}_I}^{\star\star}$ in equation (53) adopts the following expression

$$\widetilde{\boldsymbol{W}_I}^{\star\star} = \begin{bmatrix} \widetilde{\boldsymbol{P}_{\boldsymbol{F}}}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_{\boldsymbol{H}}}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_J}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_{\boldsymbol{D}_0}}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_{\boldsymbol{B}_0}}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_{\boldsymbol{d}}}\boldsymbol{E}_I & \widetilde{\boldsymbol{P}_{\boldsymbol{b}}}\boldsymbol{E}_I \\ \boldsymbol{0}_{3\times3\times3\times3} & \boldsymbol{0}_{3\times3\times3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} \\ \boldsymbol{0}_{3\times3\times3\times3} & \boldsymbol{0}_{3\times3\times3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} & \boldsymbol{0}_{3\times3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & 0 & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} \end{bmatrix}, \tag{B.1}$$

with

$$\begin{aligned}
\widetilde{\boldsymbol{P}_{\boldsymbol{F}}} &= W_{\boldsymbol{FF}} + \boldsymbol{F} \times W_{\boldsymbol{HF}} + \boldsymbol{H} \otimes W_{J\boldsymbol{F}} + W_{\boldsymbol{dF}} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bF}} \diamond \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_{\boldsymbol{H}}} &= W_{\boldsymbol{FH}} + \boldsymbol{F} \times W_{\boldsymbol{HH}} + \boldsymbol{H} \otimes W_{J\boldsymbol{H}} + W_{\boldsymbol{dH}} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bH}} \diamond \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_J} &= W_{\boldsymbol{F}J} + \boldsymbol{F} \times W_{\boldsymbol{H}J} + \boldsymbol{H} W_{JJ} + W_{\boldsymbol{d}J} \otimes \boldsymbol{D}_0 + W_{\boldsymbol{b}J} \otimes \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_{\boldsymbol{D}_0}} &= W_{\boldsymbol{FD}_0} + \boldsymbol{F} \times W_{\boldsymbol{HD}_0} + \boldsymbol{H} \otimes W_{J\boldsymbol{D}_0} + W_{\boldsymbol{dD}_0} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bD}_0} \diamond \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_{\boldsymbol{B}_0}} &= W_{\boldsymbol{FB}_0} + \boldsymbol{F} \times W_{\boldsymbol{HB}_0} + \boldsymbol{H} \otimes W_{J\boldsymbol{B}_0} + W_{\boldsymbol{dB}_0} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bB}_0} \diamond \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_{\boldsymbol{d}}} &= W_{\boldsymbol{Fd}} + \boldsymbol{F} \times W_{\boldsymbol{Hd}} + \boldsymbol{H} \otimes W_{J\boldsymbol{d}} + W_{\boldsymbol{dD}_0} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bD}_0} \diamond \boldsymbol{B}_0; \\
\widetilde{\boldsymbol{P}_{\boldsymbol{b}}} &= W_{\boldsymbol{Fb}} + \boldsymbol{F} \times W_{\boldsymbol{Hb}} + \boldsymbol{H} \otimes W_{J\boldsymbol{b}} + W_{\boldsymbol{db}} \diamond \boldsymbol{D}_0 + W_{\boldsymbol{bb}} \diamond \boldsymbol{B}_0,
\end{aligned} \tag{B.2}$$

where the tensor operation $(\bullet) \diamond (\bullet)$ between a third or second order tensors, $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{A}$ respectively, and a vector $\boldsymbol{V}$ yields

$$(\boldsymbol{\mathcal{A}} \diamond \boldsymbol{V})_{iImM} = \mathcal{A}_{imM} V_I; \qquad (\boldsymbol{A} \diamond \boldsymbol{V})_{iIM} = A_{iM} V_I. \tag{B.3}$$

The matrix $\widetilde{\boldsymbol{W}_I}^{\star}$ (featuring in the definition of the Jacobian matrix $\boldsymbol{\mathcal{A}}_I$ (53)) is defined so that $\widetilde{\boldsymbol{W}}^{\star}_{\boldsymbol{N}} = \widetilde{\boldsymbol{W}}^{\star}_I N_I$, with $\widetilde{\boldsymbol{W}}^{\star}_{\boldsymbol{N}}$ defined as

$$\widetilde{W}^{\star}_{\boldsymbol{N}} = \begin{bmatrix} -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{F}}\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{H}}\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_J\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{D}_0}\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{B}_0}\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{d}}\right) & -\left(\boldsymbol{W}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{b}}\right) \\ \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{F}}\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{H}}\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_J\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{D}_0}\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{B}_0}\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{d}}\right) & \left(\boldsymbol{W}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{b}}\right) \\ -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{F}}\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{H}}\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_J\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{D}_0}\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{B}_0}\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{d}}\right) & -\left(\boldsymbol{FW}\widetilde{\boldsymbol{H}_0}_{\boldsymbol{b}}\right) \\ \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{F}}\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{H}}\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_J\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{D}_0}\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{B}_0}\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{d}}\right) & \left(\boldsymbol{FW}\widetilde{\boldsymbol{E}_0}_{\boldsymbol{b}}\right) \end{bmatrix}. \tag{B.4}$$

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Notice that the second order tensor $\boldsymbol{W}$ is defined as $W_{IJ} = \mathcal{E}_{IJK} N_K$ and that the terms $\widetilde{\boldsymbol{E}_{0\boldsymbol{F}}}$, $\widetilde{\boldsymbol{E}_{0\boldsymbol{H}}}$, etc. in above equation (B.4) are defined as

$$\begin{aligned}
\widetilde{\boldsymbol{E}_{0\boldsymbol{F}}} &= W_{\boldsymbol{D}_0\boldsymbol{F}} + \boldsymbol{F}^T W_{\boldsymbol{dF}}; & \widetilde{\boldsymbol{E}_{0\boldsymbol{H}}} &= W_{\boldsymbol{D}_0\boldsymbol{H}} + \boldsymbol{F}^T W_{\boldsymbol{dH}};\\
\widetilde{\boldsymbol{E}_{0J}} &= W_{\boldsymbol{D}_0 J} + \boldsymbol{F}^T W_{\boldsymbol{d}J}; & \widetilde{\boldsymbol{E}_{0\boldsymbol{D}_0}} &= W_{\boldsymbol{D}_0\boldsymbol{D}_0} + \boldsymbol{F}^T W_{\boldsymbol{dD}_0};\\
\widetilde{\boldsymbol{E}_{0\boldsymbol{D}_0}} &= W_{\boldsymbol{D}_0\boldsymbol{B}_0} + \boldsymbol{F}^T W_{\boldsymbol{dB}_0}; & \widetilde{\boldsymbol{E}_{0\boldsymbol{d}}} &= W_{\boldsymbol{D}_0\boldsymbol{d}} + \boldsymbol{F}^T W_{\boldsymbol{dd}};\\
\widetilde{\boldsymbol{E}_{0\boldsymbol{b}}} &= W_{\boldsymbol{D}_0\boldsymbol{b}} + \boldsymbol{F}^T W_{\boldsymbol{db}}.
\end{aligned} \tag{B.5}$$

Finally, the terms $\widetilde{\boldsymbol{H}_{0\boldsymbol{F}}}$, $\widetilde{\boldsymbol{H}_{0\boldsymbol{H}}}$, etc. in above equation (B.4) are defined as

$$\begin{aligned}
\widetilde{\boldsymbol{H}_{0\boldsymbol{F}}} &= W_{\boldsymbol{B}_0\boldsymbol{F}} + \boldsymbol{F}^T W_{\boldsymbol{bF}}; & \widetilde{\boldsymbol{H}_{0\boldsymbol{H}}} &= W_{\boldsymbol{B}_0\boldsymbol{H}} + \boldsymbol{F}^T W_{\boldsymbol{bH}};\\
\widetilde{\boldsymbol{H}_{0J}} &= W_{\boldsymbol{B}_0 J} + \boldsymbol{F}^T W_{\boldsymbol{b}J}; & \widetilde{\boldsymbol{H}_{0\boldsymbol{D}_0}} &= W_{\boldsymbol{B}_0\boldsymbol{D}_0} + \boldsymbol{F}^T W_{\boldsymbol{bD}_0};\\
\widetilde{\boldsymbol{H}_{0\boldsymbol{D}_0}} &= W_{\boldsymbol{B}_0\boldsymbol{B}_0} + \boldsymbol{F}^T W_{\boldsymbol{bB}_0}; & \widetilde{\boldsymbol{H}_{0\boldsymbol{d}}} &= W_{\boldsymbol{B}_0\boldsymbol{d}} + \boldsymbol{F}^T W_{\boldsymbol{bd}};\\
\widetilde{\boldsymbol{H}_{0\boldsymbol{b}}} &= W_{\boldsymbol{B}_0\boldsymbol{b}} + \boldsymbol{F}^T W_{\boldsymbol{bb}}.
\end{aligned} \tag{B.6}$$

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Appendix C. Relationships between the Hessian operator of the Helmholtz's energy functional and that for the internal energy

The objective of this section is to express the physically meaningful constitutive tensors in equation introduced in Section 4, namely $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{T}}$, $\boldsymbol{R}$, $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$, emanating from the Hessian operator of the internal energy e, in terms of the components of the Hessian operator of the extended representation of the internal energy, namely $[\mathbb{H}_W]$.

Appendix C.1. Elasticity tensor

The fourth order elasticity tensor $\boldsymbol{\mathcal{C}}$ emanates from the second directional derivative of the internal energy e with respect to changes of the geometry as

$$D^2 e\,[\delta\boldsymbol{u};\boldsymbol{u}] = \boldsymbol{\nabla}_0\delta\boldsymbol{u} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}_0\boldsymbol{u} \Rightarrow \boldsymbol{\mathcal{C}} = \frac{\partial^2 e}{\partial \boldsymbol{\nabla}_0\boldsymbol{x}\boldsymbol{\nabla}_0\boldsymbol{x}}. \tag{C.1}$$

Alternatively, $\boldsymbol{\mathcal{C}}$ can be re-written in terms of the derivatives of the electro-kinematic variable set $\mathcal{V}$ as

$$\begin{aligned}\boldsymbol{\mathcal{C}} &= W_{\boldsymbol{FF}} + \boldsymbol{F}\times(W_{\boldsymbol{HH}}\times\boldsymbol{F}) + W_{JJ}\boldsymbol{H}\otimes\boldsymbol{H} + \boldsymbol{\mathcal{C}}_1 + \boldsymbol{\mathcal{C}}_2\\ &+ 2(W_{\boldsymbol{FH}}\times\boldsymbol{F})^{\text{sym}} + 2(W_{\boldsymbol{F}J}\otimes\boldsymbol{H})^{\text{sym}} + 2(W_{\boldsymbol{Fd}}\otimes\boldsymbol{D}_0)^{\text{sym}} + 2(W_{\boldsymbol{Fb}}\otimes\boldsymbol{B}_0)^{\text{sym}}\\ &+ 2((\boldsymbol{F}\times W_{\boldsymbol{H}J})\otimes\boldsymbol{H})^{\text{sym}} + 2((\boldsymbol{F}\times W_{\boldsymbol{Hd}})\otimes\boldsymbol{D}_0)^{\text{sym}} + 2((\boldsymbol{F}\times W_{\boldsymbol{Hb}})\otimes\boldsymbol{B}_0)^{\text{sym}}\\ &+ 2(\boldsymbol{H}\otimes(W_{J\boldsymbol{d}}\otimes\boldsymbol{D}_0))^{\text{sym}} + 2(\boldsymbol{H}\otimes(W_{J\boldsymbol{b}}\otimes\boldsymbol{B}_0))^{\text{sym}} + \boldsymbol{\mathcal{A}},\end{aligned} \tag{C.2}$$

where

$$\mathcal{A}_{iIjJ} = \mathcal{E}_{ijp}\mathcal{E}_{IJP}\left(\boldsymbol{\Sigma_H} + \Sigma_J\boldsymbol{\Sigma_H}\right)_{pP};\quad \mathcal{C}_{1,iIjJ} = \left(W_{\boldsymbol{dd}}\right)_{ij} D_{0I}D_{0J};\quad \mathcal{C}_{2,iIjJ} = \left(W_{\boldsymbol{bb}}\right)_{ij} B_{0I}B_{0J}. \tag{C.3}$$

Moreover, for any fourth order tensor $\boldsymbol{\mathcal{T}}$ included in equation (C.2), the symmetrised tensor $\boldsymbol{\mathcal{T}}^{\text{sym}}$ is defined as $\mathcal{T}^{\text{sym}}_{iIjJ} = \frac{1}{2}\left(\mathcal{T}_{iIjJ} + \mathcal{T}_{iJjI}\right)$.

Appendix C.2. Piezoelectric tensor

The third order piezoelectric tensor $\boldsymbol{\mathcal{Q}}$ emanates from the second directional derivative of the internal energy e with respect to changes in geometry and electric displacement field as

$$D^2 e\,[\delta\boldsymbol{u};\Delta\boldsymbol{D}_0] = \left(\boldsymbol{\nabla}_0\delta\boldsymbol{u} : \boldsymbol{\mathcal{Q}}^T\right)\cdot\Delta\boldsymbol{D}_0 \Rightarrow \boldsymbol{\mathcal{Q}} = \frac{\partial^2 e}{\partial\boldsymbol{\nabla}_0\boldsymbol{x}\partial\boldsymbol{D}_0}. \tag{C.4}$$

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Alternatively, the piezoelectric tensor $\boldsymbol{\mathcal{Q}}^T$ can be re-expresses in terms of the elements of the set $\mathcal{V}$ as

$$\begin{aligned}\boldsymbol{\mathcal{Q}}^T &= W_{\boldsymbol{F}\boldsymbol{D}_0} + \boldsymbol{F} \times W_{\boldsymbol{H}\boldsymbol{D}_0} + \boldsymbol{H} \otimes W_{J\boldsymbol{D}_0} + \boldsymbol{\mathcal{Q}}_1^T \\ &+ \boldsymbol{\mathcal{Q}}_2^T + \boldsymbol{\mathcal{Q}}_3^T + \boldsymbol{\mathcal{Q}}_4^T + \boldsymbol{\mathcal{Q}}_5^T + \boldsymbol{\mathcal{Q}}_6^T + \boldsymbol{\mathcal{Q}}_7^T + \boldsymbol{\Sigma}_{\boldsymbol{d}} \otimes \boldsymbol{I}.\end{aligned} \tag{C.5}$$

where the expressions for the tensors $\boldsymbol{\mathcal{Q}}_i^T$ in above equation (C.5) are given as

$$\left(\boldsymbol{\mathcal{Q}}_1^T\right)_{iIJ} = \left(W_{\boldsymbol{d}\boldsymbol{D}_0}\right)_{iJ} D_{0I}; \tag{C.6a}$$
$$\left(\boldsymbol{\mathcal{Q}}_2^T\right)_{iIJ} = \left(W_{\boldsymbol{F}\boldsymbol{d}iIj}\right) F_{jJ}; \tag{C.6b}$$
$$\left(\boldsymbol{\mathcal{Q}}_3^T\right)_{iIJ} = (\boldsymbol{F} \times W_{\boldsymbol{H}\boldsymbol{d}})_{iIj} F_{jJ}; \tag{C.6c}$$
$$\left(\boldsymbol{\mathcal{Q}}_4^T\right)_{iIJ} = (\boldsymbol{H} \otimes W_{J\boldsymbol{d}})_{iIj} F_{jJ}; \tag{C.6d}$$
$$\left(\boldsymbol{\mathcal{Q}}_5^T\right)_{iIJ} = \left(W_{\boldsymbol{d}\boldsymbol{d}}\right)_{ij} F_{jJ} D_{0I}; \tag{C.6e}$$
$$\left(\boldsymbol{\mathcal{Q}}_6^T\right)_{iIJ} = \left(W_{\boldsymbol{b}\boldsymbol{D}_0}\right)_{iJ} B_{0I}; \tag{C.6f}$$
$$\left(\boldsymbol{\mathcal{Q}}_7^T\right)_{iIJ} = \left(W_{\boldsymbol{b}\boldsymbol{d}}\right)_{ij} F_{jJ} B_{0I}. \tag{C.6g}$$

Appendix C.3. Piezomagnetic tensor

The third order piezomagnetic tensor $\boldsymbol{\mathcal{T}}$ emanates from the second directional derivative of the internal energy e with respect to changes in geometry and magnetic inductions field as

$$D^2 e\left[\delta \boldsymbol{u}; \Delta \boldsymbol{B}_0\right] = \left(\boldsymbol{\nabla}_0 \delta \boldsymbol{u} : \boldsymbol{\mathcal{T}}^T\right) \cdot \Delta \boldsymbol{B}_0 \Rightarrow \boldsymbol{\mathcal{T}} = \frac{\partial^2 e}{\partial \boldsymbol{\nabla}_0 \boldsymbol{x} \partial \boldsymbol{B}_0}. \tag{C.7}$$

The piezomagnetic tensor $\boldsymbol{\mathcal{T}}^T$ can be re-expressed in terms of the elements of the set $\mathcal{V}$ as

$$\begin{aligned}\boldsymbol{\mathcal{T}}^T &= W_{\boldsymbol{F}\boldsymbol{B}_0} + \boldsymbol{F} \times W_{\boldsymbol{H}\boldsymbol{B}_0} + \boldsymbol{H} \otimes W_{J\boldsymbol{B}_0} + \boldsymbol{\mathcal{T}}_1^T \\ &+ \boldsymbol{\mathcal{T}}_2^T + \boldsymbol{\mathcal{T}}_3^T + \boldsymbol{\mathcal{T}}_4^T + \boldsymbol{\mathcal{T}}_5^T + \boldsymbol{\mathcal{T}}_6^T + \boldsymbol{\mathcal{T}}_7^T + \boldsymbol{\Sigma}_{\boldsymbol{b}} \otimes \boldsymbol{I}.\end{aligned} \tag{C.8}$$

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with

$$\left(\boldsymbol{\mathcal{T}}_1^T\right)_{iIJ} = (W_{\boldsymbol{d}\boldsymbol{B}_0})_{iJ}\, B_{0I}; \tag{C.9a}$$
$$\left(\boldsymbol{\mathcal{T}}_2^T\right)_{iIJ} = \left(W_{\boldsymbol{F}\boldsymbol{b}iIj}\right) F_{jJ}; \tag{C.9b}$$
$$\left(\boldsymbol{\mathcal{T}}_3^T\right)_{iIJ} = (\boldsymbol{F} \times W_{\boldsymbol{H}\boldsymbol{b}})_{iIj} F_{jJ}; \tag{C.9c}$$
$$\left(\boldsymbol{\mathcal{T}}_4^T\right)_{iIJ} = (\boldsymbol{H} \otimes W_{J\boldsymbol{b}})_{iIj} F_{jJ}; \tag{C.9d}$$
$$\left(\boldsymbol{\mathcal{T}}_5^T\right)_{iIJ} = (W_{\boldsymbol{b}\boldsymbol{b}})_{ij}\, F_{jJ} B_{0I}.; \tag{C.9e}$$
$$\left(\boldsymbol{\mathcal{T}}_6^T\right)_{iIJ} = (W_{\boldsymbol{d}\boldsymbol{B}_0})_{iJ}\, D_{0I}; \tag{C.9f}$$
$$\left(\boldsymbol{\mathcal{T}}_7^T\right)_{iIJ} = (W_{\boldsymbol{d}\boldsymbol{b}})_{ij}\, F_{jJ} D_{0I}. \tag{C.9g}$$

Appendix C.4. Dielectric tensor

The second order dielectric tensor $\boldsymbol{\theta}$ emanates from the second directional derivative of the internal energy e with respect to changes in the electric displacement field as

$$D^2 e\left[\delta \boldsymbol{D}_0; \Delta \boldsymbol{D}_0\right] = \delta \boldsymbol{D}_0 \cdot \boldsymbol{\theta} \Delta \boldsymbol{D}_0 \Rightarrow \boldsymbol{\theta} = \frac{\partial^2 e}{\partial \boldsymbol{D}_0 \partial \boldsymbol{D}_0}. \tag{C.10}$$

Alternatively, the inverse of the dielectric tensor $\boldsymbol{\theta}$ can be re-expressed in terms of the elements of the set $\mathcal{V}$ as

$$\boldsymbol{\theta} = W_{\boldsymbol{D}_0\boldsymbol{D}_0} + (W_{\boldsymbol{D}_0\boldsymbol{d}}\boldsymbol{F} + \boldsymbol{F}^T W_{\boldsymbol{d}\boldsymbol{D}_0}) + \boldsymbol{F}^T W_{\boldsymbol{d}\boldsymbol{d}} \boldsymbol{F}. \tag{C.11}$$

Appendix C.5. Permeability tensor

The second order permeability tensor $\boldsymbol{\vartheta}$ emanates from the second directional derivative of the internal energy e with respect to changes in the magnetic induction field as

$$D^2 e\left[\delta \boldsymbol{B}_0; \Delta \boldsymbol{B}_0\right] = \delta \boldsymbol{B}_0 \cdot \boldsymbol{\vartheta} \Delta \boldsymbol{B}_0 \Rightarrow \boldsymbol{\vartheta} = \frac{\partial^2 e}{\partial \boldsymbol{B}_0 \partial \boldsymbol{B}_0}. \tag{C.12}$$

Alternatively, the inverse of the permeability tensor $\boldsymbol{\vartheta}$ can be re-expressed in terms of the elements of the set $\mathcal{V}$ as

$$\boldsymbol{\vartheta} = W_{\boldsymbol{B}_0\boldsymbol{B}_0} + (W_{\boldsymbol{B}_0\boldsymbol{b}}\boldsymbol{F} + \boldsymbol{F}^T W_{\boldsymbol{b}\boldsymbol{B}_0}) + \boldsymbol{F}^T W_{\boldsymbol{b}\boldsymbol{b}} \boldsymbol{F}. \tag{C.13}$$

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Appendix C.6. Magnetoelectric tensor

The second order magnetoelectric tensor $\boldsymbol{R}$ emanates from the second directional derivative of the internal energy e with respect to changes in the electric displacement field and magnetic induction field as

$$D^2 e\left[\delta \boldsymbol{D}_0; \Delta \boldsymbol{B}_0\right] = \delta \boldsymbol{D}_0 \cdot \boldsymbol{R} \cdot \Delta \boldsymbol{B}_0 \Rightarrow \boldsymbol{R} = \frac{\partial^2 e}{\partial \boldsymbol{D}_0 \partial \boldsymbol{B}_0}. \tag{C.14}$$

Alternatively, the magnetoelectric tensor $\boldsymbol{R}$ can be re-expressed in terms of the elements of the set $\mathcal{V}$ as

$$\boldsymbol{R} = W_{\boldsymbol{D}_0 \boldsymbol{B}_0} + W_{\boldsymbol{D}_0 \boldsymbol{b}} \boldsymbol{F} + \boldsymbol{F}^T W_{\boldsymbol{d} \boldsymbol{B}_0} + \boldsymbol{F}^T W_{\boldsymbol{d} \boldsymbol{b}} \boldsymbol{F}. \tag{C.15}$$

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Appendix D. Theorem relevant to the eigenvalue structure of the system of non redundant hyperbolic equations

The objective of this Section is to prove that, as stated in Section (4.1), a second order tensor $\boldsymbol{\mathcal{A}}^{\star}_{\boldsymbol{N}}$, defined via the multiplicative decomposition in equation (66) in terms of a symmetric and a symmetric positive definite second order tensor admits only real eigenvalues. This can be stated in the following theorem

Theorem Appendix D.1. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ be symmetric and symmetric positive definite matrices for any arbitrary dimension specification n. Let $\boldsymbol{C}$ be defined in terms of the following multiplicative decomposition as*

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}. \tag{D.1}$$

Then, the matrix $\boldsymbol{C}$ defined as in equation (D.1) *has real eigenvalues.*

Proof. The eigenvalue problem associated to matrix $\boldsymbol{C}$ introduced in Theorem Appendix D.1 is defined as

$$\left(c_\alpha \boldsymbol{I} - \boldsymbol{A}\boldsymbol{B}\right)\bar{\boldsymbol{m}}_\alpha = \boldsymbol{0}, \tag{D.2}$$

where $\bar{\boldsymbol{m}}_\alpha$ are the eigenvectors of $\boldsymbol{C}$. Note that $\boldsymbol{B}$ admits the following non-unique multiplicative decomposition, $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}^T$. A possible choice for $\boldsymbol{L}$ could be the symmetric matrix $\boldsymbol{L} = \sqrt{\boldsymbol{B}}$[8]. This particular choice of $\boldsymbol{L}$ enables to re-write equation (D.2) as

$$\left(c_\alpha \boldsymbol{I} - \boldsymbol{A}\sqrt{\boldsymbol{B}}\sqrt{\boldsymbol{B}}\right)\bar{\boldsymbol{m}}_\alpha = \boldsymbol{0} \tag{D.3}$$

The identification of the eigenvalues c_α in above (D.3) is equivalent to the solution of the following polynomial equation, namely $\det\boldsymbol{D} = 0$, where $\boldsymbol{D} = c_\alpha \boldsymbol{I} - \boldsymbol{A}\sqrt{\boldsymbol{B}}\sqrt{\boldsymbol{B}}$. Pre- and post-multiplication by $\sqrt{\boldsymbol{B}}$ $\sqrt{\boldsymbol{B}}^{-1}$ does not alter the solution to the polynomial equation (D.3), namely $\det(\boldsymbol{D}) = \det(\sqrt{\boldsymbol{B}}\boldsymbol{D}\sqrt{\boldsymbol{B}}^{-1})$. Therefore, the roots (the eigenvalues of $\boldsymbol{C}$) can be equivalently obtained as

$$\det\left(c_\alpha \boldsymbol{I} - \sqrt{\boldsymbol{B}}\boldsymbol{A}\sqrt{\boldsymbol{B}}\right) = 0. \tag{D.4}$$

[8]Since $\boldsymbol{B}$ is a positive definite matrix, it admits the following decomposition in terms of its eigenvalues $c_{\alpha_{\boldsymbol{B}}}$ and eigenvectors $\bar{\boldsymbol{m}}_{\alpha_{\boldsymbol{B}}}$, namely $\boldsymbol{B} = c_{\alpha_{\boldsymbol{B}}}\bar{\boldsymbol{m}}_{\alpha_{\boldsymbol{B}}} \otimes \bar{\boldsymbol{m}}_{\alpha_{\boldsymbol{B}}}$.The symmetric matrix $\sqrt{\boldsymbol{B}}$ can be defined as $\sqrt{\boldsymbol{B}} = \sqrt{c_{\alpha_{\boldsymbol{B}}}}\bar{\boldsymbol{m}}_{\alpha_{\boldsymbol{B}}} \otimes \bar{\boldsymbol{m}}_{\alpha_{\boldsymbol{B}}}$.

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Equation (D.4) proves that the eigenvalues of the matrix $\boldsymbol{C}$ defined in Theorem Appendix D.1 are the same as those for the matrix defined as $\sqrt{\boldsymbol{B}}\boldsymbol{A}\sqrt{\boldsymbol{B}}$ (the same cannot be said regarding the eigenvectors). Symmetry of this new matrix $\sqrt{\boldsymbol{B}}\boldsymbol{A}\sqrt{\boldsymbol{B}}$ guarantees therefore that its eigenvalues, and therefore those for matrix $\boldsymbol{C}$ are real.

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Appendix E. Algebraic manipulations for the explicit representation of acoustic waves in dielectric elastomers

The objective of this Section is to obtain a simple expression for the second order tensor $\boldsymbol{\mathcal{C}}_{\boldsymbol{D}}$ in equation (111) which would allow to obtain an explicit representation for the speed of propagation of acoustic waves for the convex multi-variable constitutive model in equation (28), particularised to the case of electromechanics. This tensor involves computing the inverse of the constitutive tensor $\boldsymbol{\theta}$ (C.11), which is defined as

$$\boldsymbol{\theta} = \frac{1}{\varepsilon_1}\boldsymbol{I} + \frac{1}{\varepsilon_2}\boldsymbol{C}, \tag{E.1}$$

where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green deformation tensor. The inverse of every symmetric positive definite second order tensor (still a symmetric positive definite second order tensor), and in particular $\boldsymbol{\theta}^{-1}$ admits the following additive decomposition [72]

$$\boldsymbol{\theta}^{-1} = \alpha_1\boldsymbol{I} + \alpha_2\boldsymbol{C} + \alpha_3\boldsymbol{G}, \tag{E.2}$$

where $\boldsymbol{G}$ is the Co-factor of $\boldsymbol{C}$. Noting that $\boldsymbol{\theta}\boldsymbol{\theta}^{-1} = \boldsymbol{I}$ and $\boldsymbol{C}\boldsymbol{G} = J^2$ and considering equations (E.1) and (E.2) gives

$$\begin{aligned}\boldsymbol{\theta}\boldsymbol{\theta}^{-1} &= \left(\frac{1}{\varepsilon_1}\boldsymbol{I} + \frac{1}{\varepsilon_2}\boldsymbol{C}\right)(\alpha_1\boldsymbol{I} + \alpha_2\boldsymbol{C} + \alpha_3\boldsymbol{G}) \\ &= \left(\frac{\alpha_1}{\varepsilon_1} + \frac{\alpha_3 J^2}{\varepsilon_2}\right)\boldsymbol{I} + \left(\frac{\alpha_2}{\varepsilon_1} + \frac{\alpha_1}{\varepsilon_2}\right)\boldsymbol{C} + \frac{\alpha_2}{\varepsilon_2}\boldsymbol{C}^2 + \frac{\alpha_3}{\varepsilon_1}\boldsymbol{G}.\end{aligned} \tag{E.3}$$

As shown in [40], the Co-factor of a symmetric positive definite tensor, and in particular $\boldsymbol{G}$, can be alternatively expressed as

$$\boldsymbol{G} = \boldsymbol{C}^2 - I_{1\boldsymbol{C}}\boldsymbol{C} + I_{2\boldsymbol{C}}\boldsymbol{I}. \tag{E.4}$$

Introduction of the expression for $\boldsymbol{G}$ in (E.4) into equation (E.3) yields,

$$\boldsymbol{\theta}\boldsymbol{\theta}^{-1} = \left(\frac{\alpha_1}{\varepsilon_1} + \frac{\alpha_3 J^2}{\varepsilon_2} + \frac{\alpha_3 I_{2\boldsymbol{C}}}{\varepsilon_1}\right)\boldsymbol{I} + \left(\frac{\alpha_2}{\varepsilon_1} + \frac{\alpha_1}{\varepsilon_2} - \frac{\alpha_3 I_{1\boldsymbol{C}}}{\varepsilon_1}\right)\boldsymbol{C} + \left(\frac{\alpha_2}{\varepsilon_2} + \frac{\alpha_3}{\varepsilon_1}\right)\boldsymbol{C}^2 = \boldsymbol{I}. \tag{E.5}$$

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Identification of the coefficients $\{\alpha_1, \alpha_2, \alpha_3\}$ in above equation (E.5) can be obtained by solving the following system of linear equations

$$\begin{aligned} \frac{\alpha_1}{\varepsilon_1} + \frac{\alpha_3 J^2}{\varepsilon_2} + \frac{\alpha_3 I_{2\boldsymbol{C}}}{\varepsilon_1} &= 1; \\ \frac{\alpha_2}{\varepsilon_1} + \frac{\alpha_1}{\varepsilon_2} - \frac{\alpha_3 I_{1\boldsymbol{C}}}{\varepsilon_1} &= 0; \\ \frac{\alpha_2}{\varepsilon_2} + \frac{\alpha_3}{\varepsilon_1} &= 0. \end{aligned} \tag{E.6}$$

Solution of the above system of equations results in

$$\alpha_1 = \frac{\theta_1^2 + \theta_1\theta_2 I_{1\boldsymbol{C}}}{g}; \qquad \alpha_2 = -\frac{\theta_1\theta_2}{g}; \qquad \alpha_3 = \frac{\theta_2^2}{g}, \tag{E.7}$$

where $\theta_i = \frac{1}{\varepsilon_i}$, $i = \{1, 2\}$ and with

$$g = \theta_1^3 + \theta_1^2\theta_2 I_{1\boldsymbol{C}} + \theta_1\theta_2^2 I_{2\boldsymbol{C}} + \theta_2^3 J^2. \tag{E.8}$$

With the help of the result in equations (E.1) and (E.7), it is now possible to evaluate the second order $\boldsymbol{\mathcal{C}_D}$ in equation (111) and to re-express it in a more convenient manner which would enable to obtain an explicit representation of the speed of propagation of acoustic waves for the convex multi-variable constitutive model considered, and defined in equation (28). This tensor can be additively decomposed into two tensor $\boldsymbol{\mathcal{C}_{D1}}$ and $\boldsymbol{\mathcal{C}_{D2}}$ as $\boldsymbol{\mathcal{C}_D} = \boldsymbol{\mathcal{C}_{D1}} - \boldsymbol{\mathcal{C}_{D2}}$, defined as

$$\begin{aligned} \boldsymbol{\mathcal{C}_{D1}} &= \boldsymbol{F}\left(\boldsymbol{I} - \boldsymbol{N}\otimes\boldsymbol{N}\right)\boldsymbol{\theta}^{-1}\boldsymbol{F}^T; \\ \boldsymbol{\mathcal{C}_{D2}} &= \boldsymbol{F}\left(\boldsymbol{I} - \boldsymbol{N}\otimes\boldsymbol{N}\right)\frac{\boldsymbol{\theta}^{-1}\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N}}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\boldsymbol{F}^T. \end{aligned} \tag{E.9}$$

Proper re-arrangement of the expressions for $\boldsymbol{\mathcal{C}_{D1}}$ and $\boldsymbol{\mathcal{C}_{D2}}$ in above equation (E.9) yields

$$\boldsymbol{\mathcal{C}_{D1}} = \left(\sum_{\alpha=1}^{2}\lambda_\alpha \boldsymbol{t}_\alpha\otimes\boldsymbol{T}_\alpha\right)\boldsymbol{\theta}^{-1}\boldsymbol{F}^T; \tag{E.10a}$$

$$\boldsymbol{\mathcal{C}_{D2}} = \left(\sum_{\alpha=1}^{2}\lambda_\alpha \boldsymbol{t}_\alpha\otimes\boldsymbol{T}_\alpha\right)\frac{\boldsymbol{\theta}^{-1}\boldsymbol{N}\otimes\boldsymbol{\theta}^{-1}\boldsymbol{N}}{\boldsymbol{N}\cdot\boldsymbol{\theta}^{-1}\boldsymbol{N}}\boldsymbol{F}^T. \tag{E.10b}$$

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Analytical derivations of the wave speepds requires pre- and post-multiplication of the above second order tensor $\boldsymbol{\mathcal{C}}_{\boldsymbol{D1}}$ and $\boldsymbol{\mathcal{C}}_{\boldsymbol{D2}}$ by the eigenmodes $\bar{\boldsymbol{p}}_\alpha$, yielding

$$\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D1}} \bar{\boldsymbol{p}}_\alpha = \left(\sum_{\alpha=1}^{2} \lambda_\alpha \boldsymbol{t}_\alpha \cdot \bar{\boldsymbol{p}}_\alpha\right) \left(\boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha\right);$$

$$\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D2}} \bar{\boldsymbol{p}}_\alpha = \frac{\left(\sum_{\alpha=1}^{2} \lambda_\alpha \boldsymbol{t}_\alpha \cdot \bar{\boldsymbol{p}}_\alpha\right) \left(\boldsymbol{T}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N}\right) \left(\boldsymbol{\theta}^{-1} \boldsymbol{N} \cdot \boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha\right)}{\boldsymbol{N} \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N}},$$

which enables $\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \bar{\boldsymbol{p}}_\alpha$ to be finally obtained as

$$\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \bar{\boldsymbol{p}}_\alpha = \left(\sum_{\alpha=1}^{2} \lambda_\alpha \boldsymbol{t}_\alpha \cdot \bar{\boldsymbol{p}}_\alpha\right) \left(\left(\boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha\right) - \frac{\left(\boldsymbol{T}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N}\right) \left(\boldsymbol{\theta}^{-1} \boldsymbol{N} \cdot \boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha\right)}{\boldsymbol{N} \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N}}\right) \tag{E.11}$$

where $\boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha$ and $\boldsymbol{\theta}^{-1} \boldsymbol{N}$ in above equation (E.11) can be written by making use of the identities in equation (112) and of equation (E.2) for the inverse of the second order tensor $\boldsymbol{\theta}$ defined in equation (E.1) as

$$\begin{aligned} \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha &= \left(\alpha_1 + \alpha_2 \lambda_\alpha^2 + \alpha_3 \left(\frac{J}{\lambda_\alpha}\right)^2\right) \boldsymbol{T}_\alpha; \\ \boldsymbol{\theta}^{-1} \boldsymbol{N} &= \left(\alpha_1 + \alpha_2 \lambda_3^2 + \alpha_3 \left(\frac{J}{\lambda_3}\right)^2\right) \boldsymbol{N}. \end{aligned} \tag{E.12}$$

From equation (E.12), it is easy to notice that

$$\boldsymbol{T}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{N} = 0. \tag{E.13}$$

Therefore, equation (E.11) can be further simplified as

$$\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \bar{\boldsymbol{p}}_\alpha = \left(\sum_{\alpha=1}^{2} \lambda_\alpha \boldsymbol{t}_\alpha \cdot \bar{\boldsymbol{p}}_\alpha\right) \left(\boldsymbol{F}^T \bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha\right) \tag{E.14}$$

For pressure waves, characterised by $\bar{\boldsymbol{p}}_\alpha = \boldsymbol{n}$, it is easy to see that $\boldsymbol{n} \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \boldsymbol{n} = 0$. For shear waves, where $\bar{\boldsymbol{p}}_\alpha = \boldsymbol{t}_\alpha$ (with $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ two orthonormal vectors to $\boldsymbol{n}$), the expression for $\bar{\boldsymbol{p}}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \bar{\boldsymbol{p}}_\alpha$ in (E.14) becomes

$$\boldsymbol{t}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \boldsymbol{t}_\alpha = \lambda_\alpha \left(\boldsymbol{F}^T \boldsymbol{t}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha\right). \tag{E.15}$$

Finally, use of equations $(112)_b$ and $(\text{E.12})_b$ enables $\boldsymbol{t}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \boldsymbol{t}_\alpha$ to be written as

$$\boldsymbol{t}_\alpha \cdot \boldsymbol{\mathcal{C}}_{\boldsymbol{D}} \boldsymbol{t}_\alpha = \lambda_\alpha \left(\boldsymbol{F}^T \boldsymbol{t}_\alpha \cdot \boldsymbol{\theta}^{-1} \boldsymbol{T}_\alpha\right) = \lambda_\alpha \left(\alpha_1 + \alpha_2 \lambda_3^2 + \alpha_3 \left(\frac{J}{\lambda_3}\right)^2\right). \tag{E.16}$$

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