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Semiclassical stochastic mechanics for the Coulomb potential with applications to modelling dark matter

Andrew Neate[1, a)] and Aubrey Truman[1, b)]
Mathematics Department, Swansea University, Singleton Park, Swansea, SA2 8PP. UK

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Little is known about dark matter particles save that their most important interactions with ordinary matter are gravitational and that, if they exist, they are stable, slow moving and relatively massive. Based on these assumptions, a semiclassical approximation to the Schrödinger equation under the action of a Coulomb potential should be relevant for modelling their behaviour. We investigate the semiclassical limit of the Schrödinger equation for a particle of mass M under a Coulomb potential in the context of Nelson's stochastic mechanics. This is done using a Freidlin-Wentzell asymptotic series expansion in the parameter $\epsilon = \sqrt{\hbar/M}$ for the Nelson diffusion. It is shown that for wave functions $\psi \sim \exp((R + iS)/\epsilon^2)$ where R and S are real valued, the $\epsilon = 0$ behaviour is governed by a constrained Hamiltonian system with Hamiltonian H^r and constraint $H^i = 0$ where the superscripts r and i denote the real and imaginary parts of the Bohr correspondence limit of the quantum mechanical Hamiltonian, independent of Nelson's ideas. Nelson's stochastic mechanics is restored in dealing with the nodal surface singularities and by computing (correct to first order in ϵ) the relevant diffusion process in terms of Jacobi fields thereby revealing Kepler's laws in a new light. The key here is that the constrained Hamiltonian system has just two solutions corresponding to the forward and backward drifts in Nelson's stochastic mechanics. We discuss the application of this theory to modelling dark matter particles under the influence of a large gravitating point mass.

I. INTRODUCTION

Nelson's stochastic mechanics[19,20,22], is an interpretation of non-relativistic quantum mechanics in which stochastic processes are used to model the behaviour of particles. In the theory, for a given wave function ψ, one can construct a corresponding diffusion process $\boldsymbol{X}_t$ whose probability density corresponds to the quantum probability density $|\psi|^2$. One of the major achievements of Nelson's theory is the demonstration that the diffusion process $\boldsymbol{X}_t$ satisfies a stochastic extension of Newton's second law of motion which is mathematically equivalent to the Schrödinger equation, allowing a classical derivation of the Schrödinger equation. Indeed, the original interpretation of the theory takes the sample path of the diffusion $\boldsymbol{X}_t$ to represent the trajectory of the system configuration, which enables one to address problems inaccessible to the usual Schrödinger theory. However, the theory is thought by many to be unduly complicated, the physical role of the sample paths is unclear and there are certain technical problems which now mean that the theory is regarded by some as a lost cause. Even Nelson himself disowned his original interpretation[21] and he subsequently posed the question, "How can a theory be so right and yet so wrong?"[22].

Here we consider the stochastic mechanics for the Bohr correspondence limit[5] of a particular family of stationary states for the Schrödinger equation of a single particle of mass M acted on by a Coulomb potential. These stationary states are known as the *atomic elliptic states* and are concentrated on ellipses[15,16]. The wave functions ψ for these states are such

a) Electronic mail: a.d.neate@swansea.ac.uk
b) Electronic mail: a.truman@swansea.ac.uk

that $\psi \sim \exp\left((R+iS)/\epsilon^2\right)$ as $\epsilon \sim 0$, where R and S are real valued and $\epsilon^2 = \hbar/M$ where M is the mass of the particle. Since we are only considering stationary states many of the usual problems of stochastic mechanics do not apply. In what follows we do not rely heavily on Nelson's detailed assumptions, but are instead guided at every stage by conventional quantum mechanics. Our treatment reveals a surprising connection between Nelson's theory applied to the Bohr correspondence limit of these stationary states and constrained Hamiltonian systems.

More precisely, in this paper we consider a diffusion $\boldsymbol{X}_t$ in $\mathbb{R}^3$ derived from the Bohr correspondence limit of the Schrödinger equation for a Coulomb potential, with the sample paths viewed as semiclassical particle trajectories. In particular the diffusion $\boldsymbol{X}_t$ satisfies an Itô stochastic differential equation of the form,

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)\,\mathrm{d}t + \epsilon\,\mathrm{d}\boldsymbol{B}_t \tag{1}$$

where the drift $\boldsymbol{b}(\boldsymbol{x}) = (b_1(\boldsymbol{x}), b_2(\boldsymbol{x}), b_3(\boldsymbol{x}))$ in Cartesian coordinates with $\boldsymbol{x} = (x_1, x_2, x_3)$ is determined by the semiclassical state according to the rules of Nelson's theory. Here again $\epsilon^2 = \hbar/M$ where M is the mass of the particle and $\boldsymbol{B}_t$ is a three dimensional Brownian motion process. Aspects of this diffusion have been studied previously[10–12,17,18]. However, the drift $\boldsymbol{b}$ is singular not only at the origin, but also across a surface corresponding to the semiclassical limit of the nodes of the original wave functions for the atomic elliptic states. Consequently it is difficult to prove the pathwise uniqueness of solutions $\boldsymbol{X}_t$ for (1) (for results on Nelson diffusions related to nodal wave functions see the work of Carlen[6]). As the diffusion $\boldsymbol{X}_t$ is derived from a semiclassical wave function which is an approximation to the original wave function correct to terms of order $\epsilon = \sqrt{\hbar/M}$, we actually only need seek a diffusion which is itself correct to terms of order ϵ. Thus we can resort to Friedlin-Wentzell asymptotics[14] by considering an asymptotic expansion for $\boldsymbol{X}_t$ in the small parameter ϵ,

$$\boldsymbol{X}_t = \boldsymbol{X}_t^0 + \epsilon \boldsymbol{X}_t^1 + O(\epsilon^2). \tag{2}$$

In this paper we study the behaviour of the first two terms in this asymptotic expansion.

Following Freidlin-Wentzell and substituting (2) back into (1) yields the leading order term $\boldsymbol{X}_t^0$ given by the ordinary differential equation,

$$\mathrm{d}\boldsymbol{X}_t^0 = \boldsymbol{b}(\boldsymbol{X}_t^0)\,\mathrm{d}t$$

and the Gaussian correction term $\boldsymbol{X}_t^1$ given by the stochastic differential equation,

$$\mathrm{d}\boldsymbol{X}_t^1 = b'(\boldsymbol{X}_t^0)\boldsymbol{X}_t^1\,\mathrm{d}t + \mathrm{d}\boldsymbol{B}_t,$$

where b' is the matrix of first order partial derivatives,

$$(b')_{ij} = \left(\frac{\partial b_i}{\partial x_j}\right), \qquad i, j = 1, 2, 3.$$

However, we still have to consider the problem of the singularities for $\boldsymbol{X}^0$ and $\boldsymbol{X}^1$ albeit that they now appear in a more manageable form.

At its most classical level (the process $\boldsymbol{X}_t^0$), the behaviour is governed by Newtonian dynamics in a semiclassical perturbation of a Coulomb potential where the perturbation of the potential depends on the semiclassical wave function. Indeed for any wave functions $\psi \sim \exp\left((R+iS)/\epsilon^2\right)$, in the Bohr correspondence limit, we obtain a constrained Hamiltonian system whose only solutions correspond to the forward and negative backward drifts from Nelson's stochastic mechanics. Thus, Nelson's stochastic mechanics is an inevitable feature of the Bohr correspondence limit for such wave functions which abound in the quantum physics of Coulomb and harmonic oscillator potentials. Equally this makes our particular treatment of the singularities seem inevitable in the semiclassical limit. We also uncover some striking new semiclassical phenomena some of which arise from elementary symmetries and some from the inherent singularity structure due to the nodal surfaces of the Schrödinger wave functions.

From an alternative perspective, Nelson's stochastic mechanics can be viewed as a stochastic perturbation of classical Newtonian mechanics, which happens to agree with the predictions of standard quantum mechanics. In this sense the equations we consider here can be viewed as a stochastic perturbation of a classical two body problem, and several such perturbations and their application to problems in astronomy have been studied previously in the literature for instance[1,7,8,17,23,28]. Here we have to emphasise that the Bohr correspondence limit for our family of wave functions gives rise to an additional Bohm type potential proportional to the modulus squared of the logarithmic derivative of $|\psi|^2$. This surprising feature does not come from Nelson's ideas but the Bohr correspondence limit. Our Bohm potential drives our putative particle trajectories in the long time limit to the classical two body problem orbits and, as we shall see, has a profound effect on the usual constants of the motion. This is what has prompted the suggested applications here in.

Indeed here we propose a new application of stochastic mechanics, to modelling the behaviour of cold dark matter particles. There is strong evidence for the existence of dark matter, yet no definitive theory of dark matter particles exists. The dynamics of the best candidates for dark matter particles, weakly interacting massive particles (WIMPs), is still unknown except in the broadest of outlines[2,27]. Here we borrow this nomenclature and refer to our dark matter particle as a WIMP-like particle.

Our putative WIMP-like particles have a large mass and, apart from their gravitational effects, interact with ordinary matter only through the short range weak force. Moreover, the clumping of galaxies indicates that these particles must be stable and slow moving, suggesting that a model for their behaviour can neglect relativistic effects. Here we propose a semiclassical model for the behaviour of a WIMP-like particle based on the Schrödinger equation (as the only Galilean invariant quantum theory) under a Coulomb potential associated with the gravitational field of a point mass representing a distant star or gas giant. That is we propose to model the trajectory of a WIMP-like particle using our diffusion $\boldsymbol{X}_t$. As we shall see, a wide range of behaviours are possible even in this simple model, depending upon the mass of the WIMP-like particle and its initial conditions.

The relevance of our results to this proposed model depends critically on the accuracy with which the quantum Hamiltonian describes the WIMP-like particle system. We make no apology here for applying our theory to systems on astronomical length scales, realising that our model only predicts WIMP-like particle behaviour on these same scales for both time and space. For such particles in proto-solar or proto-ring nebulae dominated by the gravitational force due to a point mass, taking the classical limit of coherent quantum states we feel is amply justified. Indeed, MacCullagh's formula for distant, slowly varying mass distributions suggests that our model should be much more widely applicable. As we shall see, in this setting, Nelson's stochastic mechanics reveals the classical Keplerian orbits in a totally new light.

Our model also provides a natural route to the well known Burgers equation models for the behaviour of cold dark matter which originate in the work of Zeldovich[29]. Nelson's theory is closely related to these equations and as has been shown previously, considerations of the semi-classical Nelson diffusions lead naturally to Burgers equations[17]. Indeed the hope here is that one could include the weak interactions by an averaging process reflecting the presence of Baryonic matter in a perturbation theoretic framework.

In the next section we collect together some of the background results we need in the main part of the paper. Then in Section 3 we show that our diffusion satisfies an underlying variational principle using ideas of Bismut. In sections 4 and 5 we investigate the behaviour of the zero and first order approximations $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^1$.

II. BACKGROUND

We begin with some background on the quantum Coulomb problem. For the principal quantum number $n = 0, 1, 2, 3, \ldots$, angular momentum quantum number $l = 0, 1, 2, \ldots, (n-1)$ and magnetic quantum number $m = -l, -l+1, \ldots, (l-1), l$, let ψ_{nlm} denote the usual nodal eigenfunction for the quantum Coulomb Hamiltonian,

$$H = \frac{\boldsymbol{P}^2}{2} - \frac{\mu}{|\boldsymbol{Q}|},$$

$\boldsymbol{P} = (P_1, P_2, P_3)$ being the particle momentum operator, $\boldsymbol{Q} = (Q_1, Q_2, Q_3)$ the position operator and $\mu > 0$. Then for the orbital angular momentum $\boldsymbol{L} = (L_1, L_2, L_3) := \boldsymbol{Q} \wedge \boldsymbol{P}$ and $|\boldsymbol{L}|^2 = L_1^2 + L_2^2 + L_3^2$,

$$H\psi_{nlm} = E_n \psi_{nlm}, \tag{3}$$
$$|\boldsymbol{L}|^2 \psi_{nlm} = l(l+1)\hbar^2 \psi_{nlm}, \tag{4}$$
$$L_3 \psi_{nlm} = m\hbar \psi_{nlm}, \tag{5}$$

where $E_n = -\mu^2 (2n^2\hbar^2)^{-1}$. It is well known that the quantum state $\psi_{n,n-1,n-1}(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, x_2, x_3)$ in Cartesians, is concentrated with minimal uncertainty on the Keplerian circular orbit situated in the plane $x_3 = 0$, centred at the origin with radius $-\mu/(2E_n)$, as is its conjugate $\psi_{n,n-1,1-n}$[26].

Now let $\boldsymbol{A} = (A_1, A_2, A_3)$ denote the quantum Lenz-Runge vector,

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{P} \wedge \boldsymbol{L} - \boldsymbol{L} \wedge \boldsymbol{P}) - \frac{\mu \boldsymbol{Q}}{|\boldsymbol{Q}|}.$$

Then following Pauli[24] the usual $SO(4)$ algebra gives,

$$(L_1 + iL_2)\psi_{n,n-1,n-1} = 0, \tag{6}$$
$$(A_1 + iA_2)\psi_{n,n-1,n-1} = 0, \tag{7}$$
$$L_3 \psi_{n,n-1,n-1} = \hbar(n-1)\psi_{n,n-1,n-1}. \tag{8}$$

Less well known is the fact that,

$$A_3 \psi_{n,n-1,n-1} = 0. \tag{9}$$

For $0 < e < 1$ and $\sin\theta = e$ we can define the *atomic elliptic state* $\psi_{n,\theta}$ following[15,16] as,

$$\psi_{n,\theta}(\boldsymbol{x}) = \exp\left(-\frac{i\theta}{\hbar} A_2\right) \psi_{n,n-1,n-1}(\boldsymbol{x}),$$

where in Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$. For each quantum observable F we denote by $\langle F \rangle_{n,\theta}$ its average value in the state $\psi_{n,\theta}$, defined as,

$$\langle F \rangle_{n,\theta} = \langle \psi_{n,\theta}, F\psi_{n,\theta} \rangle.$$

It can be shown that $\psi_{n,\theta}(\boldsymbol{x})$ is a coherent state which minimises the angular momentum uncertainty relations and is concentrated on the Kepler ellipse $\mathcal{K}$ with eccentricity e, in the plane $x_3 = 0$, with focus at the origin O, semimajor axis of length $-\mu/(2E_n)$ parallel to the x_1 axis, given by the equation,

$$\tilde{\vartheta} = \pi/2, \qquad -\frac{\mu(1-e^2)}{2E_n} = r\left(1 + e\cos\tilde{\phi}\right),$$

where $(r, \tilde{\vartheta}, \tilde{\phi})$ denote spherical polar coordinates. This behaviour follows from the relations,

$$\langle A_1 \rangle_{n,\theta} = (n-1)\hbar \sin\theta, \qquad \langle A_2 \rangle_{n,\theta} = 0, \qquad \langle A_3 \rangle_{n,\theta} = 0, \tag{10}$$

and

$$\langle L_1 \rangle_{n,\theta} = 0, \qquad \langle L_2 \rangle_{n,\theta} = 0, \qquad \langle L_3 \rangle_{n,\theta} = \hbar(n-1)\cos\theta, \tag{11}$$

which can be deduced from (3)–(9) (see[18]). It can also be shown that the state $\psi^*_{n,\theta}$ (the complex conjugate) which corresponds to the elliptical orbit being described in the opposite sense (as in $\psi_{n,n-1,1-n}$) is also concentrated on $\mathcal{K}$ with minimal uncertainty.

It was shown in[16] that the atomic elliptic state has the Cartesian representation,

$$\psi_{n,\theta}(\boldsymbol{x}) = C \exp\left(-\frac{\mu|\boldsymbol{x}|}{n\hbar^2}\right) L_{n-1}(n\nu(\boldsymbol{x})), \tag{12}$$

where C is a normalisation constant, L_n is a Laguerre polynomial and,

$$\nu(\boldsymbol{x}) = \frac{\mu}{n^2\hbar^2}\left(|\boldsymbol{x}| - \frac{x_1}{e} - \frac{ix_2}{e}\sqrt{1-e^2}\right), \qquad |\boldsymbol{x}| = \sqrt{x_1^2+x_2^2+x_3^2}, \tag{13}$$

$\boldsymbol{x} = (x_1, x_2, x_3)$ in Cartesians.

We have proven previously[10] that the semiclassical wave function found by taking the correspondence limit $n \to \infty$, $\hbar \to 0$ with $\lambda = n\hbar$ fixed in (12) is,

$$\psi_{\text{s.c.}}(\boldsymbol{x}) = C \exp\left(\frac{1}{\epsilon^2}(R(\boldsymbol{x}) + iS(\boldsymbol{x}))\right), \tag{14}$$

where $R, S : \mathbb{R}^3 \to \mathbb{R}$, $\epsilon^2 = \hbar$ and C is some normalisation constant with $\epsilon^2 \sim 0$. The expressions for $R + iS$ can be shown to be given by,

$$R + iS = -\frac{\mu}{\lambda}|\boldsymbol{x}| + \frac{\lambda\nu}{2}\left(1 - \sqrt{1-\frac{4}{\nu}}\right) - \lambda\ln\nu - 2\lambda\ln\left(1 - \sqrt{1-\frac{4}{\nu}}\right). \tag{15}$$

Here $\sqrt{1-\frac{4}{\nu}}$ is an analytic function of ν in the complex ν plane, cut from $\nu = 0$ to $\nu = 4$, the branch of the square root being chosen to be positive on the positive reals.

The role of the Schrödinger equation (3) in this semiclassical limit is summarised in the following lemma:

Lemma II.1 ([10])**.** *If R and S are as defined as in* (15) *then,*

$$\nabla R \cdot \nabla S = 0, \qquad \frac{1}{2}\left(|\nabla S|^2 - |\nabla R|^2\right) - \frac{\mu}{|\boldsymbol{x}|} = -\frac{\mu^2}{2\lambda^2}.$$

Following Nelson[19], corresponding to the above semiclassical wave function $\psi_{\text{s.c.}}$, we can associate a diffusion $\boldsymbol{X}_t$ defined by,

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)\,\mathrm{d}t + \epsilon\,\mathrm{d}\boldsymbol{B}_t, \qquad \boldsymbol{X}_0 = \boldsymbol{x}_0 \in \mathbb{R}^3, \tag{16}$$

where $\boldsymbol{b} = (b_1, b_2, b_3) = \nabla(R + S)$ and $\boldsymbol{B}_t$ is a three dimensional Brownian motion.

The drift $\boldsymbol{b}$ is known as the mean forward velocity and can be viewed as the conditional expectation (conditioned on the event $\boldsymbol{X}_t \approx \boldsymbol{x}$) of the velocity of departure from $\boldsymbol{x}$ of the diffusing semiclassical particle. We will consider the sample paths of $\boldsymbol{X}_t$ as representing the semiclassical particle trajectories for a WIMP-like particle associated to the semiclassical state $\psi_{\text{s.c.}}$.

We can also associate a "reflected" process,

$$\mathrm{d}\boldsymbol{X}^*_t = -\boldsymbol{b}^*(\boldsymbol{X}^*_t)\,\mathrm{d}t + \epsilon\,\mathrm{d}\boldsymbol{B}_t, \qquad \boldsymbol{X}^*_0 = \boldsymbol{x}_0 \in \mathbb{R}^3,$$

where $\boldsymbol{b}^* = (b^*_1, b^*_2, b^*_3) = \nabla(S - R)$ is known as the mean backward drift which can be viewed as the conditional expectation of the velocity of arrival of a particle at $\boldsymbol{x}$. Clearly the process $\boldsymbol{X}^*_t$ is associated with $\psi^*_{\text{s.c.}}$ (where $*$ denotes complex conjugate) in the same manner that $\boldsymbol{X}_t$ is associated with $\psi_{\text{s.c.}}$.

We note that the equations in Lemma II.1 have the obvious symmetry $(R, S) \mapsto (\pm R, \pm S)$ (any combination of $\pm$ signs) which also preserves the corresponding $|\boldsymbol{b}|^2$. We also note that

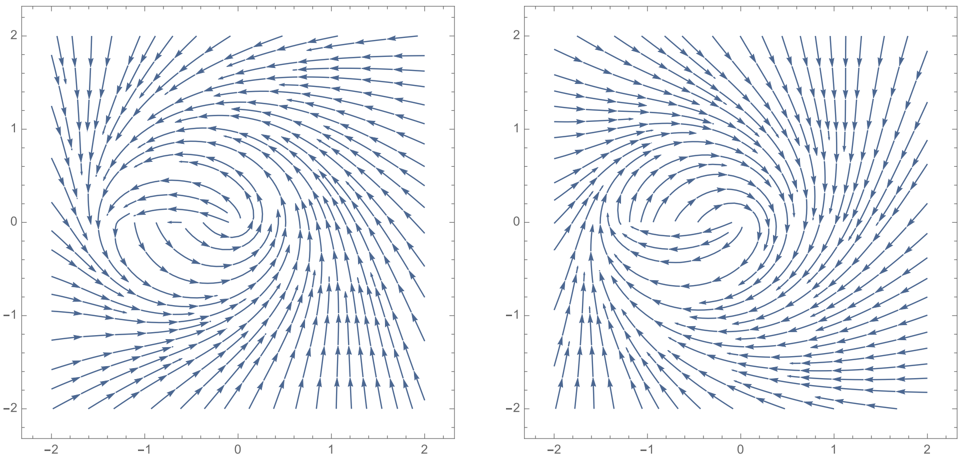

FIG. 1. Stream lines in the plane $x_3 = 0$ for the vector fields $\boldsymbol{b}$ (left) and $-\boldsymbol{b}^*$ (right) for $\lambda = \mu = 1$ and $e = 1/2$.

$(R(\boldsymbol{x}), S(\boldsymbol{x})) \mapsto (R(\boldsymbol{x}'), S(\boldsymbol{x}'))$ where $\boldsymbol{x}' = \mathcal{R}\boldsymbol{x}$ for some rotation/reflection of the Kepler ellipse $\mathcal{K}$ is also a symmetry of these equations. These transformations correspond to $\psi_{\text{s.c.}} \mapsto \psi^*_{\text{s.c.}}$, $\psi_{\text{s.c.}} \mapsto \psi^{-1}_{\text{s.c.}}$ and $\psi_{\text{s.c.}}(\boldsymbol{x}) \mapsto \psi_{\text{s.c.}}(\mathcal{R}\boldsymbol{x})$. The first and third of these are symmetries of the full Schrödinger equation, the second symmetry appears at the semiclassical level. In this work we only consider $\mathcal{R} = 1$ and the reflection $\boldsymbol{x}' = \mathcal{R}\boldsymbol{x} = (x, -y, z)$ or $\psi_{\text{s.c.}} \mapsto \psi^*_{\text{s.c.}}$ with $\boldsymbol{b} \mapsto -\boldsymbol{b}^*$, the negative backward Nelson drift. In particular we note that,

$$-\boldsymbol{b}^*(x, y, z) = (b_1(x, -y, z), -b_2(x, -y, z), b_3(x, -y, z))\,.$$

This is illustrated in the stream plots corresponding to the fields $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ shown in Figure 1.

As we shall see the diffusions $\boldsymbol{X}_t$ and $\boldsymbol{X}^*_t$ share many interesting properties.

III. A STOCHASTIC HAMILTONIAN FRAMEWORK

We begin by showing that our diffusions $\boldsymbol{X}_t$ and $\boldsymbol{X}^*_t$ can be related to a variational principle. We recall a definition of Bismut[3]. Let $\boldsymbol{Q}_t = (Q^1_t, Q^2_t, Q^3_t)$ and $\boldsymbol{P}_t = (P^1_1, P^2_t, P^3_t)$ in Cartesians be two diffusion processes.

Definition III.1. The diffusion $(\boldsymbol{Q}_t, \boldsymbol{P}_t)$ is said to be a (3 dimensional) *stochastic Hamiltonian system* if there exists a family of smooth functions $H_j : \mathbb{R}^7 \to \mathbb{R}$ for $j = 0, 1, 2, 3$ where each $H_j = H_j(t, \boldsymbol{p}, \boldsymbol{x})$ such that,

$$\mathrm{d}P^i_t = -\frac{\partial H_0}{\partial x_i}(t, \boldsymbol{P}_t, \boldsymbol{Q}_t)\,\mathrm{d}t - \sum_{j=1}^{3} \frac{\partial H_j}{\partial x^i}(t, \boldsymbol{P}_t, \boldsymbol{Q}_t) \circ \partial B^j_t,$$

$$\mathrm{d}Q^i_t = \frac{\partial H_0}{\partial p_i}(t, \boldsymbol{P}_t, \boldsymbol{Q}_t)\,\mathrm{d}t + \sum_{j=1}^{3} \frac{\partial H_j}{\partial p_i}(t, \boldsymbol{P}_t, \boldsymbol{Q}_t) \circ \partial B^j_t,$$

for each $i = 1, 2, 3$ where $\boldsymbol{B}_t = (B^1_t, B^2_t, B^3_t)$ is a 3 dimensional Brownian motion and $\circ\partial$ denotes a Stratonovich integral.

Consider the diffusion,

$$\mathrm{d}X_t = \boldsymbol{b}(\boldsymbol{X}_t)\,\mathrm{d}t + \epsilon\,\mathrm{d}\boldsymbol{B}_t,$$

where $\boldsymbol{b} = \nabla(R+S)$. Recall we define the matrix,

$$(b')_{ij} = \frac{\partial b_i}{\partial x_j}, \qquad i,j = 1,2,3.$$

If we set $\boldsymbol{Q}_t = \boldsymbol{X}_t$ and $\boldsymbol{P}_t = \boldsymbol{b}(\boldsymbol{X}_t)$ then we have,

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{P}_t\,\mathrm{d}t + \epsilon \circ \partial \boldsymbol{B}_t,$$

and by the chain rule for Stratonovich integrals,

$$\mathrm{d}\boldsymbol{P}_t = b' \circ \partial \boldsymbol{X}_t = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\,\mathrm{d}t + \epsilon b' \circ \partial \boldsymbol{B}_t.$$

Now since $\boldsymbol{b}$ is irrotational it follows that,

$$(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = \frac{1}{2}\nabla|\boldsymbol{b}|^2.$$

Moreover, from Lemma II.1 we see that,

$$\frac{1}{2}\nabla|\boldsymbol{b}|^2 = \nabla\left(\frac{1}{|\boldsymbol{x}|} + |\nabla R|^2\right). \tag{17}$$

Thus it follows that:

Theorem III.1. *The semiclassical Nelson diffusion process $(\boldsymbol{X}_t, \boldsymbol{P}_t)$ defines a 3 dimensional stochastic Hamiltonian system whose stochastic flow preserves the symplectic form with Hamiltonian family,*

$$H_0(t,\boldsymbol{p},\boldsymbol{x}) = \frac{1}{2}|\boldsymbol{p}|^2 + V_{\text{eff}}(\boldsymbol{x}), \quad H_j(t,\boldsymbol{p},\boldsymbol{x}) = \epsilon\left(p_j - b_j(\boldsymbol{x})\right),$$

where $V_{\text{eff}}(\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|} - |\nabla R|^2(\boldsymbol{x})$ is the effective potential incorporating a Bohmian potential $-|\nabla R|^2$.

A similar result holds for $\boldsymbol{X}_t^*$ with $\boldsymbol{b}$ replaced by $-\boldsymbol{b}^*$. We note that the Hamiltonian H_0 remains unchanged. The effective potential V_{eff} will play an important role in what follows.

Standard results of Bismut now show that there is an underlying variational principle:

Theorem III.2. *The process $(\boldsymbol{X}_s, \boldsymbol{P}_s)$ for $s \in (0,t)$ satisfies Hamilton's principle in that it corresponds to a critical point of Bismut's action functional,*

$$\mathcal{A}[\boldsymbol{X},\boldsymbol{P};t] := \int_0^t H_0(\boldsymbol{X}_s,\boldsymbol{P}_s)\,\mathrm{d}s + \sum_{i=1}^{3}\int_0^t H_i(\boldsymbol{X}_s,\boldsymbol{P}_s)\circ\partial B_s^i,$$

for variations with fixed end points.

Moreover, the corresponding Hamilton-Jacobi equation,

$$\frac{\partial \mathcal{S}}{\partial t}(\boldsymbol{x},t) + H_0(\boldsymbol{x},\nabla\mathcal{S}(\boldsymbol{x},t)) + \epsilon\left(\nabla\mathcal{S}(\boldsymbol{x},t) - \boldsymbol{b}(\boldsymbol{x})\right)\circ\partial\boldsymbol{B}_t = 0,$$

has stationary state solution,

$$\mathcal{S}(\boldsymbol{x},t) = R(\boldsymbol{x}) + S(\boldsymbol{x}) - Et,$$

where $E = -\mu^2/(2\lambda^2)$.

Proof. See Bismut[3]. □

We note that these properties do not hold for all stochastic perturbations of two body problems. For instance the Sharma-Parthasarathy two body problem applied to solar dust has recently been shown not to form a stochastic Hamiltonian system[7]. This system also has strong first integrals arising from the constants of the motion which will be discussed in Section IV B.

IV. $\boldsymbol{X}_t^0$ – THE LEADING ORDER BEHAVIOUR

Sadly, apart from a few special cases, the diffusion process $\boldsymbol{X}_t$ is rather intractable; for example one cannot easily prove pathwise uniqueness of $\boldsymbol{X}_t$. For this reason we now consider an asymptotic approximation to $\boldsymbol{X}_t$. For simplicity we work in natural units in which the semimajor axis of the Kepler ellipse is 1 and the energy $E = -\frac{1}{2}$ (i.e. we set $\mu = \lambda = 1$). Recall that following Freidlin and Wentzell we can write $\boldsymbol{X}_t$ as an asymptotic series in ϵ,

$$\boldsymbol{X}_t = \boldsymbol{X}_t^0 + \epsilon \boldsymbol{X}_t^1 + \epsilon^2 \boldsymbol{X}_t^2 + \dots,$$

In this section we look at the zeroth order term in this series, and in the subsequent section we will look at the first order quantum correction term $\boldsymbol{X}_t^1$.

A. Newtonian dynamics under a semiclassical perturbation of the Coulomb potential

We begin by considering the zeroth order term in our asymptotic expansion. That is the process $\boldsymbol{X}_t^0$ defined by,

$$\mathrm{d}\boldsymbol{X}_t^0 = \boldsymbol{b}(\boldsymbol{X}_t^0)\,\mathrm{d}t = \nabla(R+S)(\boldsymbol{X}_t^0)\,\mathrm{d}t, \qquad \boldsymbol{X}_{t=0}^0 = \boldsymbol{x}_0.$$

This provides a zeroth order approximation to the particle trajectory for our WIMP-like particle.

We will also consider the zero order approximation to the reflected Nelson diffusion,

$$\mathrm{d}\boldsymbol{X}_t^{0*} = -\boldsymbol{b}^*(\boldsymbol{X}_t^{0*})\,\mathrm{d}t = \nabla(R-S)(\boldsymbol{X}_t^{0*})\,\mathrm{d}t, \qquad \boldsymbol{X}_{t=0}^{0*} = \boldsymbol{x}_0,$$

which will play a role when we consider what happens when the particle reaches a point where the drift $\boldsymbol{b}$ is singular.

The pathological behaviour of $\boldsymbol{X}_t$ is a result of the singularities in the drift $\boldsymbol{b}$ which can be read off from equations (13) and (15). Here, the most important are three singularities accessible to the semiclassical limit $\boldsymbol{X}_t^0$ and how we can choose initial conditions to avoid them. The singularities are,

1. the surface Σ where $0 < \nu \leq 4$ in the plane $x_2 = 0$,
2. the curve $\mathcal{C}$ where $\nu = 0$ and $x_2 = 0$,
3. the point $|\boldsymbol{x}| = 0$.

We note that the singularity $0 < \nu < 2$ in the plane $x_2 = 0$ is not accessible to the motion unless we start the process here. We do not consider this case here.

Before seeing how to avoid these singularities we need the following result which asserts that the processes $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ correspond to the classical mechanics for a semiclassical perturbation of the Coulomb potential.

In this connection, we consider the constrained Hamiltonian system $(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^6$ with Hamiltonian,

$$H(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^2 + V_{\text{eff}}(\boldsymbol{x}), \qquad V_{\text{eff}}(\boldsymbol{x}) = -\frac{1}{|\boldsymbol{x}|} - |\nabla R(\boldsymbol{x})|^2, \qquad \boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^3, \tag{18}$$

given $H = -1/2$ with the constraint,

$$(\boldsymbol{p} - \nabla R(\boldsymbol{x})) \cdot \nabla R(\boldsymbol{x}) = 0. \tag{19}$$

For this system Hamilton's equations read,

$$\dot{\boldsymbol{x}} = \boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = -\nabla V_{\text{eff}}(\boldsymbol{x}), \qquad \Leftrightarrow \qquad \ddot{\boldsymbol{x}} = -\nabla V_{\text{eff}}(\boldsymbol{x}). \tag{20}$$

Lipschitz continuity gives the existence and uniqueness for this Hamiltonian system (without the constraints) for initial conditions $(\boldsymbol{x}_0, \boldsymbol{p}_0)$ up to the first arrival time at the singularities such as Σ. In light of Lemma II.1, the most general constrained solution would have to satisfy,

$$\boldsymbol{p} = (\nabla R)(\boldsymbol{x}) + (c_1 \nabla S)(\boldsymbol{x}) + (c_2 \nabla R \wedge \nabla S)(\boldsymbol{x}) \tag{21}$$

for some real valued scalar functions of position c_1 and c_2. Now, from energy conservation and Lemma II.1 again, it follows that,

$$\left((c_1^2 - 1) + c_2^2 |\nabla R|^2\right) |\nabla S|^2 = 0.$$

The interest here clearly centres on the case $c_1^2 = 1$ with $c_2 = 0$ which give us $\boldsymbol{p} = \nabla R(\boldsymbol{x}) \pm \nabla S(\boldsymbol{x})$.

However, from the chain rule (see equation (17)),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\nabla R \pm \nabla S)(\boldsymbol{x})\right) = -\nabla V_{\mathrm{eff}}(\boldsymbol{x}). \tag{22}$$

Thus, if we differentiate the constrained solution (21) with respect to t it follows from (22) and Hamilton's equations that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{c}}(\boldsymbol{x}) = (\boldsymbol{p} \cdot \nabla)\tilde{\boldsymbol{c}}(\boldsymbol{x}) = 0,$$

where,

$$\tilde{\boldsymbol{c}}(\boldsymbol{x}) = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) = (c_1 \mp 1)\nabla S(\boldsymbol{x}) + c_2(\nabla R \wedge \nabla S)(\boldsymbol{x}) \neq 0,$$

if we want $c_1^2 \neq 1$ and $c_2 \neq 0$. From energy conservation we have,

$$\frac{\tilde{\boldsymbol{c}}}{|\nabla S|^2} = (c_1 \mp 1)^2 + 1 - c_1^2$$

with $|c_1| < 1$. So,

$$c_1 = 1 - \frac{|\tilde{\boldsymbol{c}}|^2}{2|\nabla S|^2}, \qquad c_2 = \pm\sqrt{\frac{1 - c_1^2}{|\nabla R|^2}},$$

where

$$(\boldsymbol{p} \cdot \nabla)\tilde{\boldsymbol{c}}^2 = (\boldsymbol{p} \cdot \nabla)\tilde{c}_i = 0$$

for $i = 1, 2, 3$. Then it follows that c_1 and c_2 must satisfy,

$$\nabla\tilde{c}_1 \cdot (\nabla\tilde{c}_2 \wedge \nabla\tilde{c}_3) = 0,$$

which fixes $|\tilde{\boldsymbol{c}}|^2$. However we still have to satisfy $(\boldsymbol{p} \cdot \nabla)\tilde{c}_i = 0$ for $i = 1, 2, 3$. So unless $|\tilde{\boldsymbol{c}}|^2 = 0$ it is overdetermined. We have therefore proved that:-

Theorem IV.1. *The unique solutions of the Hamiltonian system* (20)*, in the form of a stationary vector field for the momentum* $\boldsymbol{p}$*, corresponding to the Hamiltonian* (18) *satisfying the constraint* (19) *are given by,*

$$\boldsymbol{p}(\boldsymbol{x}) = \nabla(R \pm S)(\boldsymbol{x}).$$

Thus we have shown that the constraint (19) gives rise to Nelson's $\epsilon = 0$ dynamics. We will see shortly that the constraint (19) arises naturally from the semiclassical limit of the atomic elliptic state. We note that Hamiltonian systems with constraints have been extensively studied following ideas of Dirac[25].

We emphasise here that the perturbation of the potential comes from the density associated with the semi-classical wave function which is inherently quantum mechanical in origin. Needless to say the Bohm potential gradient $-\nabla\left(|\nabla R|^2\right)$ is singular on Σ and $\mathcal{C}$. On Σ we expect the Bohmian force field to be repulsive since Σ is a limit of nodal curves[18]. It is not difficult to show that as we cross Σ from $x_2 > 0$ to $x_2 < 0$ we have $\sqrt{1-\frac{4}{\nu}} \mapsto -\sqrt{1-\frac{4}{\nu}}$ giving a jump discontinuity in $\boldsymbol{b}$. The next lemma tells us how to avoid the singularities:

Lemma IV.1. *Assuming that $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ are well defined (and avoid the singularities) for $t > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{X}_t^0) = |\nabla R(\boldsymbol{X}_t^0)|^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{X}_t^{0*}) = |\nabla R(\boldsymbol{X}_t^{0*})|^2.$$

Proof. A simple calculation using the chain rule and (19). □

Corollary IV.1. *Assuming that $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ are well defined (and avoid the singularities) and both $\nabla R(\boldsymbol{X}_{t=0}^0) \neq 0$ and $\nabla R(\boldsymbol{X}_{t=0}^{0*}) \neq 0$ then $R(\boldsymbol{X}_t^0)$ and $R(\boldsymbol{X}_t^{0*})$ are both monotone increasing in t.*

We also need to introduce real valued functions $\alpha, \beta : \mathbb{R}^3 \to \mathbb{R}$ such that,

$$\alpha(\boldsymbol{x}) + i\beta(\boldsymbol{x}) = \sqrt{1 - \frac{4}{\nu(\boldsymbol{x})}}.$$

On Σ we have $\alpha(\boldsymbol{x}) = x_2 = 0$ and as we cross Σ we have $\beta \mapsto -\beta$, $\beta(\boldsymbol{x})$ being negative for $x_2 > 0$[10].

Lemma IV.2 ([12])**.** *For any $\boldsymbol{x} \in \mathbb{R}^3$,*

$$|\nabla R(\boldsymbol{x})|^2 = 0 \qquad \Leftrightarrow \qquad \tilde{\vartheta} = \frac{\pi}{2}, \quad \frac{(1-e^2)}{r} = 1 + e\cos\tilde{\phi},$$

where $(r, \tilde{\vartheta}, \tilde{\phi})$ are the spherical polar coordinates of the point with position vector $\boldsymbol{x}$. i.e. $|\nabla R(\boldsymbol{x})|^2 = 0$ if and only if $\boldsymbol{x} \in \mathcal{K}$, the Kepler ellipse.

Corollary IV.2. *In the infinite time limit $t \to \infty$ both $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ converge to the Kepler ellipse $\mathcal{K}$ as long as they avoid the singularities.*

Proof. Clearly R attains its global maximum on the Kepler ellipse $\mathcal{K}$ giving the result. □

We note that in Cartesian coordinates we can write $\boldsymbol{b} = (b_1, b_2, b_3)$ where[10],

$$\begin{aligned}
b_1 &= \frac{1}{2}\left((\alpha+\beta-1)\frac{1}{e} - (\alpha+\beta+1)\frac{x_1}{|\boldsymbol{x}|}\right), \\
b_2 &= \frac{1}{2}\left((\alpha-\beta-1)\frac{\sqrt{1-e^2}}{e} - (\alpha+\beta+1)\frac{x_2}{|\boldsymbol{x}|}\right), \\
b_3 &= -\frac{1}{2}(\alpha+\beta+1)\frac{x_3}{|\boldsymbol{x}|}.
\end{aligned}$$

As we shall see in the next section, these cumbersome expressions satisfy some remarkable identities associated with semiclassical versions of the constants of the motion.

We add here a remark about the existence of $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$. By inspection, the drift $\boldsymbol{b} = \nabla(R+S)$ and the real symmetric matrix b' given by,

$$(b')_{ij} = \frac{\partial b^i}{\partial x^j} = \frac{\partial^2}{\partial x^i \partial x^j}(R+S),$$

are such that $|\boldsymbol{b}| < \infty$ and $\|b'\| < \infty$ where $|\cdot|$ and $\|\cdot\|$ denote the vector and matrix norms respectively, on any compact subset of $\mathbb{R}^3 \setminus \Sigma$. Thus, by Lipschitz continuity, the solution of the equation,

$$\dot{\boldsymbol{X}}_t^0(\boldsymbol{x}_0) = \boldsymbol{b}(\boldsymbol{X}_t^0(\boldsymbol{x}_0)), \qquad \boldsymbol{X}_0^0(\boldsymbol{x}_0) = \boldsymbol{x}_0,$$

exists for all finite positive times and is unique even for $\boldsymbol{x}_0 \in \Sigma$ (if we specify $\boldsymbol{b}(\boldsymbol{x}_0)$) unless there exists a time $t_0 > 0$ such that $\boldsymbol{X}_t(\boldsymbol{x}_0) \to \boldsymbol{x}$ for some $\boldsymbol{x} \in \Sigma$ as $t \to t_0$. (Clearly the same holds for $\boldsymbol{X}_t^{0*}$.) We explain how to resolve this difficulty in the next sections. Before doing so, we discuss semiclassical conservation laws and constants of the motion. This will explain the origin of the constraint (24).

B. Semiclassical versions of the constants of the motion

As we have seen, $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ both satisfy Newton's equation for a semiclassical perturbation of the Coulomb potential. Here we consider the classical mechanics associated with such a problem given by,

$$\ddot{\boldsymbol{x}} = -\nabla\left(-\frac{1}{|\boldsymbol{x}|} - |\nabla R(\boldsymbol{x})|^2\right) = -\nabla V_{\text{eff}}(\boldsymbol{x}). \tag{23}$$

Recall that for the classical unperturbed Coulomb problem the motion is determined by the seven constants $H', \boldsymbol{l}'$ and $\boldsymbol{a}'$ giving the energy, angular momentum and Lenz-Runge vector respectively where,

$$H' = \frac{|\boldsymbol{p}|^2}{2} - \frac{1}{|\boldsymbol{x}|}, \qquad \boldsymbol{l}' = (\boldsymbol{x} \wedge \boldsymbol{p}), \qquad \boldsymbol{a}' = \boldsymbol{p} \wedge \boldsymbol{l}' - \frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

Note that we use primes here to denote that these are the classical quantities ($\boldsymbol{x}$ and $\boldsymbol{p}$ denote the classical concepts of position and momentum). These provide five independent constants via the constraint equations,

$$\boldsymbol{l}' \cdot \boldsymbol{a}' = 0, \qquad |\boldsymbol{a}'|^2 = 1 + 2H'|\boldsymbol{l}'|^2. \tag{24}$$

The geometric properties of the orbit are then summarised in the relations,

$$a_1' = e, \quad a_2' = a_3' = 0, \qquad l_1' = l_2' = 0, \quad l_3' = \sqrt{1 - e^2}.$$

In the quantum problem the analogues of these quantities are the observables,

$$H = \frac{|\boldsymbol{P}|^2}{2} - \frac{1}{|\boldsymbol{Q}|}, \qquad \boldsymbol{L} = (\boldsymbol{Q} \wedge \boldsymbol{P}), \qquad \boldsymbol{A} = \frac{1}{2}\left(\boldsymbol{P} \wedge \boldsymbol{L} - \boldsymbol{L} \wedge \boldsymbol{P}\right) - \frac{\boldsymbol{Q}}{|\boldsymbol{Q}|},$$

which are related via,

$$\boldsymbol{L} \cdot \boldsymbol{A} = 0, \qquad |\boldsymbol{A}|^2 = 1 + 2H\left(|\boldsymbol{L}|^2 + \hbar^2\right). \tag{25}$$

We can define semiclassical versions of $\boldsymbol{l}'$ and $\boldsymbol{a}'$ in the state $\psi_{n,\theta}$, which we denote by $\boldsymbol{l}$ and $\boldsymbol{a}$, by taking pointwise limits of the quantum observables $\boldsymbol{L}$ and $\boldsymbol{A}$ where,

$$a_i(\boldsymbol{x}) = \lim_{\substack{n\to\infty,\hbar\to 0\\ \lambda = n\hbar}} \frac{A_i \psi_{n,\theta}(\boldsymbol{x})}{\psi_{n,\theta}(\boldsymbol{x})}, \qquad l_i(\boldsymbol{x}) = \lim_{\substack{n\to\infty,\hbar\to 0\\ \lambda = n\hbar}} \frac{L_i \psi_{n,\theta}(\boldsymbol{x})}{\psi_{n,\theta}(\boldsymbol{x})},$$

for $i = 1, 2, 3$, In general these will be complex valued quantities and so we write,

$$\boldsymbol{l} = \boldsymbol{l}^r + i\boldsymbol{l}^i, \qquad \boldsymbol{a} = \boldsymbol{a}^r + i\boldsymbol{a}^i$$

for real valued vectors $\boldsymbol{l}^r, \boldsymbol{l}^i, \boldsymbol{a}^r$ and $\boldsymbol{a}^i$.

A simple calculation allows us to write these in the phase space $(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^6$ as,

$$\begin{aligned}
\boldsymbol{l}^r(\boldsymbol{x}, \boldsymbol{p}) &= \boldsymbol{x} \wedge (\boldsymbol{p} - \nabla R(\boldsymbol{x})), \\
\boldsymbol{l}^i(\boldsymbol{x}, \boldsymbol{p}) &= -\boldsymbol{x} \wedge \nabla R(\boldsymbol{x}), \\
\boldsymbol{a}^r(\boldsymbol{x}, \boldsymbol{p}) &= (\boldsymbol{p} - \nabla R(\boldsymbol{x})) \wedge \boldsymbol{l}^r + \nabla R(\boldsymbol{x}) \wedge \boldsymbol{l}^i - \frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \\
\boldsymbol{a}^i(\boldsymbol{x}, \boldsymbol{p}) &= (\boldsymbol{p} - \nabla R) \wedge \boldsymbol{l}^i - \nabla R(\boldsymbol{x}) \wedge \boldsymbol{l}^r.
\end{aligned}$$

By considering the formal limit of the Schrödinger equation we are also lead to define the quantities,

$$\begin{aligned}
H^r(\boldsymbol{x}, \boldsymbol{p}) &= \frac{1}{2}|\boldsymbol{p} - \nabla R(\boldsymbol{x})|^2 - \frac{1}{2}|\nabla R(\boldsymbol{x})|^2 - \frac{1}{|\boldsymbol{x}|}, \\
H^i(\boldsymbol{x}, \boldsymbol{p}) &= (\boldsymbol{p} - \nabla R(\boldsymbol{x})) \cdot \nabla R(\boldsymbol{x}).
\end{aligned}$$

It is a simple exercise in vector algebra to show that for any $(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^6$,

$$\boldsymbol{a}^r \cdot \boldsymbol{l}^r - \boldsymbol{a}^i \cdot \boldsymbol{l}^i = 0, \tag{26}$$

$$\boldsymbol{a}^i \cdot \boldsymbol{l}^r + \boldsymbol{a}^r \cdot \boldsymbol{l}^i = 0, \tag{27}$$

$$|\boldsymbol{a}^r|^2 - |\boldsymbol{a}^i|^2 = 1 + 2H^r\left(|\boldsymbol{l}^r|^2 - |\boldsymbol{l}^i|^2\right) - 4H^i\left(\boldsymbol{l}^r \cdot \boldsymbol{l}^i\right), \tag{28}$$

$$\boldsymbol{a}^r \cdot \boldsymbol{a}^i = 2H^r \boldsymbol{l}^r \cdot \boldsymbol{l}^i + 2H^i\left(|\boldsymbol{l}^r|^2 - |\boldsymbol{l}^i|^2\right), \tag{29}$$

generalising the classical constraint equations (24). It should be noted that the identities (26)–(29) would actually hold for any vector field ∇R. We also note that when $\nabla R \equiv 0$ it follows that $\boldsymbol{l}^i = \boldsymbol{a}^i = 0$ and $H^i = 0$, and then these identities reduce to the classical equations (24).

Thus we see that of the 14 semiclassical observables $\boldsymbol{a}^l, \boldsymbol{a}^i, \boldsymbol{l}^r, \boldsymbol{l}^i, H^r$ and H^i there are at most 10 independent observables.

Clearly it follows from (23) that for any solution,

$$\frac{1}{2}|\dot{\boldsymbol{x}}_t|^2 + V_{\text{eff}}(\boldsymbol{x}_t) = E$$

for some constant E. Moreover, if we want our solution to converge to Keplerian motion on the Kepler ellipse we must demand that $E = -1/2$ in our natural units. The additional constraint,

$$H^i(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$$

has already arisen in Theorem IV.1 where its significance in recovering Nelson's stochastic mechanics in the $\epsilon = 0$ case has already been shown. However this constraint also leads to the following result:-

Lemma IV.3. *Suppose that $\boldsymbol{x}_t$ is a well defined solution for (23) such that H^i is a constant of the motion with $H^i(\boldsymbol{x}_t, \dot{\boldsymbol{x}}_t) = 0$ and such that $|\nabla R(\boldsymbol{x}_{t=0})| \neq 0$. Then $R(\boldsymbol{x}_t)$ is a monotone increasing function of t.*

Proof. Since $H^i = 0$ we can orthogonally decompose $\dot{\boldsymbol{x}}_t$ by writing

$$\dot{\boldsymbol{x}}_t = \boldsymbol{v}_t + \nabla R(\boldsymbol{x}_t),$$

where $\boldsymbol{v}_t \cdot \nabla R(\boldsymbol{x}_t) = 0$ for all times t. It follows that,

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{x}_t) = \nabla R(\boldsymbol{x}_t) \cdot \dot{\boldsymbol{x}}_t = |\nabla R(\boldsymbol{x}_t)|^2,$$

giving the result. □

It should be noted that on the constraint manifold,

$$\Gamma = \{(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^6 | \quad H^i(\boldsymbol{x}, \boldsymbol{p}) = 0\}$$

we have,

$$H^r(\boldsymbol{x}, \boldsymbol{p}) = H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2}|\boldsymbol{p}|^2 + V_{\text{eff}}(\boldsymbol{x}).$$

That is, in the nomenclature of Dirac, H^r and H are weakly equal (see[25]). Consequently any solution $\boldsymbol{x}_t$ which satisfies the constraint must satisfy $H^r(\boldsymbol{x}_t, \dot{\boldsymbol{x}}_t) = -1/2$.

The quantities $\boldsymbol{l}$ and $\boldsymbol{a}$ are not themselves constants of the motion for our semiclassical perturbation of the Coulomb problem. However these quantities satisfy remarkable equations echoing the classical problem. Following[18], using Pauli's identities (6)-(9) we find that $\boldsymbol{X}_t^0$ satisfies,

$$\begin{aligned}
l_3^r \cos\theta + a_1^r \sin\theta &= 1, & l_1^r \cos\theta - l_2^i - a_3^r \sin\theta &= 0, && (30)\\
l_3^i \cos\theta + a_1^i \sin\theta &= 0, & l_1^i \cos\theta + l_2^r - a_3^i \sin\theta &= 0, && (31)\\
a_3^r \cos\theta + l_1^r \sin\theta &= 0, & a_1^r \cos\theta - a_2^i - l_3^r \sin\theta &= 0, && (32)\\
a_3^i \cos\theta + l_1^i \sin\theta &= 0, & a_1^i \cos\theta + a_2^r - l_3^i \sin\theta &= 0, && (33)
\end{aligned}$$

where we recall $\sin\theta = e$. These 8 equations coupled with the constraints $H^i = 0$ and $H^r = -1/2$ fully determined the values of the observables $\boldsymbol{a}, \boldsymbol{l}, H^r$ and H^i. Indeed these relations can be rewritten in the form,

$$\begin{aligned}
&l_3^r \cos\theta + a_1^r \sin\theta = 1, && a_1^r \cos\theta - a_2^i - l_3^r \sin\theta = 0,\\
&\frac{l_3^i}{a_1^i} = \frac{a_3^r}{l_1^r} = \frac{a_3^i}{l_1^i} = -\tan\theta, && -\frac{l_2^r}{l_1^i} = \frac{l_2^i}{l_1^r} = -\frac{a_2^r}{a_1^i} = \frac{1}{\cos\theta}.
\end{aligned}$$

However it is important to note that the equations (30)–(33) are not constants of the motion for the classical mechanics associated with the potential V_{eff} as we have another solution for this problem which does not satisfy these equations. Indeed if we observe that,

$$\begin{aligned}
\boldsymbol{l}^r(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= -\boldsymbol{l}^r(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), & \boldsymbol{l}^i(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= \boldsymbol{l}^i(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), && (34)\\
\boldsymbol{a}^r(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= \boldsymbol{a}^r(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), & \boldsymbol{a}^i(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= -\boldsymbol{a}^i(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), && (35)\\
H^r(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= H^r(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), & H^i(\boldsymbol{x}, \boldsymbol{b}(\boldsymbol{x})) &= -H^i(\boldsymbol{x}, -\boldsymbol{b}_*(\boldsymbol{x})), && (36)
\end{aligned}$$

then we see that the solution $\boldsymbol{X}_t^{0*}$ satisfies,

$$\begin{aligned}
-l_3^r \cos\theta + a_1^r \sin\theta &= 1, & -l_1^r \cos\theta - l_2^i - a_3^r \sin\theta &= 0, && (37)\\
l_3^i \cos\theta - a_1^i \sin\theta &= 0, & l_1^i \cos\theta - l_2^r + a_3^i \sin\theta &= 0, && (38)\\
a_3^r \cos\theta - l_1^r \sin\theta &= 0, & a_1^r \cos\theta + a_2^i + l_3^r \sin\theta &= 0, && (39)\\
-a_3^i \cos\theta + l_1^i \sin\theta &= 0, & -a_1^i \cos\theta + a_2^r - l_3^i \sin\theta &= 0. && (40)
\end{aligned}$$

However in the rearranged form we have,

$$\begin{aligned}
&-l_3^r \cos\theta + a_1^r \sin\theta = 1, && a_1^r \cos\theta + a_2^i + l_3^r \sin\theta = 0,\\
&\frac{l_3^i}{a_1^i} = \frac{a_3^r}{l_1^r} = \frac{a_3^i}{l_1^i} = \tan\theta, && -\frac{l_2^r}{l_1^i} = \frac{l_2^i}{l_1^r} = -\frac{a_2^r}{a_1^i} = -\frac{1}{\cos\theta}
\end{aligned}$$

Thus we see that,

$$\frac{l_3^i}{a_1^i} = \frac{a_3^r}{l_1^r} = \frac{a_3^i}{l_1^i}, \qquad -\frac{l_2^r}{l_1^i} = \frac{l_2^i}{l_1^r} = -\frac{a_2^r}{a_1^i},$$

define classical constants of the motion for our constrained Hamiltonian system.

Given the complexity of the expressions for ∇R and ∇S we think these identities are quite remarkable. They show how the quantum modification of Newtonian dynamics presented here inherits some of the $SO(4)$ algebra of the quantum theoretical problem.

It follows from Lemma IV.3 that any solution $\boldsymbol{x}_t$ for (23) with the constraints $H^i = 0$ and $H^r = -1/2$ will converge to the Kepler ellipse $\mathcal{K}$ in the long time limit since R is increasing and has its only manifold of critical points on $\mathcal{K}$. Moreover we also have $|\nabla R(\boldsymbol{x}_t)|^2 \to 0$ as $t \to \infty$ and so,

$$\boldsymbol{l}^i \to 0, \qquad \boldsymbol{a}^i \to 0.$$

It would then follow from (30)–(33) that,

$$\boldsymbol{l}^r \to \boldsymbol{l}' = (0, 0, \sqrt{1-e^2}), \qquad \boldsymbol{a}^r \to \boldsymbol{a}' = (e, 0, 0).$$

Similarly from (37)–(40) it would follow that,

$$\boldsymbol{l}^r \to \boldsymbol{l}' = (0, 0, -\sqrt{1-e^2}), \qquad \boldsymbol{a}^r \to \boldsymbol{a}' = (e, 0, 0).$$

Thus we see that in the long time limit $\boldsymbol{X}_t^0$ and $\boldsymbol{X}_t^{0*}$ correspond to Keplerian orbits with reversed directions.

As a final remark we note that these properties give us first integrals for the full diffusion processes $\boldsymbol{X}_t$ and $\boldsymbol{X}_t^*$. Following Bismut we define a strong first integral for a stochastic process $\boldsymbol{X}_t$ as a function $I : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{d}I(\boldsymbol{X}_t) = 0$. (A weak constant is a function $I : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{E}(I(X_t))$ is constant.) Clearly it follows from the above that strong first integrals for $\boldsymbol{X}_t$ and $\boldsymbol{X}_t^*$ can be constructed from (30)–(33) and (37)–(40) respectively.

C. Avoiding the Singularity

The Bohr correspondence limit of the atomic elliptic state has lead us via Nelson's stochastic mechanics to the Newtonian dynamics of a particle in the effective potential V_{eff},

$$V_{\text{eff}}(\boldsymbol{x}) = -\frac{1}{|\boldsymbol{x}|} - |\nabla R(\boldsymbol{x})|^2,$$

under the constraint,

$$(\boldsymbol{p} - \nabla R(\boldsymbol{x})) \cdot \nabla R(\boldsymbol{x}) = 0,$$

with energy in natural units given by

$$\frac{1}{2}|\boldsymbol{p}|^2 + V_{\text{eff}}(\boldsymbol{x}) = -\frac{1}{2}.$$

As we have seen, $\boldsymbol{b}$ is singular on Σ where $x_2 = 0$ and $2 \leq \nu \leq 4$, the accessible part of the singularity for $x_2 > 0$ where $b_2 < 0$. We now see how we can avoid this singularity.

We work in spherical polar coordinates $(r, \tilde{\vartheta}, \tilde{\phi})$. Firstly observe that on $x_3 = 0$ ($\tilde{\vartheta} = \pi/2$),

$$\nu = \frac{r}{e}\left(e - \cos\tilde{\phi} - i\sin\tilde{\phi}\sqrt{1-e^2}\right)$$

and

$$|\nu| = \frac{r}{e}\left(1 - e\cos\tilde{\phi}\right).$$

We also note that,

$$\alpha^2 - \beta^2 + 2i\alpha\beta = 1 - 4\frac{\nu^*}{|\nu|^2},$$

where ν^* denotes the complex conjugate of ν.

On $\mathcal{K}$,

$$r = \frac{(1-e^2)}{(1+e\cos\tilde{\phi})}$$

and so substituting for r gives,

$$\alpha = \frac{1+e\cos\tilde{\phi}}{1-e\cos\tilde{\phi}}, \qquad \beta = -\frac{2e\sin\tilde{\phi}}{\sqrt{1-e^2}(1-e\cos\tilde{\phi})}.$$

Referring to Lemma IV.2, we see that R is a constant on $\mathcal{K}$ where it attains its global maximum,

$$R(\boldsymbol{X}_t^0) = R_{\max} = -\ln 4 - \ln e,$$

for $\boldsymbol{X}_t^0 \in \mathcal{K}$.

On Σ, $\sqrt{1-\frac{4}{\nu}}$ is pure imaginary and so letting $|x_2| \to 0$ we see that on Σ, $1+\beta^2 = 4/\nu$ giving,

$$R = -r + \frac{\nu}{2} - \ln 4$$

and R is continuous as we cross Σ where $\nu = r - \frac{x_1}{e}$.

Now on the singularity Σ the level curves of ν are hyperbolas of eccentricity $1/e$,

$$\nu_0 = r\left(1 - \frac{\cos\tilde{\vartheta}}{e}\right) \tag{41}$$

for constants $0 \le \nu_0 \le 4$ in terms of polar coordinates $(r,\tilde{\vartheta})$ in the (x_1,x_3) plane. Therefore we see that for a fixed value of ν_0 the minimum value of r occurs when $\tilde{\vartheta} = \pi$ when,

$$r = \frac{\nu_0 e}{(e+1)}$$

For a fixed value of ν_0 the largest value of R is given by,

$$R = -\frac{\nu_0 e}{(e+1)} + \frac{\nu_0}{2} - \ln 4.$$

The maximum value of $\nu_0 = 4$ and so on Σ

$$(R|_\Sigma)_{\max} = \frac{2(1-e)}{(1+e)} - \ln 4.$$

Proposition IV.1. *For any $e \in (0,1)$ there exists an open set $U \subset \mathbb{R}^3$ such that for all $\boldsymbol{x}_0 \in U$ the process $\boldsymbol{X}_t^0(x_0)$ is a C^2 solution for,*

$$\frac{\mathrm{d}\boldsymbol{X}_t^0}{\mathrm{d}t} = \boldsymbol{b}(\boldsymbol{X}_t^0),$$

which converges to the Kepler ellipse $\mathcal{K}$ in the infinite time limit.

Proof. Take U to be the set of $\boldsymbol{x} \in \mathbb{R}^3$ defined by,

$$\lambda\left(\frac{2(1-e)}{(1+e)} - \ln 4\right) \le R(\boldsymbol{x}) \le \lambda\left(-\ln 4 - \ln e\right).$$

Note that for all $e \in (0,1)$,

$$\frac{2(1-e)}{(1+e)} + \ln e < 0. \qquad \square$$

D. Small eccentricities

We now consider the limiting case $e \sim 0$.

Proposition IV.2.

$$R + iS = -r + \ln(x_1 + ix_2) + O(e), \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

as $e \sim 0$ with Σ (the singularity) being $x_1 = x_2 = 0$.

This suggests we take a closer look at the complex singularity at $r = 0$ as well as the singularity $\nu = 0$. Rather surprisingly the latter is not a real singularity at all for our modified Newtonian dynamics in two dimensions, the point being that the logarithmic terms can be recombined to give

$$-\ln\left(2\nu - 4 - 2\nu\sqrt{1 - \frac{4}{\nu}}\right),$$

and $\nabla\nu$ is well defined in 2 dimensions. But,

$$R(0) + iS(0) = -\ln 4 - i\pi,$$

and $R|_{\nu=0} \leq R(0)$. So in any case 0 is not accessible to $\boldsymbol{X}_t^0$ in three dimensions if $\boldsymbol{x}_0$ is such that $R(\boldsymbol{x}_0) > -\ln 4$ and neither is $\nu = 0$, $R(\boldsymbol{X}_t^0)$ being increasing with t. We assume in what follows that $R(\boldsymbol{x}_0) > -\ln 4$.

Therefore we have proved:-

Theorem IV.2. *The complex singularity at $r = 0$ and $\nu = 0$ are not accessible to $\boldsymbol{X}_t^0$ for $t > 0$ as long as $R(\boldsymbol{x}_0) > -\ln 4$.*

From Proposition IV.2 it follows that as $e \to 0$,

$$R + iS \to -r + \ln(x_1 + ix_2)$$

and

$$\boldsymbol{b}(\boldsymbol{x}) = \left(\frac{(x_1 - x_2)}{x_1^2 + x_2^2} - \frac{x_1}{r}, \frac{(x_1 + x_2)}{x_1^2 + x_2^2} - \frac{x_2}{r}, -\frac{x_3}{r}\right)$$

(this agrees with the expressions for the circular case in[13,18]). This defines the quantum modification of Newtonian dynamics for the circular orbit, in spherical polar coordinates $(r, \tilde{\vartheta}, \tilde{\phi})$:-

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r} - 1, \qquad \frac{\mathrm{d}\tilde{\vartheta}}{\mathrm{d}t} = \frac{\cot\tilde{\vartheta}}{r^2}, \qquad \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t} = \frac{\csc^2\tilde{\vartheta}}{r^2},$$

with initial conditions $(r_0, \tilde{\vartheta}_0, \tilde{\phi}_0)$.

Proposition IV.3. *For $r_0 \neq 1$,*

$$-r_t + r_0 - \ln\left(\frac{1 - r_t}{1 - r_0}\right) = t$$

and $r_t \to 1$ as $t \to \infty$. If $r_0 > 1$,

$$r_t - 1 = \sum_{n=1}^{\infty} \frac{\left((r_0 - 1)e^{r_0}e^{-t}\right)^n}{n!}(-n)^{n-1},$$

for sufficiently large $t > 0$ and

$$T_t = \int_0^t \frac{\mathrm{d}s}{r_s^2}, \qquad \cos\vartheta_t = \exp(-T_t)\cos\tilde{\vartheta}_0 \to 0$$

and $\dot{\tilde{\phi}}_t \to 1$ as $t \to \infty$.

Proof. Since

$$\int \frac{r\,\mathrm{d}r}{(1-r)} = \int \mathrm{d}t,$$

it follows that,

$$r_0 - r_t = \ln\left(\frac{1-r_t}{1-r_0}\right) + t$$

thus,

$$(1-r_0)\exp\left(r_0 - t\right) = (1-r_t)\exp\left(r_t\right),$$

so assuming that $r_0 > 1$,

$$\exp\left(c-t\right) = (r_t - 1)\exp\left(r_t\right),$$

c being constant. It follows that for $T = \exp\left(c-t\right)$ and $f(r) = e^{-r}$,

$$r_t = 1 + Tf(r_t), \qquad |Tf(r_t)| \le |r_t - 1|,$$

for sufficiently large $t > 0$. Lagrange's expansion theorem gives for sufficiently large t,

$$r_t = 1 + \sum_{n=1}^{\infty} \frac{T^n}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}} \left(f^n(a)\right)|_{a=1},$$

and desired result follows. The rest of the proposition follows easily from the above equations in polar coordinates. □

We leave the reader to derive the corresponding result for $r_0 < 1$.

We can improve on the above as we now see.

For small eccentricity $e \sim 0$ with $x_2 > 0$ and $|\nu| > 1$,

$$\sqrt{1 - \frac{4}{\nu}} = 1 - 2\sum_{k=0}^{\infty} \frac{(2k)!}{k!(k+1)!} \frac{1}{\nu^{k+1}},$$

combining the logarithm terms as above we obtain,

$$R + iS = -r + \sum_{k=0}^{\infty} \frac{(2k)!}{k!(k+1)!} \frac{1}{\nu^k} - \ln\left(\sum_{k=1}^{\infty} \frac{(2k)!}{k!(k+1)!} \frac{1}{\nu^k}\right).$$

Thus we find:-

Theorem IV.3. *For small eccentricity $e \sim 0$ with $x_2 > 0$, $|\nu| > 1$,*

$$R + iS = -r - \ln\left(\frac{4}{\nu}\right) + \sum_{k=0}^{\infty} \frac{(2k)!}{k!(k+1)!} \frac{1}{\nu^k} - \ln\left(1 + \sum_{k=2}^{\infty} \frac{(2k)!}{k!(k+1)!} \frac{1}{\nu^{k-1}}\right).$$

This expression is useful in studying small eccentricity orbits.

Example IV.1. Ignoring normalisation constants for $e \sim 0$ and $x_2 > 0$, $|\nu| > 1$, we have,

$$R + iS = -r + \ln(x_1 + ix_2) + \frac{e(1-r)}{x_1 + ix_2} + O(e^2),$$

in Cartesians. This gives in the infinite time limit $\tilde{\vartheta} \to \pi/2$ and $r \to 1 + r_1$ where $r_1 = O(e)$,

$$\dot{r} = -1 + \frac{1}{r} - \frac{e}{r^2}(\cos\tilde{\phi} - \sin\tilde{\phi})$$

and

$$\dot{\tilde{\phi}} = \frac{1}{r^2}\left(1 + e\left(1 - \frac{1}{r}\right)(\cos\tilde{\phi} + \sin\tilde{\phi})\right).$$

So,

$$r_1 + \frac{\mathrm{d}r_1}{\mathrm{d}\tilde{\phi}} = -e(\cos\tilde{\phi} - \sin\tilde{\phi}),$$

i.e. $r - 1 = -e\cos\tilde{\phi}$, or $\frac{1}{r} = 1 + e\cos\tilde{\phi}$ correct to first order in e since the semi-latus rectum $l = 1$ to this order. It also follows that to this order $r^2\dot{\tilde{\phi}} = 1$ as expected.

Finally we estimate how rapidly $\boldsymbol{X}^0$ converges to $x_3 = 0$ plane as $e \sim 0$.

Proposition IV.4. *Assume that* $\boldsymbol{x}_0$ *is such that* $R(\boldsymbol{x}_0) > -\ln 4$. *Then as eccentricity* $e \sim 0$,

$$x_3(t) \sim \exp\left(-\int_0^t \frac{\mathrm{d}s}{r(s)}\right) x_3(0),$$

where $r(s) = |\boldsymbol{X}^0_{x_0}(s))|$, $0 < r(s) < \infty$, *for particles avoiding* Σ.

Proof. The point is that $\nu \neq 0$ and so for $r = |\boldsymbol{x}|$ we have $R = -r - \ln 4$, so if R is increasing $\nu = 0$ and $r = 0$ are inaccessible to the motion if $R(\boldsymbol{x}_0) > -\ln 4$. Also as $r \sim \infty$ it is easy to show that ,

$$\boldsymbol{b}(\boldsymbol{x}) \sim -\hat{\boldsymbol{x}},$$

so

$$\boldsymbol{b}\cdot\hat{\boldsymbol{x}} = \dot{r} \sim -1$$

as $r \sim \infty$. It follows that r must be bounded if Σ is avoided. Now

$$\alpha + i\beta = \sqrt{1 - \frac{4}{\nu}} = \sqrt{\left(1 - \frac{4e}{(er - x_1 - ix_2\sqrt{1 - e^2})}\right)}.$$

So as $e \sim 0$, $\alpha \sim 1$ and $\beta \sim 0$. i.e. $\alpha + \beta + 1 \sim 2$. But for $t > 0$,

$$\dot{x}_3(t) = -\frac{\alpha + \beta + 1}{2} \times \frac{x_3(t)}{r(t)},$$

from which desired result follows. □

E. Singular behaviour of $\boldsymbol{X}^0$

We now discuss what happens when our zeroth order approximation to the WIMP-like particle trajectory $\boldsymbol{X}^0_t$ impacts upon Σ from $x_2 > 0$ the accessible part of the singularity where $b_2 \leq 0$ in the limit $x_2 \to 0_+$. Here on Σ we have $\alpha = x_2 = 0$,

$$|\boldsymbol{b}|^2 = \frac{1}{4}\left(\frac{(\beta - 1)^2}{e^2} + (\beta + 1)^2 - 2\frac{(\beta^2 - 1)x_1}{er} + (\beta + 1)^2\frac{1 - e^2}{e^2}\right).$$

It is easy to check that the linear terms in β cancel. So for $\boldsymbol{x} \in \Sigma$ defining,

$$\boldsymbol{b}_\pm(\boldsymbol{x}) = \lim_{\delta\to 0} \nabla(R + S)(x_1, \pm\delta, x_3),$$

we obtain $|\boldsymbol{b}_+|^2 = |\boldsymbol{b}_-|^2$, so that energy is conserved as we cross Σ. It follows that the Bohm potential $-|\nabla R|^2$ is continuous as we cross Σ. Unfortunately the corresponding force field $\nabla(|\nabla R|^2)$ has a jump discontinuity across Σ and is undefined on Σ.

Recall from the proof of Lemma 2 in[18] that the nodes of the atomic elliptic state $\psi_{n,\theta}$ (i.e. the points $\boldsymbol{x}$ where $\psi_{n,\theta}(\boldsymbol{x}) = 0$) are given by the equations

$$\frac{1}{n} z(m, n-1) = \nu(\boldsymbol{x}), \qquad x_2 = 0,$$

where $z(m, n-1)$ is the m^{th} real root of the Laguerre poynomial $L_{n-1}(\cdot) = 0$ for $m = 1, 2, \ldots, n-1$ and that the singularity Σ is a limit of such curves as $n \to \infty$. Indeed these nodal curves correspond to setting $\nu_0 = z(m, n-1)$ in (41). Thus, if for certain parts of Σ the corresponding $\nu = \nu_0$ values are well approximated by the above limit as $n \to \infty$ for fixed m, then you would expect that the curves $x_2 = 0$, $\nu = \nu_0$ should repel or reflect the WIMP-like quantum particles. Given that this family of curves is sufficiently dense in Σ, one would expect that there would be a positive probability of our semiclassical WIMP-like particle trajectory $\boldsymbol{X}_t^0$ being reflected at Σ. To be consistent with our semiclassical equations for R and S, the simplest way to effect this would be for $\psi_{\text{s.c.}}(\boldsymbol{x}) \to \psi^*_{\text{s.c.}}(\boldsymbol{x})$ or $\psi_{\text{s.c.}}(\boldsymbol{x}) \to \psi_{\text{s.c.}}(\mathcal{R}\boldsymbol{x})$ where $\mathcal{R}\boldsymbol{x} = (x, -y, z)$ for $\boldsymbol{x} = (x, y, z)$ in Cartesians. In terms of the drift $\boldsymbol{b}$ for our Keplerian diffusion on impact with Σ, with this probability $\mathcal{Q}$,

$$(b_1, b_2, b_3)(x_1, x_2, x_3) \mapsto (b_1(x_1, -x_2, x_3), -b_2(x_1, -x_2, x_3), b_3(x_1, x_2, x_3)), \quad \boldsymbol{b} \mapsto -\boldsymbol{b}^*,$$

until next impact with Σ, in line with our results for the constrained Hamiltonian system. Thus we have some value $\mathcal{Q}$ giving the probability of reflection and then $\mathcal{P} = (1 - \mathcal{Q})$ represents the probability of particle passing through Σ. As the trajectory passes through Σ there will be a jump discontinuity in the value of the drift $\boldsymbol{b}$ since $\beta \to -\beta$ as we cross Σ. Effectively when a particle impacts on Σ it restarts from that point either following the trajectory $\boldsymbol{X}_t^0$ or $\boldsymbol{X}_t^{0*}$ depending on whether it has been transmitted or reflected. As we have seen, both of these trajectories will drive the particle towards the Kepler ellipse in the infinite time limit and to the usual classical planetary motion, the only change being the sense of rotation about the Kepler ellipse. The values of $\mathcal{Q}$ and $\mathcal{P}$ should depend on the point of impact and are the same for drifts $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ crossing Σ in opposite directions. This we believe is the best way for our model to incorporate classical as well as quantum behaviours demanding that the trajectory includes these instantaneous reflections and refractions. At the end of this paper we indicate how to estimate $\mathcal{Q}$ and $\mathcal{P}$ in the limit of the eccentricity $e \sim 0$ and why $\mathcal{Q}$ and $\mathcal{P}$ are the same for $\boldsymbol{b}$ and $-\boldsymbol{b}^*$.

In summary if we consider the path $\boldsymbol{X}_s^0(\boldsymbol{x}_0)$ for $s \in [0, t]$ starting at $\boldsymbol{x}_0 \notin \Sigma$ first impacting on Σ from $x_2 > 0$ at $\boldsymbol{X}_t^0(\boldsymbol{x}_0)$. We have to assign 2 probabilities $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{Q}(\boldsymbol{x})$ for $\boldsymbol{x} \in \Sigma$ with $\mathcal{P} + \mathcal{Q} = 1$ for 2 different outcomes:

1. $\mathcal{P}(\boldsymbol{X}_t^0(\boldsymbol{x}_0))$ the probability of the particle passing straight through Σ, the drift $\boldsymbol{b}$ having a jump discontinuity as a result of $\beta \mapsto -\beta$, ν crossing the cut, x_2 becoming negative.

2. $\mathcal{Q}(\boldsymbol{X}_t^0(\boldsymbol{x}_0))$ the probability of the particle being reflected with $\psi_{\text{s.c.}} \mapsto \psi^*_{\text{s.c.}}$, and $\boldsymbol{b} \mapsto \nabla(R - S)$ instantaneously.

Remark IV.1. Prescription 1 preserves the semiclassical quantities $\boldsymbol{l}$ and $\boldsymbol{a}$ while 2 only preserves the energy, with $\boldsymbol{a} \mapsto \boldsymbol{a}^*$ and $\boldsymbol{l} \mapsto -\boldsymbol{l}^*$ where $*$ denotes the complex conjugate as identified in (34)–(36).

Remark IV.2. For the hyperbolic branch of Σ given by $\nu = 2$ in the plane $x_2 = 0$, it is interesting to note that

$$|\boldsymbol{b}_+| = |\boldsymbol{b}_-| = \frac{1}{e},$$

so the effective potential is constant on this curve.

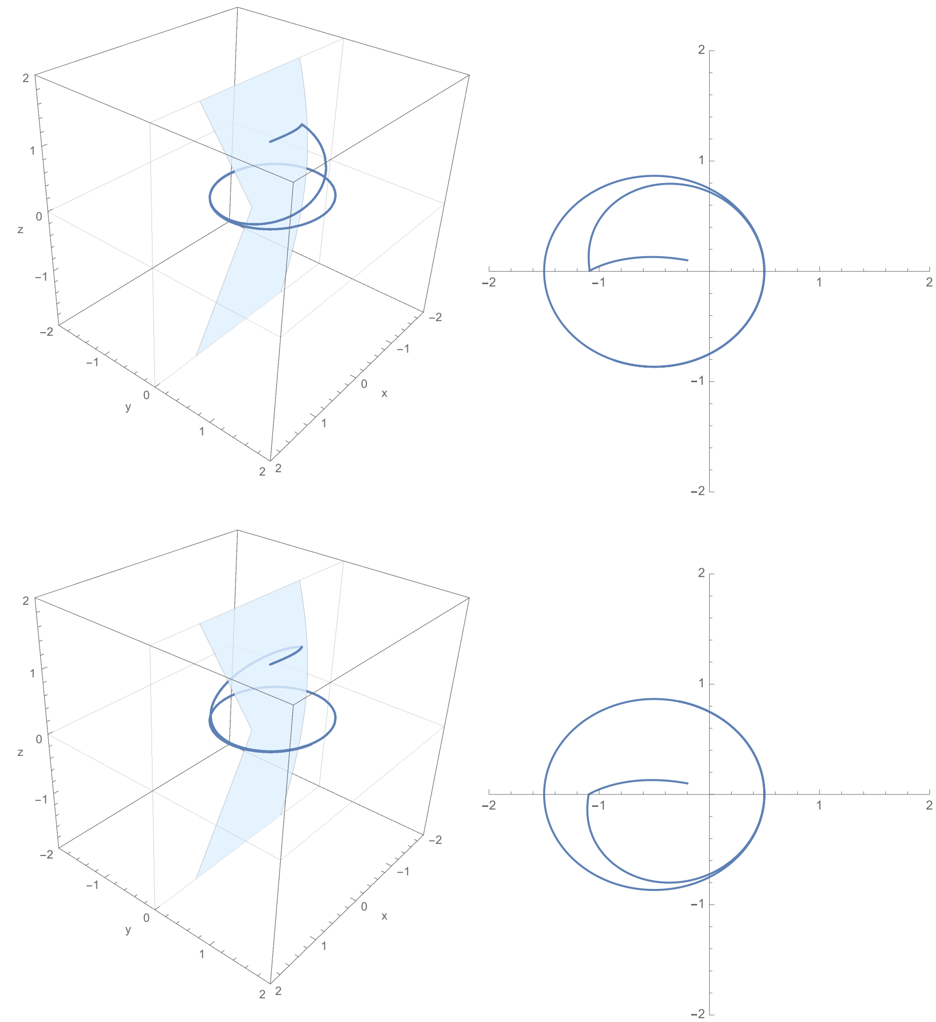

FIG. 2. A trajectory for $\boldsymbol{X}_t^0$ which reaches the singularity and either passes through (bottom row) or is reflected (top row), shown in three dimensions with the singularity (first column) and in projection on the (x_1, x_2) plane (second column).

Remark IV.3. Since $-\boldsymbol{b}^*$ is the negative backward drift, we feel that our prescription for this instantaneous reflection at Σ is true to the spirit of Nelson's stochastic mechanics. It is remarkable that our constrained Hamiltonian system gives this as the only other possible drift.

A pair of trajectories being reflected and refracted on Σ is shown in Figure 2.

V. $\boldsymbol{X}_t^1$ – THE GAUSSIAN QUANTUM CORRECTION

Now let us consider the Gaussian correction term $\boldsymbol{X}_t^1$ defined by,

$$\mathrm{d}\boldsymbol{X}_t^1 = b'(\boldsymbol{X}_t^0)\boldsymbol{X}_t^1\,\mathrm{d}t + \mathrm{d}\boldsymbol{B}_t,$$

where b' is the matrix,

$$(b')_{ij} = \left(\frac{\partial b_i}{\partial x_j} \right).$$

Consider the family of uniformly bounded continuous matrix-valued functions,

$$\mathcal{M}_T := \left\{ A : [0,T) \to \mathbb{R}^{d\times d} \middle| A(t) \text{ continuous, } \|A\| := \sup_{s\in[0,T)} \|A(s)\| < \infty \right\},$$

and define the time ordered products,

$$\mathcal{T}_+ \left(\int_0^t A(s)\,\mathrm{d}s \right) = I + \int_0^t A(t_1)\,\mathrm{d}t_1 + \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\, A(t_1)A(t_2) + \ldots,$$
$$\mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) = I + \int_0^t A(t_1)\,\mathrm{d}t_1 + \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\, A(t_2)A(t_1) + \ldots,$$

where in $\mathcal{T}_+$ the arguments in the terms in the products in the multiple integrals are ordered strictly increasing and in $\mathcal{T}_-$ they are strictly decreasing from left to right.

Lemma V.1. *Suppose that $A \in \mathcal{M}_T$, then the infinite series $\mathcal{T}_\pm \left(\int_0^t A(s)\,\mathrm{d}s \right)$ converges uniformly for $t \in [0,T)$ and solves the initial value problems,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_+ \left(\int_0^t A(s)\,\mathrm{d}s \right) = \mathcal{T}_+ \left(\int_0^t A(s)\,\mathrm{d}s \right) A(t), \qquad \mathcal{T}_+ (0) = I,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) = A(t)\mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right), \qquad \mathcal{T}_- (0) = I.$$

Proof. See for instance[9]. □

Corollary V.1. *Suppose that $A \in \mathcal{M}_T$, then*

$$\mathcal{T}_+ \left(-\int_0^t A(s)\,\mathrm{d}s \right) \mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) = \mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) \mathcal{T}_+ \left(-\int_0^t A(s)\,\mathrm{d}s \right) = I.$$

If we define the matrix A by,

$$A(t) = b'(X_t^0),$$

it follows that,

$$\mathrm{d}_s \left(\mathcal{T}_+ \left(\int_0^s A(u)\,\mathrm{d}u \right) \boldsymbol{X}_s^1 \right) = \mathcal{T}_+ \left(-\int_0^s A(u)\,\mathrm{d}u \right) \mathrm{d}\boldsymbol{B}_s.$$

Assuming now that $\boldsymbol{X}_{s=0}^1 = 0$, we find,

$$\boldsymbol{X}_t^1 = \mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) \int_0^t \mathcal{T}_+ \left(-\int_0^s A(u)\,\mathrm{d}u \right) \mathrm{d}\boldsymbol{B}_s.$$

Thus, we obtain,

$$\mathbb{E}\left(|\boldsymbol{X}_t^1|^2\right) = \int_0^t \mathrm{Tr} \left[\mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) \mathcal{T}_+ \left(-\int_0^s A(u)\,\mathrm{d}u \right) \right.$$
$$\left. \times\, \mathcal{T}_- \left(-\int_0^s A(u)\,\mathrm{d}u \right) \mathcal{T}_+ \left(\int_0^t A(s)\,\mathrm{d}s \right) \right] \mathrm{d}s,$$

where $A = b'(\boldsymbol{X}^0)$. To emphasise the dependence of results on the initial position $\boldsymbol{X}_{t=0}^0 = \boldsymbol{x}_0$ we write $\boldsymbol{X}_t^i = \boldsymbol{X}_t^i(\boldsymbol{x}_0)$. We have proved that:

Theorem V.1. *There exists an open set $U \subset \mathbb{R}^3$ such that for all $\boldsymbol{x}_0 \in U$ the trajectory $\boldsymbol{X}_t^0(\boldsymbol{x}_0) \notin \Sigma$ for all $t > 0$ and in the infinite time limit $\boldsymbol{X}_t^0(\boldsymbol{x}_0)$ converges to Keplerian motion on the Kepler ellipse. Moreover, for all $\boldsymbol{x}_0 \in U$ the formal asymptotic expansion of the correspondence limit of the Nelson diffusion is given by,*

$$\boldsymbol{X}_t(\boldsymbol{x}_0) = \boldsymbol{X}_t^0(\boldsymbol{x}_0) + \epsilon \boldsymbol{X}_t^1 + O(\epsilon^2), \qquad t > 0,$$

where $\boldsymbol{X}_t^1$ is the Gaussian,

$$\boldsymbol{X}_t^1 = \mathcal{T}_- \left(\int_0^t A(s)\,\mathrm{d}s \right) \int_0^t \mathcal{T}_+ \left(-\int_0^s A(u)\,\mathrm{d}u \right) \mathrm{d}\boldsymbol{B}_s,$$

where $A(s) = b'(\boldsymbol{X}_s^0(\boldsymbol{x}_0))$.

In the case when the trajectory of $\boldsymbol{X}_t^0$ reaches the singularity at some time t_0, then with probability $\mathcal{P}$ the trajectory $\boldsymbol{X}^0$ crosses Σ from $x_2 > 0$ to $x_2 < 0$ and $\boldsymbol{b}$ undergoes an instantaneous jump in line with $\sqrt{1 - \frac{4}{\nu}} \mapsto -\sqrt{1 - \frac{4}{\nu}}$ corresponding to a refraction of the particle trajectory, and with probability $\mathcal{Q}$ the trajectory is reflected at Σ and $\boldsymbol{b} \mapsto -\boldsymbol{b}_*$. With these probabilities for $t > t_0$, the classical particle motion is governed by $\boldsymbol{b}$ or $-\boldsymbol{b}_*$ until the particle next impacts with Σ. Here it is important to note that for motions generated by $-\boldsymbol{b}_*$ singularity can only be approached from $y < 0$ crossing with $(-\boldsymbol{b}_*)_y > 0$ (see Figure 1). With these adjustments to $\boldsymbol{b}$ and b' the last formula continues to be valid until the next impact with Σ.

This formula is easily extended to be valid for multiple transmissions and reflections. Needless to say the result is highly dependent upon $\boldsymbol{x}_0$ and the eccentricity e. A very complicated pattern of behaviour emerges as we promised, especially if we take into account collisions with equal mass WIMP-like particles with drifts $\nabla(\pm R \pm S)$ as the semiclassical symmetries suggest, with particles spiralling inward and outward to and from the Keplerian orbit.

Consider the behaviour of the classical limit $\boldsymbol{X}^0$ for the our semiclassical perturbation of the Coulomb potential starting in the region $x_2 > 0$. For small eccentricities e one expects that when the number of visits to Σ (if there are any) is finite, if the last impact results in the drift being $-\boldsymbol{b}_*$ or $\boldsymbol{b}$ then in the infinite time limit the WIMP-like particle will spiral towards Keplerian motion on the ellipse $\mathcal{K}$ but with opposite senses of rotation. It would be interesting to compute which $\boldsymbol{x}_0$'s give rise to different motions on impact with respective probabilities in terms of $\mathcal{P}$ and $\mathcal{Q}$.

We conclude with a brief discussion of what becomes of our results when $\boldsymbol{X}_t^0(\boldsymbol{x}_0)$ is periodic in t with small time period τ. Since the convergence of the trajectory to periodic Keplerian motion is very rapid when it occurs; this will give us the qualitative behaviour of our model.

For a real symmetric continuous matrix A with matrix norm $\|A(\cdot)\|$ uniformly bounded we denote the $(n+1)^{\text{th}}$ term of $\mathcal{T}_+ \left(\int_0^t A(s)\,\mathrm{d}s \right)$ as $A_n(t)$ where,

$$A_n(t) = \int_0^t \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_{n-1} \ldots \int_0^{t_2} \mathrm{d}t_1\, A(t_1)A(t_2)\ldots A(t_n).$$

Now suppose that A is periodic with period τ assumed small. Let ΠC_t denote the n cell, $0 < t_1 < t_2 < \ldots < t_n < t$. Then for each j by periodicity,

$$A(t_1)A(t_2)\ldots A(t_{j-1})A(t_j + \tau)A(t_{j+1})\ldots A(t_n) = A(t_1)A(t_2)\ldots A(t_n).$$

Thus the values of this matrix product are determined by the values in the small hypercube, H_τ, given by $0 \le t_j \le \tau$ for $j = 1, 2, \ldots, n$. The number of cells in ΠC_t as $\tau \sim 0$ is given by N where,

$$N \cong \frac{\mathrm{Vol}(\pi(t))}{\mathrm{Vol}(H_\tau)} = \frac{\mathrm{Vol}(\pi(t))}{n!\tau^n} = \frac{t^n}{n!\tau^n},$$

so as $\tau \sim 0$,

$$A_n(t) \sim \frac{t^n}{n!}(\bar{A}(\tau))^n,$$

where

$$\bar{A}(\tau) = \frac{1}{\tau}\int_0^\tau A(s)\,\mathrm{d}s.$$

So the leading behaviour of our time ordered exponential as $\tau \sim 0$ is

$$\mathcal{T}_+\left(\int_0^t A(s)\,\mathrm{d}s\right) \sim \exp\left(t\bar{A}(\tau)\right),$$

with a similar result for $\mathcal{T}_-\left(\int_0^t A(s)\,\mathrm{d}s\right)$. For any $t > u$ it is easy to prove from the above that

$$X_t^1 = \mathcal{T}_-\left(\int_u^t b'(\boldsymbol{X}_s^0))\,\mathrm{d}s\right)\boldsymbol{X}_u^1 + \mathcal{T}_-\left(\int_u^t b'(\boldsymbol{X}_s^0)\,\mathrm{d}s\right)\int_u^t \mathcal{T}_+\left(-\int_u^t b'(\boldsymbol{X}_s^0)\,\mathrm{d}u\right)\mathrm{d}\boldsymbol{B}_s.$$

Let us assume that in the infinite time limit $\boldsymbol{X}^0$ converges to Keplerian motion on $\mathcal{K}$.

Now let u be the first hitting time of $\mathcal{K}$ for X^0. Thus,

$$u = \inf\left\{s : R(\boldsymbol{X}_s^0(\boldsymbol{x}_0)) \geq -\ln(4e)\right\}.$$

Then as $\tau \sim 0$, $t > u$,

$$\begin{aligned}\boldsymbol{X}_t^1 \sim{}& \exp((t-u)\bar{b}'_\tau)\left(\mathcal{T}_-\left(\int_0^u b'(\boldsymbol{X}_v^0)\,\mathrm{d}v\right)\int_0^u \mathcal{T}_+\left(-\int_0^s b'(\boldsymbol{X}_v^0)\,\mathrm{d}v\right)\mathrm{d}\boldsymbol{B}_s\right)\\ &+\int_u^t \exp\left((t-s)\bar{b}'_\tau\right)\mathrm{d}\boldsymbol{B}_s,\end{aligned}$$

i.e. the Gaussian correction consists of two terms one coming from the transient motion as the particle spirals towards the Kepler ellipse $\mathcal{K}$ and the other from the persistent periodic motion on the Kepler ellipse.

To conclude we content ourselves by quoting b' and $\bar{b}'_\tau$ for the circular orbit of unit radius in natural units. In Cartesians with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\rho = \sqrt{x_1^2 + x_2^2}$,

$$b' = \begin{pmatrix} \frac{1}{\rho^2} - \frac{1}{r} + x_1^2\left(\frac{1}{r^3} - \frac{2}{\rho^4}\right) + \frac{2x_1x_2}{\rho^4} & -x_1x_2\left(\frac{2}{\rho^4} - \frac{1}{r^3}\right) + \frac{x_2^2 - x_1^2}{\rho^4} & \frac{x_1x_3}{r^3} \\ -x_1x_2\left(\frac{2}{\rho^4} - \frac{1}{r^3}\right) + \frac{x_2^2 - x_1^2}{\rho^4} & \frac{1}{\rho^2} - \frac{1}{r} + x_2^2\left(\frac{1}{r^3} - \frac{2}{\rho^4}\right) + \frac{2x_1x_2}{\rho^4} & \frac{x_3x_2}{r^3} \\ \frac{x_1x_3}{r^3} & \frac{x_2x_3}{r^3} & -\frac{1}{r} + \frac{x_3^2}{r^3} \end{pmatrix}$$

and finally

$$\bar{b}'_\tau = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

So the quantum correction from the persistent periodic motion is in this case in Cartesians, $(W_1(1-e^{-t}), W_2(1-e^{-t}), W_3(1-e^{-2t})/2)$, W_1, W_2, W_3 being independent Brownian motions. Needless to say the first noise term is more difficult to compute. To analyse the persistent periodic noise term for $e \neq 0$ should not be too difficult.

To emphasise the dependence of results on the initial position $\boldsymbol{x}_0 = \boldsymbol{X}_{t=0}^\epsilon$ we write $\boldsymbol{X}_t^\epsilon = \boldsymbol{X}^\epsilon(t, \boldsymbol{x}_0, \omega)$ and suppress ω when $\epsilon = 0$. Assuming that for sufficiently small t for

$s \in [0,t]$ the map $\boldsymbol{x}_0 \mapsto \boldsymbol{X}^0(s,\boldsymbol{x}_0)$ is locally injective and C^1, then if $u \mapsto b'(\boldsymbol{X}^0(u,\boldsymbol{x}_0)$ is a continuous map of time to 3×3 matrices, for $u \in [0,s]$ for $R(\boldsymbol{x}_0) > -\ln 4$,

$$\mathcal{T}_- \left(\int_0^s b'(\boldsymbol{X}^0(u,\boldsymbol{x}_0)\,\mathrm{d}u \right) = \left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(s,\boldsymbol{x}_0) \right).$$

This follows from differentiaiting the equation for $\boldsymbol{X}^0$ giving,

$$\frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(s,\boldsymbol{x}_0) \right) = b'(\boldsymbol{X}^0(s,\boldsymbol{x}_0)) \left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(s,\boldsymbol{x}_0) \right)$$

and the uniqueness of solutions to ODEs. But in any case the last identity implies that,

$$\det\left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(s,\boldsymbol{x}_0) \right) = \exp\left(\int_0^s \mathrm{Tr}\left(b'(\boldsymbol{X}^0(u,\boldsymbol{x}_0)) \right) \mathrm{d}u \right).$$

The local inverse function theorem implies locally injective behaviour of our map, since it follows from above that

$$\det\left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(s,\boldsymbol{x}_0) \right) \neq 0, \qquad s \in [0,t].$$

Hence $\boldsymbol{x}_0 \mapsto \boldsymbol{X}^0(s,\boldsymbol{x}_0)$ will have a unique local inverse if $R(\boldsymbol{x}_0) > -\ln 4$ given by $\boldsymbol{x}_0 = \boldsymbol{x}_0(s,\boldsymbol{X}^0)$.

Corollary V.2. *For $R(\boldsymbol{x}_0) > -\ln 4$ and for sufficiently small $t > 0$ that for $s \in [0,t]$ $\boldsymbol{X}^0(s,\boldsymbol{x}_0)$ avoids Σ the first order quantum correction to $\boldsymbol{X}^0$ is the Gaussian,*

$$\epsilon \boldsymbol{X}_t^1(\boldsymbol{x}_0) = \epsilon \left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0} \right)(t,\boldsymbol{x}_0) \left\{ \int_0^t \left(\frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{X}^0}(s,\boldsymbol{X}^0) \right)\Bigg|_{\boldsymbol{X}^0=\boldsymbol{X}^0(s,\boldsymbol{x}_0)} \mathrm{d}\boldsymbol{B}_s + \boldsymbol{X}_{t=0}^1(\boldsymbol{x}_0) \right\}.$$

Corollary V.3. *When $\boldsymbol{X}_s^0(\boldsymbol{x}_0)$ first arrives at Σ, say at $\boldsymbol{x}_0(\Sigma)$ at time $t_0 = t_0(\Sigma,\boldsymbol{x}_0)$, in the case of reflection/transmission at Σ with probability $\mathcal{Q}/\mathcal{P}$, $\boldsymbol{b} \mapsto \tilde{\boldsymbol{b}}/\boldsymbol{b}_-$, $\boldsymbol{X}^0 \mapsto \tilde{\boldsymbol{X}}^0$ for the new drifts for $t_0(\Sigma,\boldsymbol{x}_0) < t < t_0(\Sigma,\boldsymbol{x}_0(\Sigma))$, time of return visit to Σ,*

$$\begin{aligned}\boldsymbol{X}_t^1(\boldsymbol{x}_0) = &\left(\frac{\partial \hat{\boldsymbol{X}}_0}{\partial \boldsymbol{x}_0(\Sigma)}(t,\boldsymbol{x}_0(\Sigma)) \right) \\ &\times \left\{ \int_{t_0}^t \left(\frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{X}^0}(s,\boldsymbol{X}^0) \right)\Bigg|_{\boldsymbol{X}^0=\boldsymbol{X}^0(s,\boldsymbol{x}_0(\Sigma))} \mathrm{d}\boldsymbol{B}_s \right. \\ &\left. \quad + \left(\frac{\partial \boldsymbol{X}^0}{\partial \boldsymbol{x}_0}(t_0,\boldsymbol{x}_0) \right) \int_0^{t_0} \left(\frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{X}^0}(s,\boldsymbol{X}^0) \right)\Bigg|_{\boldsymbol{X}^0=\boldsymbol{X}^0(s,\boldsymbol{x}_0)} \mathrm{d}\boldsymbol{B}_s \right\}.\end{aligned}$$

This formula is valid in the neighbourhood of $\boldsymbol{x}_0$ where the local inverse functions exist as long as $R(\boldsymbol{x}_0) > -\ln 4$. We hope to remove this restriction in a future publication on global results for this problem. The formula illustrates very nicely the importance of Jacobi fields in this setting which seems to be a new feature.

VI. DISCUSSION AND CONCLUSION

A Galilean invariant quantum theory for WIMP-like particles inevitably leads one to consider the Bohr correspondence limit of the Schrödinger equation in a Coulomb potential, at least to a first approximation. In this paper we have seen how the Bohr correspondence limit of the atomic elliptic states in the setting of Nelson's stochastic mechanics gives a semiclassical dynamics for the Kepler/Coulomb problem. Moreover, we have seen how this semiclassical dynamics for any state $\psi \sim \exp((R+iS)/\hbar)$ is derivable from the limiting quantum particle density $\exp(2R/\epsilon^2)$ as $\epsilon \sim 0$, as a constrained Hamiltonian system with Hamiltonian,

$$H(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^2 + V_{\mathrm{eff}}(\boldsymbol{x}), \qquad V_{\mathrm{eff}}(\boldsymbol{x}) = -\frac{1}{|\boldsymbol{x}|} - |\nabla R(\boldsymbol{x})|^2, \qquad \boldsymbol{x},\boldsymbol{p} \in \mathbb{R}^3,$$

with natural units $H = -1/2$, with the constraint,

$$(\boldsymbol{p} - \nabla R(\boldsymbol{x})) \cdot \nabla R(\boldsymbol{x}) = 0.$$

This constraint merely fixes the semiclassical Hamiltonian here to be real and equal to H. To this extent these results for $\epsilon = 0$ transcend the framework of Nelson's mechanics.

Here nodal surfaces of wave-functions appear as singularities of Nelson's forward and negative backward drifts $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ which also arise as the unique solutions to our constrained Hamiltonian system (showing how Nelson's mechanics naturally arises from our constraint in the semiclassical limit.) We have regularised these singularities by remaining true to Nelson's ideas about forward and backward drifts by demanding that the WIMP-like particle is instantly reflected or transmitted at the singularity surfaces, with drifts $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ (revealed by the constrained Hamiltonian system as the only possibilities) with definite probabilities. In this way we can obtain a stochastic Hamiltonian system in the sense of Bismut, with semiclassical constants or conservation laws appearing as strong first integrals of the stochastic system, recapitulating the dynamical symmetry group $SO(4)$. This setup reveals Kepler's laws for WIMP-like particles in the infinite time limit of our system in a probabilistic setting, the drifts $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ (which emerge naturally from the constrained Hamiltonian) being the main ingredients of the theory. The resulting equations have explicit solutions in terms of Jacobi fields, up to leading behaviour in ϵ, detailed herein.

Our thesis depends critically upon the symmetry between $\boldsymbol{b}$ and $-\boldsymbol{b}^*$ (or ψ and ψ^*) and the fact that a Hamiltonian description of our system should accurately describe the physics of this system over long time periods. We advance two simple quantum mechanical arguments supporting these ideas to conclude this paper. These are of some independent interest.

Firstly a few remarks about why our model should give an accurate picture of WIMP-like particle behaviour over long periods. For a quantum mechanical particle in the initial normalised state ψ_i the probability P of finding this particle in the normalised state ψ_0 at time $t > 0$ is given in terms of the $L^2(\mathbb{R}^3)$ inner product $\langle \cdot, \cdot \rangle$ as,

$$P = \left| \left\langle \psi_0, \exp\left(-\frac{it}{\hbar}H\right)\psi_i \right\rangle \right|^2,$$

H being the quantum Hamiltonian operator. So if $H\psi_0 = E_0\psi_0$ for some real constant E_0 then,

$$P = \left|\langle \psi_0, \psi_i \rangle\right|^2.$$

If ψ_0 is the normalised atomic elliptic state and H the Coulomb Hamiltonian and $\psi_i = (\psi_0)_{\mathrm{s.c.}}$, the Bohr correspondence limit, then $P \sim \left|\langle (\psi_0)_{\mathrm{s.c.}}, (\psi_0)_{\mathrm{s.c.}} \rangle\right|^2 = 1$ for all times $t > 0$. So, as long as the Bohr correspondence limit is valid and the Coulomb Hamiltonian gives the correct dynamics, our model should be valid for long times. The next lemma makes these ideas more precise.

The lifetime up to time T of normalised state ψ_i, $\tau_T(\psi_i)$, is defined by,

$$\tau_T(\psi_i) = \int_0^T \mathrm{d}t \left| \left\langle \psi_i, \exp\left(-\frac{itH}{\hbar}\right)\psi_i \right\rangle \right|^2 \leq T.$$

Lemma VI.1. *Let ψ_i be given by,*

$$\psi_i = \sum_{n=0}^{\infty} c_n \phi_n, \qquad c_n = \langle \phi_n, \psi_i \rangle, \qquad \sum_{n=0}^{\infty} |c_n|^2 = 1,$$

and $\{\phi_n\}$ form a complete orthonormal system of eigenfunctions of H,

$$H\phi_n = E_n \phi_n, \quad n = 0, 1, 2, \ldots$$

Let $|c_0| = \max_j |c_j|$. Then,

$$\left| \frac{\tau_T(\psi_i)}{T} - \sum_{n=0}^{\infty} |c_n|^4 \right| \le \frac{2\hbar}{T\Delta E}(1 - |c_0|^2),$$

where $\Delta E = \inf_{m>n}(E_m - E_n)$ which is assumed positive.

Proof. Clearly,

$$\tau_T(\psi_i) = \frac{1}{2} \int_{-T}^{T} \mathrm{d}t \left| \left\langle \psi_i, \exp\left(-\frac{itH}{\hbar} \right) \psi_i \right\rangle \right|^2,$$

and so an elementary calculation yields,

$$\tau_T(\psi_i) = \frac{1}{2} \int_{-T}^{T} \sum_{m,n} \exp\left(\frac{it}{\hbar}(E_n - E_m) \right) |c_n|^2 |c_m|^2 \, \mathrm{d}t.$$

Therefore, interchanging orders of integration and summation yields,

$$\left| \frac{\tau_T(\psi_i)}{T} - \sum_{n=0}^{\infty} |c_n|^4 \right| \le \frac{2\hbar}{T\Delta E} \sum_{m>n} |c_n|^2 |c_m|^2,$$

from which the result follows. □

We conclude with a simple quantum mechanical calculation (in the correspondence limit), of some independent interest for the momentum distribution of WIMP-like particles in the atomic circular states. This result is mathematically striking so we present it as a lemma. We begin with the elementary result in two dimensions for the momentum space wave functions for the circular state,

$$\tilde{\psi}_2(p) \propto \int_0^{\infty} r \, \mathrm{d}r \exp\left(\frac{-r + \ln r + 1}{\epsilon^2} \right) \int_0^{2\pi} \mathrm{d}\theta \exp\left(\frac{i(\theta - pr\sin\theta)}{\epsilon^2} \right),$$

where $p = |\boldsymbol{P}|$ the magnitude of the momentum. This is an easy consequence of above.

Lemma VI.2. *For the atomic elliptic state in two dimensions, in the correspondence limit the relative probability of finding the momentum $\boldsymbol{p} \in \mathrm{d}^2 p$ is $|\tilde{\psi}_2(\boldsymbol{p})|^2 \, \mathrm{d}^2 p$ where,*

$$\tilde{\psi}_2(p) = \begin{cases} \sqrt{\frac{2}{\pi n}} (p^2 - 1)^{-1/4} \cos\left(n\left((p^2-1)^{1/2} - \arccos(p^{-1}) \right) - \frac{\pi}{4} \right), & p > 1, \\ \sqrt{\frac{1}{2\pi n}} (1 - p^2)^{-1/4} \cosh\left(n(1-p^2)^{1/2} - \operatorname{arccosh}(p^{-1}) \right), & p < 1, \end{cases}$$

where $p = |\boldsymbol{p}|$ as $n \sim \infty$, $\epsilon \sim 0$ with $\epsilon^2 n = 1$ fixed.

Proof. This is a simple consequence of the known asymptotics of the Bessel function,

$$J_n(z) = n^{-1} \int_0^{\pi} \cos(z\sin\theta - n\theta) \, \mathrm{d}\theta,$$

namely,

$$J_n(\operatorname{sech}\alpha) \sim \frac{\exp\left(n(\tanh\alpha - \alpha)\right)}{\sqrt{2\pi n \tanh\alpha}}(1 + O(n^{-1})), \qquad \alpha > 0,$$

and

$$J_n(\sec\beta) \sim \frac{\sqrt{2}}{\sqrt{\pi n \tan\beta}} \cos\left(n(\tan\beta - \beta) - \frac{\pi}{4}\right)(1 + O(n^{-1})), \qquad 0 < \beta < \pi/2,$$

and

$$J_n(n) \sim \frac{\Gamma(1/2)2^{1/3}}{2\pi 3^{1/6} n^{1/3}}, \qquad n \sim \infty.$$

This is a standard exercise using the method of steepest descent and Laplace's result on asymptotic expansions[4]. □

Corollary VI.1. *For the atomic circular state in 3 dimensions with the plane of the circle being (x, y) plane in Cartesians, in the correspondence limit,*

$$\tilde{\psi}(p) \sim \exp\left(\frac{-p_z^2}{2\epsilon^2}\right) \tilde{\psi}_2(\boldsymbol{p}_2),$$

where $\boldsymbol{p}_2 = (p_x, p_y)$, $p = |\boldsymbol{p}_2|$ and $\boldsymbol{p} = (p_x, p_y, p_z)$.

The above analysis shows that the probability distribution for momentum $p = |\boldsymbol{p}_2|$ is much more smeared out when $p > 1$ than it is when $p < 1$ where it is exponentially vanishingly small. This is of course a consequence of the Heisenberg uncertainty principle in this setting since in 2 dimensions $p > 1$ represents the interior of the circular orbit and $p < 1$ the exterior. Here we have the limiting case for the momentum distribution for the atomic elliptic state as the eccentricity $e \sim 0$ where the singularity surface Σ is in the plane $y = 0$ with $x < 0$ and

$$2e \leq e\sqrt{x^2 + z^2} + |\boldsymbol{x}| \leq 4e.$$

Now consider $\left(\tilde{\psi^*}\right)(\boldsymbol{p})$ the momentum space wave function corresponding to the configuration space wave function $\psi^*(\boldsymbol{x})$. We obtain easily that for $\boldsymbol{p} \in \mathbb{R}^3$,

$$\left(\tilde{\psi^*}\right)(\boldsymbol{p}) = \left(\tilde{\psi^*}\right)(-\boldsymbol{p}) = \tilde{\psi}(\boldsymbol{p}),$$

the right hand side being momentum space wave function corresponding to ψ. So from this symmetry we expect that as $e \sim 0$ then $\mathcal{P} \sim \mathcal{Q} \sim \frac{1}{2}$.

The observant reader will have noticed that the limiting momentum space wave function is not L^2-integrable over the whole of $\mathbb{R}^3$ so that the range of integration in $\boldsymbol{p}$ has to be restricted. Clearly $p = 1$ is just the classical hodograph for the circular orbit and for non zero probability $p > 1$. Working in polar coordinates $|p\Delta p| = O(L_3^2 \Delta r r^{-3}) = O(\epsilon)$ when $p \sim 1$ giving $\Delta p = O(\epsilon)$ for a circular orbit of unit radius. This gives the range of p integration.

We hope to be able to repeat this analysis for the general elliptical case. A final remark: a full quantum mechanical treatment of this problem would have to incorporate a measuring device determining the first time when the quantum particle arrived at Σ taking into account the lifetime of the state ψ,

$$\tau(\psi) = \frac{1}{2}\int_{-\infty}^{\infty} \left|\left\langle \psi, \exp\left(\frac{-itH}{\hbar}\right)\psi\right\rangle\right|^2 \mathrm{d}t$$

and the support of ψ in the Hamiltonian special representation $(E_0 - \Delta E, E_0 + \Delta E)$. This treatment would be much more long winded than the present approach based on the limit of Nelson's stochastic mechanics and the Bohm potential of nodal surfaces.

DEDICATION

We would like to dedicate this paper to Professor Neville Temperley on the occasion of his 100th birthday. He was the leading light of mathematical physics in Wales for many years.

[1]S. Albeverio, Ph. Blanchard, and R. Høegh-Krohn. A stochastic model for the orbits of planets and satellites: an interpretation of Titius-Bode law. *Exposition. Math.*, 1(4):365–373, 1983.
[2]J. Binney and S. Tremaine. *Galactic Dynamics: (Second Edition)*. Princeton Series in Astrophysics. Princeton University Press, 2011.
[3]J.M. Bismut. Mécanique aléatoire, volume 866 of *Lecture Notes in Math.*, Springer, Berlin, 1981.
[4]N. Bleistein and R. A. Handelsman. *Asymptotic Expansions of Integrals.* Dover Publications, 1975.
[5]N. Bohr. Uber die Anwendung der Quantentheorie auf den Atombau. I. Die Grundpostulaten der Quantentheorie. *Z. Phys.*, 13:117–165, 1923.
[6]E. Carlen. Conservative diffusions. *Comm. Math. Phys.*, 94(3), 293?315, 1984.
[7]J. Cresson, F. Pierret, and B. Puig. The Sharma-Parthasarathy stochastic two-body problem. *J. Math. Phys.*, 56(3):–, 2015.
[8]J. Cresson. The stochastisation hypothesis and the spacing of planetary systems. *J. Math. Phys.*, 52(11):113502, 20, 2011.
[9]J. D. Dollard, and C. N. Friedman. Product integration with applications to differential equations. Addison Wesley, 1979.
[10]R. Durran, A. Neate, and A. Truman. The divine clockwork: Bohr's correspondence principle and Nelson's stochastic mechanics for the atomic elliptic state. *J. Math. Phys.*, 49(3):032102, 30, 2008.
[11]R. Durran, A. Neate, A. Truman, and F.-Y. Wang. The correspondence limit of the atomic elliptic state. *EPL (Europhysics Letters)*, 84(3):30003, 2008.
[12]R. Durran, A. Neate, A. Truman, and F.-Y. Wang. On the divine clockwork: the spectral gap for the correspondence limit of the Nelson diffusion generator for the atomic elliptic state. *J. Math. Phys.*, 49(10):102103, 22, 2008.
[13]R. Durran and A. Truman. Planetesimal diffusions. In *Stochastic mechanics and stochastic processes (Swansea, 1986)*, volume 1325 of *Lecture Notes in Math.*, pages 76–88. Springer, Berlin, 1988.
[14]M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.
[15]J. C. Gay, D. Delande, and A. Bommier. Atomic quantum states with maximum localization on classical elliptical orbits. *Phys. Rev. A*, 39:6587–6590, Jun 1989.
[16]C. Lena, D. Delande, and J. C. Gay. Wave functions of atomic elliptic states. *Europhys. Lett.*, 15(7):697–702, 1991.
[17]A. Neate and A. Truman. A stochastic Burgers-Zeldovich model for the formation of planetary ring systems and the satellites of jupiter and saturn. *J. Math. Phys.*, 54(3):–, 2013.
[18]A. Neate and A. Truman. Semiclassical wave functions and semiclassical dynamics for the Kepler/Coulomb problem. *J. Phys. A*, 47(22):225302, 2014.
[19]E. Nelson. *Dynamical theories of Brownian motion.* Princeton University Press, Princeton, N.J., 1967.
[20]E. Nelson. *Quantum Fluctuations.* Princeton University Press, Princeton, N.J., 1985.
[21]E. Nelson. Field theory and the future of stochastic mechanics. In *Stochastic Processes in Classical and Quantum Systems (Proceedings, Ascona, Switzerland, 1985))*, volume 262 of *Lecture Notes in Phys.*, pages 438–69. Springer, Berlin, 1986.
[22]E. Nelson. Review of stochastic mechanics. *Journal of Physics: Conference Series*, 361(1):012011, 2012.
[23]L. Nottale. The quantization of the solar system. *Astron. Astrophys.*, 315, 1996.
[24]W. Pauli. Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik. *Z. Phys.*, 36: 336-363, 1926.
[25]Rothe, H.J. and Rothe, K.D. *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, World Scientific lecture notes in physics, World Scientific, 2010.
[26]W. Thirring. *Quantum mathematical physics.* Springer-Verlag, Berlin, second edition, 2002. Atoms, molecules and large systems, Translated from the 1979 and 1980 German originals by Evans M. Harrell II.
[27]S. Weinberg. *Cosmology.* OUP Oxford, 2008.
[28]J.-C. Zambrini. Variational processes and stochastic versions of mechanics. *J. Math. Phys.*, 27(9):2307–2330, 1986.
[29]Y. B.Zel'dovich. Gravitational instability: An approximate theory for large density perturbations. *Astron. Astrophys.*, 5: 84-89, 1970.