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SELF-SIMILAR FAST-REACTION LIMITS FOR REACTION-DIFFUSION SYSTEMS ON UNBOUNDED DOMAINS

E.C.M. CROOKS AND D. HILHORST

Abstract. We present a unified approach to characterising fast-reaction limits of systems of either two reaction-diffusion equations, or one reaction-diffusion equation and one ordinary differential equation, on unbounded domains, motivated by models of fast chemical reactions where either one or both reactant(s) is/are mobile. For appropriate initial data, solutions of four classes of problems each converge in the fast-reaction limit $k \to \infty$ to a self-similar limit profile that has one of four forms, depending on how many components diffuse and whether the spatial domain is a half or whole line. For fixed k, long-time convergence to these same self-similar profiles is also established, thanks to a scaling argument of Kamin. Our results generalise earlier work of Hilhorst, van der Hout and Peletier to a much wider class of problems, and provide a quantitative description of the penetration of one substance into another in both the fast-reaction and long-time regimes.

1. Introduction

Systems of the form

$$
(1.1)\qquad
\begin{aligned}
u_t &= d_u u_{xx} - kuv, && (x,t) \in (0,\infty) \times (0,T),\\
v_t &= -kuv, && (x,t) \in (0,\infty) \times (0,T),\\
u(0,t) &= U_0, && \text{for} \quad t \in (0,T),\\
u(x,0) &= 0, \quad v(x,0) = V_0 && \text{for} \quad x \in (0,\infty),
\end{aligned}
$$

arise in modelling chemical reactions

$$A + B \xrightarrow{k} C$$

taking place in a semi-infinite region, modelled for simplicity by the one-dimensional spatial domain $(0,\infty)$ with surface $x = 0$. Here u and v represent concentrations of a mobile chemical A and immobile substrate B respectively, U_0 and V_0 are positive constants, and k is the (positive) rate constant of the reaction. The mobile reactant u is initially not present in the domain $(0,\infty)$, the concentration of u outside the domain imposes a boundary condition $u = U_0$ at $x = 0$, and the immobile substrate is assumed initially to have uniform concentration V_0 throughout $(0,\infty)$. Examples of where such systems can arise include modelling the penetration of radio-labelled antibodies into tumourous tissue, or of carbonic acid into porous rock. The fast-reaction $k \to \infty$ limit of solutions of (1.1) is both physically relevant, since, for example, the attachment of antibodies to tissue can be very fast whereas the fact that antibodies are often relatively large makes diffusion typically slow, and mathematically useful and interesting. In [5], it was established by Hilhorst, van der Hout and Peletier that $k-$dependent solutions (u^k, v^k) of (1.1) converge as $k \to \infty$ on bounded time intervals $[0,T]$ to self-similar limit profiles $(u,v)(x/\sqrt{t})$ that satisfy a free boundary problem. This free boundary has the form $x = a\sqrt{t}$ where a is a positive constant, and separates the region in which the mobile chemical A is present from that where it is absent, thus characterising the rate at which, in the limit of fast reaction, A invades the immobile substrate B. Such information about how one substance penetrates into another has key applications to, for example, assessment of the effectiveness of radiotherapy or prediction of rates of carbon dioxide sequestration.

The modelling of other physical problems can clearly give rise to systems related to, but different from, (1.1), for which the fast-reaction limit and characterisation of rates at which one substance invades another are again of interest. For instance, if both a reactant u and substrate v are mobile, such as when carbonic acid penetrates into water instead of rock, the substrate will diffuse, introducing a term $d_v v_{xx}$ into the model, and typically satisfy a zero flux boundary condition $v_x = 0$ at the surface $x = 0$. Similar models but with the half-line spatial domain $(0, \infty)$ replaced by the whole line $\mathbb{R}$, can arise, for example in neutralisation reactions where u is the concentration of an acid, v the concentration of a base, either both mobile or one mobile and one immobile, and the two initially separated chemicals are brought together to react [11, 12, 13, 14]. The form of reaction can also be much more general than in (1.1), because, for instance, chemicals A and B may react in the form

$$mA + nB \xrightarrow{k} C$$

where the stoichiometric coefficients $m, n \in \mathbb{R}$ are positive, which gives rise to interaction terms $-ku^m v^n$ instead of $-kuv$, or more generally, $-kF(u, v)$, with suitable hypotheses on F. Since reactions can exhibit fractional order kinetics [11, 13], m and n need not be integers, and thus it is important to allow the interaction term to be not necessarily Lipschitz continuous on $[0, \infty) \times [0, \infty)$. Initial conditions can also be more complicated than the simple piecewise constant functions in (1.1).

Here we present a unified approach to characterising the self-similar fast-reaction limits for four different classes of problem covering all of the physical models above, with either one or both reactants mobile and spatial domain either the whole line $\mathbb{R}$ or the half-line $(0, \infty)$, and with sets of conditions encompassing a broad range of both interaction terms and initial conditions. Our framework includes as a special case the results of [5] for the prototype problem (1.1), and also the first extension of [5] in [6] that allows more general forms of F than $-kuv$. Note that the simple form of reaction and single mobile reactant in (1.1) actually enables this particular problem to be transformed to a single parabolic equation, whereas both [5] and the general framework presented here need alternative, more widely applicable ideas. Additionally, we exploit a scaling argument to apply our results on convergence to self-similar limit profiles as $k \to \infty$ to show that for fixed k, solutions converge in the long-time limit $t \to \infty$ to these same self-similar limit profiles in a certain average sense. This enables us also to provide rigorous justification for some limiting self-similar profiles derived previously by asymptotic methods by Trevelyan et al [14] in the context of long-time behaviour of reaction fronts in two-layer systems, and in fact, the asymptotic work of [11, 12, 13, 14], together with [5, 6], was central to the motivation for our work.

We treat two pairs of problems, depending on whether the spatial domain is $\mathbb{R}$ or $(0, \infty)$. The first pair is defined on the strip $Q_T = \{(x, t) : x \in \mathbb{R}, 0 < t < T\}$, and the system considered is

$$(P_1^k) \begin{cases} u_t &= d_u u_{xx} - kF(u, v) & \text{in } Q_T, \\ v_t &= d_v v_{xx} - kF(u, v) & \text{in } Q_T, \\ u(x, 0) &= u_0^k(x), \quad v(x, 0) = v_0^k(x) & \text{for } \ x \in \mathbb{R}, \end{cases}$$

where we define

$$u_0^\infty := \begin{cases} U_0 & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \quad , \quad v_0^\infty := \begin{cases} 0 & \text{for } x < 0, \\ V_0 & \text{for } x > 0, \end{cases}$$

with U_0, V_0 positive constants, and choose the initial data $u_0^k, v_0^k \in C^2(\mathbb{R})$ such that $0 \leq u_0^k \leq M$, $0 \leq v_0^k \leq M$ for some $M \geq \max\{U_0, V_0\}$,

$$u_0^k(x) \to U_0, 0 \ \text{ and } \ v_0^k(x) \to 0, V_0 \ \text{ as } \ x \to -\infty, \infty \ \text{ resp.,}$$

$$\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R})} < \infty, \ \ \|v_0^k - v_0^\infty\|_{L^1(\mathbb{R})} < \infty,$$

$$k \mapsto u_0^k - u_0^\infty, \;\; k \mapsto v_0^k - v_0^\infty \;\text{ belong to }\; C(\mathbb{R}^+, L^1(\mathbb{R})),$$

$$u_0^k \to u_0^\infty, \;\; v_0^k \to v_0^\infty \;\text{ in }\; L^1(\mathbb{R}) \text{ as } k \to \infty,$$

and there exists a continuous function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(\mu) \to 0$ as $\mu \to 0$ and

$$\|u_0^k(\cdot + \xi) - u_0^k(\cdot)\|_{L^1(\mathbb{R})} + \|v_0^k(\cdot + \xi) - v_0^k(\cdot)\|_{L^1(\mathbb{R})} \;\leq\; \omega(|\xi|) \;\text{ for all } k > 0, \; \xi \in \mathbb{R}.$$

The parameter k is positive and the interaction function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is such that

(i) there exists $\alpha > 0$ such that $F \in C^{0,\alpha}(\mathbb{R}^+ \times \mathbb{R}^+)$,
(ii) $F(u, 0) = F(0, v) = 0$ for all $u, v \in \mathbb{R}^+$ and $F(u, v) > 0$ for $(u, v) \in (0, \infty) \times (0, \infty)$,
(iii) $F(\cdot, v)$ and $F(u, \cdot)$ are non-decreasing for all $u, v \in \mathbb{R}^+$.

Two cases for (P_1^k) are considered, when the diffusion coefficients d_u and d_v are both strictly positive (two mobile reactants), and when $d_u > 0$ and $d_v = 0$ (one mobile and one immobile reactant).

The second pair of problems is defined on the half-strip $S_T = \{(x, t) : 0 < x < \infty, 0 < t < T\}$, and we consider the system

$$(P_2^k) \begin{cases} u_t &= d_u u_{xx} - kF(u, v) & \text{in } S_T, \\ v_t &= d_v v_{xx} - kF(u, v) & \text{in } S_T, \\ u(0, t) &= U_0, \quad d_v v_x(0, t) = 0 & \text{for } \; t \in (0, T), \\ u(x, 0) &= u_0^k(x), \quad v(x, 0) = v_0^k(x) & \text{for } \; x \in \mathbb{R}, \end{cases}$$

where $u_0^k, v_0^k \in C^2(\mathbb{R}^+)$ are such that $0 \leq u_0^k \leq M$, $0 \leq v_0^k \leq M$ for some $M \geq \max\{U_0, V_0\}$, and now

$$u_0^k(x) \to 0 \;\text{ and }\; v_0^k(x) \to V_0 \;\text{ as }\; x \to \infty,$$

$$\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R}^+)} < \infty, \;\; \|v_0^k - v_0^\infty\|_{L^1(\mathbb{R}^+)} < \infty,$$

$$u_0^k \to u_0^\infty, \;\; v_0^k \to v_0^\infty \;\text{ in }\; L^1(\mathbb{R}^+) \text{ as } k \to \infty,$$

for each $r > 0$, there exists a continuous function $\omega_r : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega_r(\mu) \to 0$ as $\mu \to 0$ and

$$\|u_0^k(\cdot + \xi) - u_0^k(\cdot)\|_{L^1((r,\infty))} + \|v_0^k(\cdot + \xi) - v_0^k(\cdot)\|_{L^1((r,\infty))} \;\leq\; \omega_r(|\xi|) \;\text{ for all } k > 0, \; |\xi| < r/4,$$

and k and F are as in problem (P_1^k). We again consider both the case of two mobile reactants, where the diffusion coeffcients d_u and d_v are both strictly positive, and the case of one mobile and one immobile reactant, when $d_u > 0$ and $d_v = 0$.

For each of these four problems, we prove the convergence of solutions (u^k, v^k) on bounded time intervals $(0, T)$ as $k \to \infty$ to a self-similar profile (u, v) in which u and v are segregated, separated by a free boundary. In each case, the limits u of u^k and v of v^k are given by the positive and negative parts respectively of a function w, that is,

$$u = w^+ \quad \text{and} \quad v = -w^-,$$

where $s^+ = \max\{0, s\}$ and $s^- = \min\{0, s\}$. This limit function w has one of four self-similar forms, depending on whether (u^k, v^k) satisfy (P_1^k) or (P_2^k), and on whether $d_v > 0$ or $d_v = 0$. If (u^k, v^k) satisfies (P_1^k), there exists a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that

$w(x,t) = f\left(x/\sqrt{t}\right)$ for $(x,t) \in Q_T$; if $d_v > 0$, then $a \in \mathbb{R}$ is the unique root of the equation $d_u U_0 \int_a^\infty e^{\frac{a^2-s^2}{4d_v}} ds = d_v V_0 \int_{-\infty}^a e^{\frac{a^2-s^2}{4d_u}} ds$, and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_{-\infty}^\eta e^{-\frac{s^2}{4d_u}} ds}{\int_{-\infty}^a e^{-\frac{s^2}{4d_u}} ds}\right), & \text{if } \eta \leq a, \\ -V_0 \left(1 - \dfrac{\int_\eta^\infty e^{-\frac{s^2}{4d_v}} ds}{\int_a^\infty e^{-\frac{s^2}{4d_v}} ds}\right), & \text{if } \eta > a, \end{cases}$$

whereas if $d_v = 0$, then $a > 0$ is the unique root of the equation $U_0 = \frac{V_0 a}{2d_u} \int_{-\infty}^a e^{\frac{a^2-s^2}{4d_u}} ds$, and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_{-\infty}^\eta e^{-\frac{s^2}{4d_u}} ds}{\int_{-\infty}^a e^{-\frac{s^2}{4d_u}} ds}\right), & \text{if } \eta \leq a, \\ -V_0, & \text{if } \eta > a. \end{cases}$$

On the other hand, if (u^k, v^k) satisfies (P_2^k), there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ and a positive constant $a > 0$ such that $w(x,t) = f\left(x/\sqrt{t}\right)$ for $(x,t) \in S_T$; if $d_v > 0$, then $a > 0$ is the unique root of the equation $d_u U_0 \int_a^\infty e^{\frac{a^2-s^2}{4d_v}} ds = d_v V_0 \int_0^a e^{\frac{a^2-s^2}{4d_u}}$, and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_0^\eta e^{-\frac{s^2}{4d_u}} ds}{\int_0^a e^{-\frac{s^2}{4d_u}} ds}\right), & \text{if } \eta \leq a, \\ -V_0 \left(1 - \dfrac{\int_\eta^\infty e^{-\frac{s^2}{4d_v}} ds}{\int_a^\infty e^{-\frac{s^2}{4d_v}} ds}\right), & \text{if } \eta > a, \end{cases}$$

whereas if $d_v = 0$, then $a > 0$ is the unique root of the equation $U_0 = \frac{V_0 a}{2d_u} \int_0^a e^{\frac{a^2-s^2}{4d_u}} ds$, and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_0^\eta e^{-\frac{s^2}{4d_u}} ds}{\int_0^a e^{-\frac{s^2}{4d_u}} ds}\right), & \text{if } \eta \leq a, \\ -V_0, & \text{if } \eta > a. \end{cases}$$

Clearly, in all four cases, a free boundary is given by the set where f equals zero, which has the form $x = a\sqrt{t}$ where the constant a is determined by a different equation for each problem. Note that only when (u^k, v^k) satisfies (P_1^k) with $d_u > 0$ and $d_v > 0$ is the constant a in the corresponding limit problem not necessarily strictly positive, and hence only for this problem is it possible for v to invade u instead of vice versa. Sufficient conditions ensuring $a > 0$, $a < 0$ or $a = 0$ in this case are given in Proposition 2.27.

In the last section of the paper, we fix k and initial conditions u_0 and v_0 such that

$$\|u_0 - u_0^\infty\|_{L^1} < \infty, \quad \|v_0 - v_0^\infty\|_{L^1} < \infty,$$

and either

$$u_0(x) \to U_0, 0 \text{ as } x \to -\infty, \infty \text{ and } v_0(x) \to 0, V_0 \text{ as } x \to -\infty, \infty,$$

in the case of the two full-line problems (P_1^k), or

$$u_0(x) \to 0 \text{ as } x \to \infty \text{ and } v_0(x) \to V_0 \text{ as } x \to \infty,$$

in the case of the two half-line problems (P_2^k), and then show, by exploiting the $k \to \infty$ results already established, that as $t \to \infty$ along a subsequence, $u(\cdot, t)$ and $v(\cdot, t)$ converge, in a certain average sense, to the appropriate one of the same four self-similar profiles. The proof uses a scaling argument originally due to Kamin [8].

This paper extends the earlier work of [5, 6] both by treating the case of two mobile reactants ($d_u > 0$, $d_v > 0$) in addition to that of one mobile reactant ($d_u > 0$, $d_v = 0$), and in considering the whole-line problem (P_1^k) in addition to the half-line problem (P_2^k). Importantly, we also allow significantly more general initial data than previous work. In [5, 6], the initial conditions for (P_2^k) are taken to be constant on the half-line $\mathbb{R}^+$, in fact equal to the initial data for the limiting self-similar solution, $u_0^\infty|_{\mathbb{R}^+}$, $v_0^\infty|_{\mathbb{R}^+}$. This implies monotonicity properties in space and time of solutions (u^k, v^k) of (P_2^k) that are exploited in [5, 6] to obtain some compactness of sequences $\{(u^k, v^k)\}_{k>0}$. Here, on the other hand, the initial data (u_0^k, v_0^k) is only supposed to satisfy the hypotheses listed above, and u_0^k, v_0^k may be non-monotonic in space and can even exceed U_0, V_0 on parts of the domain. For such initial conditions, monotonicity properties of (u^k, v^k) are no longer expected, of course, and alternative methods are needed. We exploit some ideas used previously in [4], [7] and [10], keeping in mind that here, in contrast to [4] and [10], our domains are unbounded. Note further that, motivated by the desire to include reaction dynamics of the form $F(u, v) = u^m v^n$ with $0 \le m < 1$, $0 \le n < 1$ (see [13], for example), we do not assume that F is Lipschitz continuous. Instead, as in [6], F is assumed to satisfy monotonicity hypotheses that suffice to establish comparison theorems (see Lemmas 2.10 and 3.2) in the absence of Lipschitz continuity. These monotonicity properties of F also enable the proof of L^1-contraction properties (see Lemma 2.15 and Lemma 3.7) giving bounds on differences of space translates, independently of d_v sufficiently small and of k, that yield sufficient compactness to pass to the limits both as $k \to \infty$ and as $d_v \to 0$.

We remark that the form of the self-similar solutions obtained here is clearly due to the presence of the heat operator and the fact that the same interaction term, $-kF(u, v)$, occurs in each equation in both (P_1^k) and (P_2^k). In fact, identical limit profiles are obtained for a relatively wide class of interaction terms $-kF(u, v)$ under suitable conditions on F, such as positivity and monotonicity, that suffice to ensure segregation of the two components and compactness properties of sets of solutions $\{(u^k, v^k)\}_{k>0}$. Interesting potential extensions of this work include investigating possible convergence to other types of self-similar solutions when the diffusion terms u_{xx}, v_{xx} are replaced by nonlinear diffusion terms, and also problems on multi-dimensional spatial domains.

The rest of the paper is organised as follows. In Section 2, we study the whole-line problem (P_1^k), starting with existence and uniqueness of solutions for (P_1^k), first when $d_u > 0$ and $d_v > 0$, and then, via some a priori estimates that are also useful in passing to the limit as $k \to \infty$, when $d_u > 0$ and $d_v = 0$. A key bound on $kF(u^k, v^k)$ in $L^1(Q_T)$, independent of k and $d_v \ge 0$, is given in Theorem 2.12. The last part of Section 2 is concerned with the limit of solutions (u^k, v^k) of (P_1^k) as $k \to \infty$, which is characterised as a self-similar solution in Theorem 2.26. This self-similar solution has one of two forms, depending on whether $d_v > 0$ or $d_v = 0$. Section 3 is devoted to corresponding results for the half-line problem (P_2^k), for which some different arguments are required on account of the boundary at $x = 0$. Theorem 3.4 is the half-line counterpart of Theorem 2.12. The two limiting self-similar solutions in this case, one for $d_v > 0$ and the other for $d_v = 0$, are given in Theorem 3.16. Finally, in Section 4, the results of the previous sections are used to deduce long-time convergence of solutions of (P_1^k) and (P_2^k) to the appropriate one of the four self-similar solutions.

Note that since we are interested in taking limits as $d_v \to 0$, when we write that a given bound is independent of d_v, we always mean that the bound is independent of $d_v \le D$ for some $D > 0$, *i.e.* that the bound is independent of d_v sufficiently small. Note also that throughout the paper, our notion of solution of (P_1^k) and (P_2^k) depends on whether $d_v > 0$ or $d_v = 0$, being classical and weak respectively, and is made precise in Theorems 2.9, 2.18, 3.1 and 3.10 below. Various results, such as the comparison principles Lemma 2.10, 3.2, a priori bounds Lemma 2.12, 3.4, etc., hold both when $d_v > 0$ and $d_v = 0$, with almost identical proofs, and so to avoid duplication, we will present results for $d_v \geq 0$ and understand an appropriate notion of solution in each case. Additionally, we adopt the notational convention that terms multiplied by d_v, such as $d_v v_{xx}$, for example, are understood to be simply absent when $d_v = 0$.

2. The whole-line case: problem (P_1^k)

2.1. Existence and uniqueness of solutions for (P_1^k) when $d_u > 0$ and $d_v > 0$. We consider first an approximate problem $(P_1^{R,\mu})$ to (P_1^k). Choose $M \geq \max\{U_0, V_0\}$, let $R > 1$, and consider the problem

$$(P_1^{R,\mu}) \begin{cases} u_t &= d_u u_{xx} - kF_\mu(u,v) \qquad \text{in } (-R,R)\times(0,T), \\ v_t &= d_v v_{xx} - kF_\mu(u,v) \qquad \text{in } (-R,R)\times(0,T), \\ u_x(-R,t) &= u_x(R,t) = 0 \qquad \text{for } t \in (0,T), \\ v_x(-R,t) &= v_x(R,t) = 0 \qquad \text{for } t \in (0,T), \\ u(x,0) &= u_{0,R}^k(x), \quad v(x,0) = v_{0,R}^k(x) \qquad \text{for } x \in (-R,R), \end{cases}$$

where $u_{0,R}^k, v_{0,R}^k \in C^2(\mathbb{R})$ are such that $0 \leq u_{0,R}^k \leq M$, $0 \leq v_{0,R}^k \leq M$ and

$$u_{0,R}^k(x) = 0 \text{ for } x > \left(1 - \frac{1}{R}\right)R, \quad u_{0,R}^k(x) = U_0 \text{ for } x < -\left(1 - \frac{1}{R}\right)R, \tag{2.1}$$

$$v_{0,R}^k(x) = V_0 \text{ for } x > \left(1 - \frac{1}{R}\right)R, \quad v_{0,R}^k(x) = 0 \text{ for } x < -\left(1 - \frac{1}{R}\right)R, \tag{2.2}$$

which defines the functions $u_{0,R}^k, v_{0,R}^k$ on the whole real line. We suppose also that the diffusion coefficients d_u and d_v are both strictly positive. The function F_μ is a regularisation of F, such that $F_\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(i) $F_\mu \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$,
(ii) $F_\mu(u,0) = F_\mu(0,v) = 0$ for all $u, v \in \mathbb{R}^+$, and $F_\mu(u,v) > 0$ for $(u,v) \in (0,\infty)\times(0,\infty)$,
(iii) $F_\mu(\cdot, v)$ and $F_\mu(u,\cdot)$ are non-decreasing for all $u, v \in \mathbb{R}^+$,
(iv) $F_\mu \to F$ in $L^\infty_{loc}(\mathbb{R}\times\mathbb{R})$ as $\mu \to 0$.

By a solution of $(P_1^{R,\mu})$ we mean a pair (u,v) such that $u, v \in C^{2,1}(([-R,R]\times[\delta,T]) \cap C^0([-R,R]\times[0,T])$ for each $\delta > 0$ and satisfy $(P_1^{R,\mu})$.

Lemma 2.1. *Let u, v and $\tilde{u}, \tilde{v}$ be two solutions of $(P_1^{R,\mu})$ whose initial data satisfy*

$$u(\cdot,0) \leq \tilde{u}(\cdot,0), \quad v(\cdot,0) \geq \tilde{v}(\cdot,0) \quad \text{in } (-R,R). \tag{2.3}$$

Then

$$u(\cdot,t) \leq \tilde{u}(\cdot,t), \quad v(\cdot,t) \geq \tilde{v}(\cdot,t) \quad \text{in } (-R,R)\times(0,T). \tag{2.4}$$

Proof. This follows from [15, p 241, Lem. 5.2 and p 244, Thm. 5.5] applied to the new system obtained from $(P_1^{R,\mu})$ under the change of variables $u \mapsto u$ and $v \mapsto V_0 - v$ (note that in the notation of [15], $u = (u_1, u_2)$ is a vector). □

The following corollary is immediate from Lemma 2.1.

Corollary 2.2. *For given initial data $u_{0,R}^k$, $v_{0,R}^k$, there is at most one solution $(u_{R,\mu}^k, v_{R,\mu}^k)$ of $(P_1^{R,\mu})$.*

We also have the following bound, which is easily proved using the scalar maximum principle.

Lemma 2.3. *Let $(u^k_{R,\mu}, v^k_{R,\mu})$ be a solution of $(P_1^{R,\mu})$. Then*

$$0 \le u^k_{R,\mu} \le M, \;\; 0 \le v^k_{R,\mu} \le M \quad \text{on} \quad (-R,R)\times(0,T). \tag{2.5}$$

Proof. We define

$$\begin{aligned} \mathcal{L}_1(u) &:= u_t - d_u u_{xx} + kF(u,v), \\ \mathcal{L}_2(v) &:= v_t - d_v v_{xx} + kF(u,v). \end{aligned}$$

Since $\mathcal{L}_i(0) = 0$ and $\mathcal{L}_i(M) \ge 0$ for $i = 1, 2$, the assertion follows from the maximum principle. □

Lemma 2.4. *There exists a unique solution $(u^k_{R,\mu}, v^k_{R,\mu})$ of $(P_1^{R,\mu})$.*

Proof. It follows from Lunardi [9, Prop. 7.3.2] that there exist $u^k_{R,\mu}, v^k_{R,\mu}$ and $T^* \in (0,T]$ such that $u^k_{R,\mu}, v^k_{R,\mu} \in C^{2,1}(([\delta,T]\times[-R,R]) \cap C^0([-R,R]\times[0,T])$ for each $\delta > 0$ and satisfy $(P_1^{R,\mu})$ with T replaced by T^*. That in fact we can take $T^* = T$ is a consequence of Lemma 2.3, and uniqueness of the solution is given by Corollary 2.2. □

We now introduce a class of cut-off functions. First define an even, non-negative cut-off function $\psi^1 \in C^\infty(\mathbb{R})$ such that $0 \le \psi^1(x) \le 1$ for all $x \in \mathbb{R}$, $\psi^1(x) = 1$ when $|x| \le 1$, and $\psi^1(x) = 0$ when $|x| \ge 2$. Then given $L \ge 1$, define the family of cut-off functions $\psi^L \in C^\infty(\mathbb{R})$ by $\psi^L(x) = 1$ when $|x| \le L$ and $\psi^L(x) = \psi^1(|x| + 1 - L)$ when $|x| \ge L$. Clearly ψ^L, ψ^L_x and ψ^L_{xx} are bounded in $L^\infty(\mathbb{R})$ independently of L, and ψ^L_x and ψ^L_{xx}, being supported on sets of measure at most two, are also bounded in $L^1(\mathbb{R})$ independently of L. Let $Q_{L,T}$ denote the truncated space-time domain $(-L,L)\times(-T,T)$. In the following, $C(L)$ denotes some L-dependent constant which varies according to context.

Lemma 2.5. *Let $L > 0$. Then there exists a constant $C(L)$ such that if $R > L + 1$, then*

$$k \iint_{Q_{L,T}} F_\mu(u^k_{R,\mu}, v^k_{R,\mu})\, dxdt \;\le\; C(L), \tag{2.6}$$

for all $k, \mu > 0$.

Proof. Multiplying the equation for $u^k_{R,\mu}$ in $(P_1^{R,\mu})$ by ψ^L and integrating over $Q_{L+1,T}$ gives that

$$\int_{-L-1}^{L+1} \psi^L\{u^k_{R,\mu}(\cdot,T) - u^k_{0,R}(\cdot)\}\, dx = d_u \iint_{Q_{L+1,T}} u^k_{R,\mu}\psi^L_{xx}\, dxdt - k \iint_{Q_{L+1,T}} F_\mu(u^k_{R,\mu}, v^k_{R,\mu})\psi^L\, dxdt,$$

which, together with Lemma 2.3 and the definition of ψ^L, yields (2.6). □

Lemma 2.6. *The solutions $u^k_{R,\mu}, v^k_{R,\mu}$ are bounded in $L^2(0,T;H^1_{loc}(\mathbb{R}))$ independently of k, R, μ.*

Proof. We prove the bound for $u^k_{R,\mu}$. Suppose that $R > L+1$. Then multiplying the equation for $u^k_{R,\mu}$ by $u^k_{R,\mu}\psi^L$ and integrating over $Q_{L+1,T}$ gives that

$$\begin{aligned} \frac{1}{2}\int_{-L-1}^{L+1} \psi^L\, \{u^k_{R,\mu}(\cdot,T)^2 - u^k_{0,R}(\cdot)^2\}\, dx \le -d_u \iint_{Q_{L+1,T}} (u^k_{R,\mu})^2_x\, \psi^L\, dxdt \\ + \frac{d_u}{2} \iint_{Q_{L+1,T}} (u^k_{R,\mu})^2 \psi^L_{xx}\, dxdt, \end{aligned}$$

since $F_\mu(u^k_{R,\mu}, v^k_{R,\mu}) \ge 0$. The result again follows using Lemma 2.3 and the definition of ψ^L. □

In order to prove that the sets $\{u^k_{R,\mu} : k, R, \mu > 0\}$, $\{v^k_{R,\mu} : k, R, \mu > 0\}$ are each relatively compact in $L^2_{loc}(\mathbb{R} \times (0,T))$, we first give estimates of the differences of space and time translates of $u^k_{R,\mu}$ and $v^k_{R,\mu}$.

Lemma 2.7. *For each $L > 0$, there exists a constant $C(L)$ such that*

$$\iint_{Q_{L,T}} (u^k_{R,\mu}(x+\xi,t) - u^k_{R,\mu}(x,t))^2 \, dxdt \le C(L)|\xi|^2,$$

$$\iint_{Q_{L,T}} (v^k_{R,\mu}(x+\xi,t) - v^k_{R,\mu}(x,t))^2 \, dxdt \le C(L)|\xi|^2,$$

for all $\xi \in \mathbb{R}$, $|\xi| \le L$.

Proof. As a result of the gradient bounds in Lemma 2.6, the proof of this closely follows the proof of [4, Lemma 2.6] and we omit the details. □

Lemma 2.8. *For each $L > 0$, there exists a constant $C(L)$ such that*

$$\iint_{Q_{L,T-\tau}} (u^k_{R,\mu}(x,t+\tau) - u^k_{R,\mu}(x,t))^2 \, dxdt \le C(L)\tau,$$

$$\iint_{Q_{L,T-\tau}} (v^k_{R,\mu}(x,t+\tau) - v^k_{R,\mu}(x,t))^2 \, dxdt \le C(L)\tau,$$

for all $\tau \in (0,T)$.

Proof. The gradient bounds in Lemma 2.6 together with Lemma 2.5 enable the proof of [4, Lemma 2.7] to be easily adapted. □

We can now establish the existence of a classical solution of the original problem (P^k_1) on Q_T when both diffusion coefficients d_u and d_v are strictly positive.

Theorem 2.9. *Suppose that $d_u > 0$ and $d_v > 0$. Then given $k > 0$, there exists a classical solution (u^k, v^k) of (P^k_1) such that for each $\delta > 0$, $J > 0$ and $p \ge 1$,*

$$u^k, v^k \in C^{2,1}(\mathbb{R} \times [\delta,T]) \cap C^0(\mathbb{R} \times [0,T]) \cap W^{2,1}_p((-J,J) \times (0,T)), \tag{2.7}$$

and

$$0 \le u^k \le M, \quad 0 \le v^k \le M \quad \text{on} \quad \mathbb{R} \times (0,T). \tag{2.8}$$

Proof. Let $u^k_{0,R}, v^k_{0,R}$ be as in the formulation of problem $(P^{R,\mu}_1)$ and such that as $R \to \infty$, $u^k_{0,R} \to u^k_0$ and $v^k_{0,R} \to v^k_0$ in $C^1_{loc}(\mathbb{R})$. Then given $R_n \to \infty$ and $\mu_n \downarrow 0$, it follows from the Fréchet-Kolmogorov Theorem (see, for example, [3, Corollary 4.27]) and Lemmas 2.3, 2.7 and 2.8, that there exist subsequences $\{R_{n_j}\}^\infty_{j=1}$, $\{\mu_{n_j}\}^\infty_{j=1}$ and functions $u^k \in L^\infty(Q_T)$ and $v^k \in L^\infty(Q_T)$ such that

$$u^k_{R_{n_j},\mu_{n_j}} \to u^k, \quad v^k_{R_{n_j},\mu_{n_j}} \to v^k \quad \text{strongly in } L^2_{loc}(Q_T) \text{ and } a.e. \text{ in } Q_T,$$

as $j \to \infty$. We can then easily pass to the limit in the weak form of (P^k_1). To see that the solution is in fact classical, note first that for a fixed k, the term $kF(u^k, v^k)$ is in $L^\infty(Q_T)$, which, since $u^k_0, v^k_0 \in C^2(\mathbb{R})$, implies that $u^k, v^k \in W^{2,1}_p((-J,J) \times (0,T))$ for each $J > 0$ and $p \ge 1$, and hence $u^k, v^k \in C^{1+\lambda,\frac{1+\lambda}{2}}(\mathbb{R} \times [0,T])$ for each $\lambda \in (0,1)$. Since $F \in C^{0,\alpha}(\mathbb{R}^+ \times \mathbb{R}^+)$, it then follows that $F(u^k, v^k)$ is Hölder continuous and so $u^k, v^k \in C^{2+\lambda,\frac{2+\lambda}{2}}(\mathbb{R} \times (0,T])$ for some $\lambda > 0$. The bounds (2.8) are immediate from Lemma 2.3. □

To show uniqueness, we use the following comparison theorem for (P_1^k), proved with arguments inspired by [6, Lemma 2.7]. Note that this result covers both the case $d_u > 0$, $d_v > 0$ and the case $d_u > 0$, $d_v = 0$, and the monotonicity properties of F are exploited to overcome the fact that F is not assumed to be Lipschitz continuous. For an alternative approach when F is Lipschitz and $d_v = 0$, see [10, Lemma 5].

Lemma 2.10. *Suppose that $d_u > 0$, $d_v \geq 0$, and let $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ be such that for each $J > 0$ and $p \geq 1$, $\overline{u}, \underline{u} \in L^\infty(Q_T) \cap W_p^{2,1}((-J,J)\times(0,T))$, $\overline{v}, \underline{v} \in L^\infty(Q_T) \cap W_p^{2,1}((-J,J)\times(0,T))$ if $d_v > 0$, $\overline{v}, \underline{v} \in L^\infty(Q_T) \cap W^{1,\infty}(0,T; L^\infty((-J,J)))$ if $d_v = 0$, and $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ satisfy*

$$\begin{array}{lll} \overline{u}_t \geq d_u \overline{u}_{xx} - kF(\overline{u},\overline{v}), & \underline{u}_t \leq d_u \underline{u}_{xx} - kF(\underline{u},\underline{v}), & \\ \overline{v}_t \leq d_v \overline{v}_{xx} - kF(\overline{u},\overline{v}), & \underline{v}_t \geq d_v \underline{v}_{xx} - kF(\underline{u},\underline{v}), & \text{in } Q_T, \end{array}$$

and

$$\overline{u}(\cdot,0) \geq \underline{u}(\cdot,0), \qquad \overline{v}(\cdot,0) \leq \underline{v}(\cdot,0) \quad \text{on } \mathbb{R}.$$

Then

$$\overline{u} \geq \underline{u} \quad \text{and} \quad \overline{v} \leq \underline{v} \quad \text{in } Q_T.$$

Proof. Let $u := \underline{u} - \overline{u}$, $v := \overline{v} - \underline{v}$, $u_0 := \underline{u}(\cdot,0) - \overline{u}(\cdot,0)$ and $v_0 := \overline{v}(\cdot,0) - \underline{v}(\cdot,0)$. Then

$$u_t \leq d_u u_{xx} - k\{F(\underline{u},\underline{v}) - F(\overline{u},\overline{v})\} \quad \text{in } Q_T, \tag{2.9}$$

$$v_t \leq d_v v_{xx} - k\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\} \quad \text{in } Q_T, \tag{2.10}$$

and

$$u_0 \leq 0, \quad v_0 \leq 0 \text{ on } \mathbb{R}.$$

Now take a smooth non-decreasing convex function $m^+ : \mathbb{R} \to \mathbb{R}$ with

$$m^+ \geq 0, \quad m^+(0) = 0, \quad (m^+)'(0) = 0, \quad m^+(r) \equiv 0 \text{ for } r \leq 0, \quad m^+(r) = |r| - \frac{1}{2} \text{ for } r > 1,$$

and for each $\alpha > 0$, define the functions

$$m_\alpha^+(r) := \alpha m^+\left(\frac{r}{\alpha}\right),$$

which as $\alpha \to 0$ approximate the positive part of r. Then multiplying (2.9) by $(m_\alpha^+)'(u)$ and (2.10) by $(m_\alpha^+)'(v)$ gives

$$\begin{aligned} (m_\alpha^+)'(u)u_t &\leq d_u (m_\alpha^+)'(u)u_{xx} - k(m_\alpha^+)'(u)\{F(\underline{u},\underline{v}) - F(\overline{u},\overline{v})\} \quad \text{in } Q_T, \\ (m_\alpha^+)'(v)v_t &\leq d_v (m_\alpha^+)'(v)v_{xx} - k(m_\alpha^+)'(v)\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\} \quad \text{in } Q_T, \end{aligned}$$

and it follows from adding these inequalities that

$$\begin{aligned} (m_\alpha^+)'(u)u_t + (m_\alpha^+)'(v)v_t \leq d_u(m_\alpha^+)'(u)u_{xx} + d_v(m_\alpha^+)'(v)v_{xx} \\ -k[(m_\alpha^+)'(v) - (m_\alpha^+)'(u)]\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\}. \end{aligned} \tag{2.11}$$

Now with ψ^L the cut-off functions defined before Lemma 2.5, multiplying by ψ^L and integrating over $\mathbb{R} \times (0,t_0)$, $t_0 \in (0,T]$, gives

$$\begin{aligned} \int_0^{t_0}\int_{\mathbb{R}} u_{xx}(m_\alpha^+)'(u)\psi^L \, dxdt &= -\int_0^{t_0}\int_{\mathbb{R}} u_x[(m_\alpha^+)''(u)u_x\psi^L + (m_\alpha^+)'(u)\psi_x^L] \, dxdt \\ &\leq -\int_0^{t_0}\int_{\mathbb{R}} u_x(m_\alpha^+)'(u)\psi_x^L \, dxdt = \int_0^{t_0}\int_{\mathbb{R}} m_\alpha^+(u)\psi_{xx}^L \, dxdt, \end{aligned}$$

since $(m_\alpha^+)''(u) \geq 0$ because m_α^+ is convex. So (2.11) yields

$$\int_{\mathbb{R}} \psi^L(x)[m_\alpha^+(u) + m_\alpha^+(v)](x,t_0)\ dx \leq \int_{\mathbb{R}} \psi^L(x)[m_\alpha^+(u) + m_\alpha^+(v)](x,0)\ dx$$
$$+ \int_0^{t_0}\int_{\mathbb{R}} \psi^L_{xx}(x)\{d_u m_\alpha^+(u) + d_v m_\alpha^+(v)\}\ dxdt$$
$$-k\int_0^{t_0}\int_{\mathbb{R}} \psi^L(x)[(m_\alpha^+)'(v) - (m_\alpha^+)'(u)]\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\}\ dxdt,$$

and letting $\alpha \to 0$ gives

$$\int_{\mathbb{R}} \psi^L(x)[u^+ + v^+](x,t_0)\ dx \leq \int_{\mathbb{R}} \psi^L(x)[u^+ + v^+](x,0)\ dx + \int_0^{t_0}\int_{\mathbb{R}} \psi^L_{xx}(x)\{d_u u^+ + d_v v^+\}\ dxdt$$
$$-k\int_0^{t_0}\int_{\mathbb{R}} \psi^L(x)[(\text{sgn}v)^+ - (\text{sgn}u)^+]\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\}\ dxdt,$$

where $u^+ := \max(u,0)$. Then $(u^+ + v^+)(\cdot,0) = 0$, and the expression

$$\sharp := [(\text{sgn}v)^+ - (\text{sgn}u)^+]\{F(\overline{u},\overline{v}) - F(\underline{u},\underline{v})\}$$

is non-zero only if either

(i) $(\text{sgn}v)^+ = 1$ and $(\text{sgn}u)^+ = 0$, in which case $\overline{v} \geq \underline{v}$ and $\underline{u} \leq \overline{u}$, so that $F(\overline{u},\overline{v}) - F(\underline{u},\underline{v}) \geq 0$, because $F(\cdot,v)$ and $F(u,\cdot)$ are non-decreasing for all $u,v \in \mathbb{R}^+$, and hence $\sharp \geq 0$, or

(ii) $(\text{sgn}v)^+ = 0$ and $(\text{sgn}u)^+ = 1$, in which case $\overline{v} \leq \underline{v}$ and $\underline{u} \geq \overline{u}$, so that $F(\overline{u},\overline{v}) - F(\underline{u},\underline{v}) \leq 0$, and hence, again, $\sharp \geq 0$.

Thus

$$\int_{\mathbb{R}} \psi^L[u^+ + v^+](x,t_0)\ dx \leq \int_0^{t_0}\int_{\mathbb{R}} [d_u u^+ + d_v v^+]|\psi^L_{xx}|\, dxdt. \tag{2.12}$$

Now the right-hand side of (2.12) is bounded independently of L. So by Lebesgue's monotone convergence theorem, $u^+, v^+ \in L^\infty(0,T;L^1(\mathbb{R}))$, and thus the right-hand side of (2.12) in fact tends to 0 as $L \to \infty$. Hence

$$[u^+ + v^+](\cdot,t_0) = 0 \quad \text{on } \mathbb{R},$$

and the result follows. □

The following corollary is immediate from Lemma 2.10.

Corollary 2.11. *Suppose $d_u > 0$ and $d_v > 0$. Then given $k > 0$, there is at most one solution (u^k, v^k) of (P_1^k) in $L^\infty(Q_T) \cap W_p^{2,1}((-J,J)\times(0,T))$ for all $J > 0$, $p \geq 1$.*

2.2. **Existence and uniqueness of solutions for (P_1^k) when $d_u > 0$ and $d_v = 0$.** Next we prove some preliminary estimates that will be used in the following both to prove existence of solutions of (P_1^k) when $d_u > 0$ and $d_v = 0$, and, in the next section, to study the limit of (P_1^k) as $k \to \infty$.

The following bound for $kF(u^k, v^k)$ is key. Note that $kF(u^k, v^k)$ is controlled by u^k on part of the spatial domain and by v^k on the other part, due to the fact that u_0^k is bounded in $\|\cdot\|_{L^1(\mathbb{R}^+)}$ independently of k, and v_0^k is bounded in $\|\cdot\|_{L^1(\mathbb{R}^-)}$ independently of k. A similar phenomenon occurs in the proof of the corresponding estimate in the half-line case, Lemma 3.4, in which $kF(u^k, v^k)$ is controlled by u^k on $(1,\infty)\times(0,T)$, and by v^k in the boundary region $(0,1)\times(0,T)$.

Lemma 2.12. *There exists a constant $C > 0$, independent of $d_v \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (P_1^k) satisfying (2.8), we have*

$$\int_0^T \int_{\mathbb{R}} kF(u^k, v^k)\; dx\, dt \leq C.$$

Proof. Define a cut-off function $\phi^1 \in C^\infty(\mathbb{R})$ such that $0 \leq \phi^1(x) \leq 1$ for all $x \in \mathbb{R}$, $\phi^1 = 1$ when $x \in [0,1]$, and $\phi^1(x) = 0$ when $x \notin (-1,2)$. Then given $L \geq 1$, define $\phi^L \in C^\infty(\mathbb{R})$ by $\phi^L(x) = \phi^1(x)$ if $x \leq 0$, $\phi^L(x) = 1$ when $x \in [0, L]$, and $\phi^L(x) = \phi^1(x + 1 - L)$ when $x \geq L$, and define $\tilde{\phi}^L \in C^\infty(\mathbb{R})$ by $\tilde{\phi}^L(x) = \phi^L(-x)$ for all $x \in \mathbb{R}$. Note that $0 \leq \phi^L(x), \tilde{\phi}^L(x) \leq 1$ for all $x \in \mathbb{R}$, and ϕ^L_x, ϕ^L_{xx}, $\tilde{\phi}^L_x$ and $\tilde{\phi}^L_{xx}$ are bounded in both $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ independently of L. Consider first the case when $d_u > 0$ and $d_v > 0$. Then multiplying the equation for u^k by ϕ^L and integrating over $\mathbb{R} \times (0, t_0)$, $t_0 \in (0, T]$, gives that

$$\int_{-1}^\infty \phi^L(x) u^k(x, t_0)\, dx + \int_0^{t_0} \int_{-1}^\infty \phi^L(x) kF(u^k, v^k)\, dxdt =$$

$$d_u \int_0^{t_0} \int_{-1}^\infty \phi^L_{xx}(x) u^k(x,t)\, dxdt + \int_{-1}^\infty \phi^L(x) u_0^k(x)\, dx, \tag{2.13}$$

which, since the definition of ϕ^L and the facts that $0 \leq u^k \leq M$ and $\|u_0^k\|_{L^1(\mathbb{R}^+)}$ is bounded independently of k imply that the right-hand side of (2.13) is bounded independently of L and k, gives the existence of $C > 0$ such that for all $k > 0$ and $t_0 \in (0, T]$,

$$\int_{-1}^\infty \phi^L(x) u^k(x, t_0)\, dx + \int_0^{t_0} \int_{-1}^\infty \phi^L(x) kF(u^k, v^k)\, dxdt \;\leq\; C, \tag{2.14}$$

and then, since $u^k \geq 0$, letting $L \to \infty$ using Lebesgue's monotone convergence theorem gives

$$\int_0^T \int_0^\infty kF(u^k, v^k)\, dxdt \leq C. \tag{2.15}$$

(Note that if we had $d_u = 0$ instead of $d_u > 0$, then (2.14) could be proved likewise, with the first term on the right-hand side of (2.13) absent due to the lack of diffusion term.)

Similarly, since $\|v_0^k\|_{L^1(\mathbb{R}^-)}$ is bounded independently of k, multiplying the equation for v^k by $\tilde{\phi}^L$ and integrating over $\mathbb{R} \times (0, t_0)$ yields that C can be chosen large enough that for all $L, k > 0$, we also have

$$\int_{-\infty}^1 \tilde{\phi}^L(x) v^k(x, t_0)\, dx + \int_0^{t_0} \int_{-\infty}^1 \tilde{\phi}^L(x) kF(u^k, v^k)\, dxdt \;\leq\; C, \tag{2.16}$$

and hence, since $v^k \geq 0$, letting $L \to \infty$ yields that

$$\int_0^T \int_{-\infty}^0 kF(u^k, v^k)\, dx \leq C. \tag{2.17}$$

The result then follows from (2.15) and (2.17). □

Lemma 2.13. *There exists a constant $C > 0$, independent of $d_v \geq 0$ and $k > 0$, such that for all $k > 0$ and any solution (u^k, v^k) of (P_1^k) satisfying (2.8),*

$$\|u^k(\cdot, t) - u_0^\infty\|_{L^1(\mathbb{R})} \;\leq C \qquad \textit{and} \qquad \|v^k(\cdot, t) - v_0^\infty\|_{L^1(\mathbb{R})} \leq C \quad \textit{for all } t \in [0, T].$$

Proof. Note first that it follows immediately from (2.14), (2.16) and Lebesgue's monotone convergence theorem that there exists $C > 0$, independent of $d_v \geq 0$ and $k > 0$, such that

$$\int_0^\infty u^k(x,t_0)\,dx \leq C \quad \text{and} \quad \int_{-\infty}^0 v^k(x,t_0)\,dx \leq C \quad \text{for all } t_0 \in [0,T]. \tag{2.18}$$

Now choose a smooth convex function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \geq 0, \qquad m(0) = 0, \qquad m'(0) = 0, \qquad m(r) = |r| - \frac{1}{2} \qquad \text{for } |r| > 1,$$

and for each $\alpha > 0$, define the functions

$$m_\alpha(r) := \alpha m\left(\frac{r}{\alpha}\right),$$

which approximate the modulus function as $\alpha \to 0$, and define $\hat{u}^k := U_0 - u^k$. Then

$$\begin{aligned} \hat{u}^k_t &= d_u \hat{u}^k_{xx} + kF(u^k, v^k) && \text{in } Q_T \\ \hat{u}^k(x,0) &= U_0 - u^k_0(x), && \text{for} \quad x \in \mathbb{R}. \end{aligned}$$

Now with $\tilde{\phi}^L$ as in Lemma 2.12,

$$\begin{aligned} \int_{\mathbb{R}} \hat{u}^k_{xx} m'_\alpha(\hat{u}^k)\tilde{\phi}^L\,dx &= -\int_{\mathbb{R}} \hat{u}^k_x [m''_\alpha(\hat{u}^k)\hat{u}^k_x \tilde{\phi}^L + m'_\alpha(\hat{u}^k)\tilde{\phi}^L_x]\,dx \\ &\leq -\int_{\mathbb{R}} \hat{u}^k_x m'_\alpha(\hat{u}^k)\tilde{\phi}^L_x\,dx = \int_{\mathbb{R}} m_\alpha(\hat{u}^k)\tilde{\phi}^L_{xx}\,dx, \end{aligned}$$

so multiplying the equation for $\hat{u}^k$ by $\tilde{\phi}^L m'_\alpha(\hat{u}^k)$ and integrating over $\mathbb{R} \times (0,t_0)$, $t_0 \in (0,T)$, gives that

$$\begin{aligned} \int_{\mathbb{R}} \tilde{\phi}^L m_\alpha(\hat{u}^k(x,t_0))\,dx \leq \int_{\mathbb{R}} \tilde{\phi}^L m_\alpha(\hat{u}^k(x,0))\,dx &+ d_u \int_0^{t_0}\int_{\mathbb{R}} m_\alpha(\hat{u}^k)\tilde{\phi}^L_{xx}\,dxdt \\ &+ \int_0^{t_0}\int_{\mathbb{R}} kF(u^k,v^k)m'_\alpha(\hat{u}^k)\tilde{\phi}^L\,dxdt, \end{aligned} \tag{2.19}$$

and then letting $\alpha \to 0$ in (2.19) yields

$$\begin{aligned} &\int_{\mathbb{R}} \tilde{\phi}^L |\hat{u}^k(x,t_0)|\,dx \leq \\ &\int_{\mathbb{R}} \tilde{\phi}^L |\hat{u}^k(x,0)|\,dx + d_u \int_0^{t_0}\int_{\mathbb{R}} |\hat{u}^k|\tilde{\phi}^L_{xx}\,dxdt + \int_0^{t_0}\int_{\mathbb{R}} kF(u^k,v^k)\mathrm{sgn}(\hat{u}^k)\tilde{\phi}^L\,dxdt. \end{aligned} \tag{2.20}$$

Now by Lemma 2.12, (2.8), and the fact that $\|u^k_0 - u^\infty_0\|_{L^1(\mathbb{R})}$ is bounded independently of k, the right-hand side of (2.20) is bounded independently of L and k. So it follows from (2.20) that there exists C, independent of k, such that

$$\int_{-\infty}^0 |u^k(x,t_0) - U_0|\,dx \;\leq\; C \quad \text{for all } t_0 \in (0,T). \tag{2.21}$$

Then taking ϕ^L as in Lemma 2.12, multiplying the equation satisfied by $\hat{v}^k := V_0 - v^k$ by $\phi^L m_\alpha(\hat{v}^k)$ and again integrating over $\mathbb{R} \times (0,t_0)$ gives, using a similar argument to above, that C can be chosen large enough that we also have that

$$\int_0^\infty |v^k(x,t_0) - V_0|\,dx \;\leq\; C \quad \text{for all } t_0 \in (0,T). \tag{2.22}$$

The result follows from (2.21), (2.22), and (2.18). ∎

We prove next a bound for the L^2-norm of the space derivatives u_x and v_x.

Lemma 2.14. *Suppose that $d_u > 0$ and $d_v \geq 0$. Then there exists a constant C, independent of $d_v \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (P_1^k) satisfying (2.8),*

$$d_u \int_0^T \int_{\mathbb{R}} (u_x^k)^2(x,t)\, dxdt \leq C \quad \text{and} \quad d_v \int_0^T \int_{\mathbb{R}} (v_x^k)^2(x,t)\, dxdt \leq C. \tag{2.23}$$

Proof. Let ϕ^L and $\tilde{\phi}^L$ be as in the proof of Lemma 2.12. Then multiplication of the equation for u^k by $u^k\phi^L$ and integration over Q_T gives

$$\frac{1}{2}\int_{\mathbb{R}} \phi^L(x)(u^k)^2(x,T)\, dx + d_u \int_0^T \int_{\mathbb{R}} (u_x^k)^2 \phi^L(x)\, dxdt =$$

$$\frac{1}{2}\int_{\mathbb{R}} \phi^L(x)(u^k)^2(x,0)\, dx + \frac{d_u}{2}\int_0^T \int_{\mathbb{R}} (u^k)^2(x,t)\phi_{xx}^L(x)\, dxdt - \int_0^T \int_{\mathbb{R}} ku^kF(u^k,v^k)\phi^L(x)\, dxdt$$

$$\leq \frac{1}{2}\int_{\mathbb{R}} \phi^L(x)(u^k)^2(x,0)\, dx + \frac{d_u}{2}\int_0^T \int_{\mathbb{R}} (u^k)^2(x,t)\phi_{xx}^L(x)\, dxdt, \tag{2.24}$$

since $F(u^k, v^k) \geq 0$. Now it follows from (2.8) and the definition of ϕ^L that the right-hand side of (2.24) is bounded independently of L. It thus follows that $d_u \int_0^T \int_{\mathbb{R}} (u_x^k)^2 \phi^L\, dxdt$ is bounded independently of L and hence, using Lebesgue's monotone convergence theorem to let $L \to \infty$ in $\int_0^T \int_0^\infty (u_x^k)^2 \phi^L\, dxdt$, that letting $L \to \infty$ in (2.24) implies that there exists a constant $C_1 > 0$ such that

$$d_u \int_0^T \int_0^\infty (u_x^k)^2\, dxdt \leq C_1 + \frac{1}{2}\int_0^\infty (u^k)^2(x,0)\, dx \leq \frac{C}{2},$$

where the constant C is independent of $d_u, k > 0$, by (2.8) and the fact that $\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R})}$ is bounded independently of k. Then letting $\hat{u}^k := U_0 - u^k$, multiplying the equation for $\hat{u}^k$ by $\hat{u}^k\tilde{\phi}^L$ and integrating over Q_T yields

$$\frac{1}{2}\int_{\mathbb{R}} \tilde{\phi}^L(\hat{u}^k)^2(x,T)\, dx + d_u \int_0^T \int_{\mathbb{R}} (\hat{u}_x^k)^2 \tilde{\phi}^L\, dxdt = \tag{2.25}$$

$$\frac{1}{2}\int_{\mathbb{R}} \tilde{\phi}^L(\hat{u}^k)^2(x,0)\, dx + \frac{d_u}{2}\int_0^T \int_{\mathbb{R}} (\hat{u}^k)^2(x,t)\tilde{\phi}_{xx}^L\, dxdt - \int_0^T \int_{\mathbb{R}} k\hat{u}^kF(u^k,v^k)\tilde{\phi}^L\, dxdt.$$

Now $\hat{u}^k$ may not be non-negative, but we can call on Lemma 2.12 to deduce that the right-hand side of (2.25) is bounded independently of L, so that arguing similarly to before gives that C_1 can be chosen larger if necessary so that

$$d_u \int_0^T \int_{-\infty}^0 (u_x^k)^2\, dxdt \;\leq\; C_1 + \frac{1}{2}\int_{-\infty}^1 (\hat{u}^k)^2(x,0)\, dx \;+\; M\int_0^T \int_{\mathbb{R}} kF(u^k,v^k)\, dxdt.$$

Again invoking Lemma 2.12, it then follows that C can be chosen larger if necessary, still independent of $d_u, k > 0$, so that $d_u \int_0^T \int_{-\infty}^0 (u_x^k)^2 \leq C/2$. If $d_v > 0$, the estimate for v_x^2 can be proved likewise, using the equation for v_k. $\square$

The following estimates for the differences of space and time translates of solutions will yield sufficient compactness both to obtain the existence of solutions of (P_1^k) when $d_u > 0$ and $d_v = 0$, and to study the strong-interaction limit $(k \to \infty)$ of (P_1^k). Here we want to allow $d_v = 0$ and do not have $L^2(Q_T)$ bounds for v_x^k in this case. Thus we cannot simply refer to [4, Lemma 2.6] to control the differences of space translates, as in the proof of Lemma 2.7, but instead need an alternative method. Our proof centres on showing that solutions (u^k, v^k) of (P_1^k) satisfy the L^1-contraction property (2.35). Note that the monotonicity properties of F are used here, in establishing the sign condition (2.33). See also [7, Prop. 4] and [10, Prop. 3] for some related arguments.

It is convenient to introduce a shorthand notation for space and time translates. Given a function h, let

$$S_\xi h(x,t) := h(x+\xi,t), \quad T_\tau h(x,t) := h(x,t+\tau), \tag{2.26}$$

for all (x,t) in a suitable space-time domain and appropriate ξ and τ.

Lemma 2.15. *Suppose that $d_u > 0$ and $d_v \geq 0$, and let (u^k, v^k) be a solution of (P_1^k) satisfying (2.8). Then there exists a function $G \geq 0$, independent of $d_v \geq 0$ and $k > 0$, such that $G(\xi) \to 0$ as $|\xi| \to 0$, and for all $t \in (0,T)$,*

$$\int_{\mathbb{R}} |u^k(x,t) - S_\xi u^k(x,t)| + |v^k(x,t) - S_\xi v^k(x,t)| \; dx \;\leq\; G(\xi). \tag{2.27}$$

Proof. Define

$$u := u^k - S_\xi u^k, \qquad v := v^k - S_\xi v^k, \qquad u_0 := u_0^k - S_\xi u_0^k, \qquad v_0 := v_0^k - S_\xi v_0^k, \tag{2.28}$$

so that

$$\begin{aligned} u_t \;&= d_u u_{xx} - k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} && \text{in } Q_T \\ v_t \;&= d_v v_{xx} - k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} && \text{in } Q_T \\ u(x,0) \;&= u_0^k(x) - u_0^k(x+\xi), \quad v(x,0) = v_0^k(x) - v_0^k(x+\xi) && \text{for} \quad x \in \mathbb{R}. \end{aligned}$$

Now given $L, \alpha > 0$, let ψ^L be the cut-off functions defined before Lemma 2.5, and let m_α be as defined in the proof of Lemma 2.13. Then multiplying the equation for u by $\psi^L m_\alpha'(u)$ and integrating over $\mathbb{R} \times (0,t_0)$, $t_0 \in (0,T)$, gives

$$\begin{aligned} \int_{\mathbb{R}} \psi^L(x) m_\alpha(u(x,t_0)) \; dx &= \int_{\mathbb{R}} \psi^L(x) m_\alpha(u_0(x)) \; dx - d_u \int_0^{t_0} \int_{\mathbb{R}} \psi^L m_\alpha''(u)(u_x)^2 \; dxdt \\ &\quad + d_u \int_0^{t_0} \int_{\mathbb{R}} \psi^L_{xx} m_\alpha(u) \; dxdt - \int_0^{t_0} \int_{\mathbb{R}} \psi^L m_\alpha'(u) k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} \; dxdt \\ &\leq \int_{\mathbb{R}} \psi^L(x) m_\alpha(u_0(x)) \; dx + d_u \int_0^{t_0} \int_{\mathbb{R}} \psi^L_{xx} m_\alpha(u) \; dxdt \\ &\quad - \int_0^{t_0} \int_{\mathbb{R}} \psi^L m_\alpha'(u) k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} \; dxdt, \end{aligned} \tag{2.29}$$

since $m_\alpha''(u) \geq 0$ because m_α is convex. Then letting $\alpha \to 0$ in (2.29) yields that for each $L > 0$ and each $t_0 \in (0,T)$,

$$\begin{aligned} \int_{\mathbb{R}} \psi^L(x) |u(x,t_0)| \; dx &\leq \int_{\mathbb{R}} \psi^L(x) |u_0(x)| \; dx + d_u \int_0^{t_0} \int_{\mathbb{R}} \psi^L_{xx}(x) |u(x,t)| \; dxdt \\ &\quad - \int_0^{t_0} \int_{\mathbb{R}} \psi^L(x) \operatorname{sgn}(u) k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} \; dxdt, \end{aligned} \tag{2.30}$$

and similarly,

$$\begin{aligned} \int_{\mathbb{R}} \psi^L(x) |v(x,t_0)| \; dx &\leq \int_{\mathbb{R}} \psi^L(x) |v_0(x)| \; dx + d_v \int_0^{t_0} \int_{\mathbb{R}} \psi^L_{xx}(x) |v(x,t)| \; dxdt \\ &\quad - \int_0^{t_0} \int_{\mathbb{R}} \psi^L(x) \operatorname{sgn}(v) k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} \; dxdt. \end{aligned} \tag{2.31}$$

Adding (2.30) and (2.31) then gives

$$\int_{\mathbb{R}} \psi^L(x)\{|u(x,t_0)|+|v(x,t_0)|\}\,dx \le \int_{\mathbb{R}} \psi^L(x)\{|u_0(x)|+|v_0(x)|\}\,dx$$

$$+\int_0^{t_0}\int_{\mathbb{R}} \psi^L_{xx}(x)\{d_u|u(x,t)|+d_v|v(x,t)|\}\,dxdt \tag{2.32}$$

$$-k\int_0^{t_0}\int_{\mathbb{R}} \psi^L(x)\{\mathrm{sgn}(u)+\mathrm{sgn}(v)\}\{F(u^k,v^k)-F(S_\xi u^k,S_\xi v^k)\}\,dxdt.$$

Then since $F(\cdot,v)$ and $F(u,\cdot)$ are non-decreasing for all $u,v\in\mathbb{R}^+$, we have

$$(\mathrm{sgn}(u)+\mathrm{sgn}(v))\{F(u^k,v^k)-F(S_\xi u^k,S_\xi v^k)\}\ge 0, \tag{2.33}$$

because either $\mathrm{sgn}(u)+\mathrm{sgn}(v)=0$ or else $\mathrm{sgn}(u)+\mathrm{sgn}(v)$ and $F(u^k,v^k)-F(S_\xi u^k,S_\xi v^k)$ have the same sign, and hence

$$\int_{\mathbb{R}} \psi^L(x)\{|u(x,t_0)|+|v(x,t_0)|\}\,dx \le \int_{\mathbb{R}} \psi^L(x)\{|u_0(x)|+|v_0(x)|\}\,dx \tag{2.34}$$

$$+\int_0^{t_0}\int_{\mathbb{R}} \psi^L_{xx}(x)\{d_u|u(x,t)|+d_v|v(x,t)|\}\,dxdt.$$

Now by Lemma 2.13, $u(\cdot,t)-u_0^\infty, v(\cdot,t)-v_0^\infty\in L^1(\mathbb{R})$ for each $t\in(0,T)$. Hence we can let $L\to\infty$ in (2.34) and thus obtain that for each $t_0\in(0,T)$,

$$\int_{\mathbb{R}}\{|u^k(x,t_0)-u^k(x+\xi,t_0)|+|v^k(x,t_0)-v^k(x+\xi,t_0)|\}\,dx \tag{2.35}$$

$$\le \int_{\mathbb{R}}\{|u_0^k(x)-u_0^k(x+\xi)|+|v_0^k(x)-v_0^k(x+\xi)|\}\,dx.$$

The existence of G is then immediate from the assumption that $\|u_0^k(\cdot+\xi)-u_0^k(\cdot)\|_{L^1(\mathbb{R})}+\|v_0^k(\cdot+\xi)-v_0^k(\cdot)\|_{L^1(\mathbb{R})}\le\omega(|\xi|)$ where $\omega(|\xi|)\to 0$ as $\xi\to 0$.

□

Lemma 2.16. *Suppose that $d_u>0$, $d_v\ge 0$ and let (u^k,v^k) be a solution of (P_1^k) satisfying (2.8). Then there exists $C>0$, independent of d_v and k, such that for any $\tau\in(0,T)$,*

$$\int_0^{T-\tau}\int_{\mathbb{R}}|T_\tau u^k(x,t)-u^k(x,t)|^2\,dxdt \le \tau C,$$

$$\int_0^{T-\tau}\int_{\mathbb{R}}|T_\tau v^k(x,t)-v^k(x,t)|^2\,dxdt \le \tau C.$$

Proof. The proof is similar to that of [7, Lemma 3]; see also [4, Lemma 2.6]. We sketch the key points here, focussing on the parts where our problem needs a slightly different argument. Let ψ^L

be as defined before Lemma 2.5. Then it follows using the equation for u^k that

$$\begin{aligned}
&\int_0^{T-\tau}\int_{\mathbb{R}}[T_\tau u^k(x,t)-u^k(x,t)]^2\psi^L\,dxdt\\
&= \int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}}(u^k(x,t+\tau)-u^k(x,t))[d_u u^k_{xx}(x,t+s)-kF(u^k,v^k)]\psi^L\,dxdtds\\
&= -\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}} d_u[u^k(x,t+\tau)-u^k(x,t)]u_x(x,t+s)\psi^L_x\,dxdtds\\
&\quad -\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}} d_u[u^k(x,t+\tau)-u^k(x,t)]_x u_x(x,t+s)\psi^L\,dxdtds\\
&\quad -\int_0^{\tau}\int_0^{T-\tau}\int_{\mathbb{R}}[u^k(x,t+\tau)-u^k(x,t)]kF(u^k,v^k)\,\psi^L\,dxdtds\\
&\le \big(\sup|\psi^L_x|\big)\int_0^{\tau}\int_0^{T-\tau}\int_{L\le|x|\le L+1} d_u|u^k(x,t+\tau)-u^k(x,t)||u_x(x,t+s)|\,dxdtds\\
&\quad + 2d_u\tau\int_0^T\int_{\mathbb{R}}(u^k_x)^2(x,t)\,dxdt\;+\;2M\tau\int_0^T\int_{\mathbb{R}}kF(u^k,v^k)\,dxdt. \qquad (2.36)
\end{aligned}$$

Now the mapping $(x,t)\mapsto|u^k(x,t+\tau)-u^k(x,t)||u_x(x,t+s)|$ is integrable on $\mathbb{R}\times(0,T-\tau)$, by (2.8), Lemma 2.13 and Lemma 2.14, and $\big(\sup|\psi^L_x|\big)$ is bounded independently of L. So the first term on the right-hand side of (2.36) tends to 0 as $L\to\infty$. Thus letting $L\to\infty$ yields

$$\int_0^{T-\tau}\int_{\mathbb{R}}[u^k(x,t+\tau)-u^k(x,t)]^2\,dxdt\le 2d_u\tau\int_0^T\int_{\mathbb{R}}(u^k_x)^2(x,t)\,dxdt\,+\,2M\tau\int_0^T\int_{\mathbb{R}}kF(u^k,v^k)\,dxdt,$$

from which the estimate for u^k follows using Lemma 2.12 and Lemma 2.14. When $d_v>0$, the estimate for v^k follows likewise, using the equation for v^k. When $d_v=0$, a similar but simpler argument applies, omitting the terms deriving from v^k_{xx}. □

We can now prove a convergence result for solutions (u^k,v^k) of (P_1^k) as $d_v\to 0$.

Lemma 2.17. *Let $k>0$ and $d_u>0$ be fixed and $(u^k_{d_v},v^k_{d_v})$ be solutions of (P_1^k) satisfying (2.8) with $d_v>0$. Then there exists $(u^k_*,v^k_*)\in(L^\infty(Q_T))^2$ such that up to a subsequence, for each $J>0$,*

$$\begin{aligned}
u^k_{d_v}&\to u^k_* && \text{in } L^2((-J,J)\times(0,T)),\\
v^k_{d_v}&\to v^k_* && \text{in } L^2((-J,J)\times(0,T)),\\
u^k_{d_v}-\tilde u&\rightharpoonup u^k_*-\tilde u && \text{in } L^2(0,T;H^1(\mathbb{R})),
\end{aligned}$$

as $d_v\to 0$, where $\tilde u\in C^\infty(\mathbb{R})$ is a smooth function such that $\tilde u(x)=u_0^\infty(x)$ for all $|x|\ge 1$.

Proof. It follows from Lemma 2.13 and (2.8) that $\|u^k_{d_v}-u_0^\infty\|_{L^2(Q_T)}$ and $\|v^k_{d_v}-v_0^\infty\|_{L^2(Q_T)}$ are bounded independently of d_v. So Lemmas 2.15, 2.16 and the Riesz-Fréchet-Kolmogorov Theorem [3, Theorem 4.26] yield that the sets $\{u^k_{d_v}-u_0^\infty\}_{d_v>0}$ and $\{v^k_{d_v}-v_0^\infty\}_{d_v>0}$ are each relatively compact in $L^2((-J,J)\times(0,T))$ for each $J>0$. The weak convergence of $u^k_{d_v}-\tilde u$ in $L^2(0,T;H^1(\mathbb{R}))$ follows from the fact that $\|u^k_{d_v}-u_0^\infty\|_{L^2(Q_T)}$ is bounded independently of d_v together with Lemma 2.14. □

Theorem 2.18. *Let $d_v=0$ and $k>0$. Then problem (P_1^k) has a unique weak solution*

$$(u^k,v^k)\in W^{2,1}_p((-J,J)\times(0,T))\times W^{1,\infty}(0,T;L^\infty((-J,J)))\ \text{ for each } J>0,\ \ p\ge 1,$$

where (u^k, v^k) *is a weak solution in the sense that*

$$(2.37)\qquad \iint_{Q_T} u^k \psi_t \; dxdt + \iint_{Q_T} \{d_u u^k \psi_{xx} - kF(u^k, v^k)\psi\} \; dxdt \;=\; -\int_{\mathbb{R}} u_0^k \psi(\cdot, 0) \; dx,$$

$$(2.38)\qquad \iint_{Q_T} v^k \psi_t \; dxdt - \iint_{Q_T} kF(u^k, v^k)\psi \; dxdt \;=\; -\int_{\mathbb{R}} v_0^k \psi(\cdot, 0) \; dx,$$

for all $\psi \in \mathcal{F}_T = \{\psi \in C^{2,1}(Q_T) : \psi(\cdot, T) = 0$ *and* $supp\,\psi \subset [-J, J] \times [0, T]$ *for some* $J > 0\}$*, and also satisfies* $0 \le u^k, v^k \le M$*.*

Proof. Multiplying (P_1^k) by $\psi \in \mathcal{F}_T$ and integrating over Q_T yields that for each $d_v > 0$, solutions $(u_{d_v}^k, v_{d_v}^k)$ of (P_1^k) satisfy

$$(2.39)\qquad \iint_{Q_T} u_{d_v}^k \psi_t \; dxdt + \iint_{Q_T} \{d_u u_{d_v}^k \psi_{xx} - kF(u_{d_v}^k, v_{d_v}^k)\psi\} \; dxdt \;=\; -\int_{\mathbb{R}} u_0^k \psi(\cdot, 0) \; dx,$$

$$(2.40)\qquad \iint_{Q_T} v_{d_v}^k \psi_t \; dxdt + \iint_{Q_T} \{d_v v_{d_v}^k \psi_{xx} - kF(u_{d_v}^k, v_{d_v}^k)\psi\} \; dxdt \;=\; -\int_{\mathbb{R}} v_0^k \psi(\cdot, 0) \; dx.$$

Then the existence of a solution (u^k, v^k) to (2.37)-(2.38) follows by using Lemma 2.17 to pass to the limit along a subsequence as $d_v \to 0$ in (2.39)-(2.40). The regularity of u^k follows from the fact that solutions $(u_{d_v}^k, v_{d_v}^k)$ of (P_1^k) satisfying (2.8) are such that $(u_{d_v}^k)_t - d_u(u_{d_v}^k)_{xx} = -kF(u_{d_v}^k, v_{d_v}^k)$ is bounded in $L^\infty(Q_T)$ independently of d_v, which, since $u_0^k \in C^2(\mathbb{R})$, implies that $u_{d_v}^k$ is bounded independently of $d_v > 0$ in $W_p^{2,1}((-J, J) \times (0, T))$ for each $J > 0$ and $p \ge 1$. The regularity of v^k is immediate from the fact that (2.38) implies that $v^k \in W^{1,\infty}(0, T; L^\infty((-J, J)))$ for each $J > 0$, and the uniqueness of (u^k, v^k) follows from the comparison principle in Lemma 2.10. □

2.3. **The limit problem for** (P_1^k) **as** $k \to \infty$**.** The a priori estimates of the previous section yield sufficient compactness to establish the existence of limits of solutions of (P_1^k) as $k \to \infty$, both when $d_v > 0$ and when $d_v = 0$. The proof of the following result is directly analogous to that of Lemma 2.17, using bounds independent of k in place of bounds independent of d_v, and is left to the reader.

Lemma 2.19. *Let* $d_u > 0$ *and* $d_v \ge 0$ *be fixed and* (u^k, v^k) *be solutions of* (P_1^k) *satisfying (2.8) with* $k > 0$*. Then there exists* $(u, v) \in (L^\infty(Q_T))^2$ *such that up to a subsequence, for each* $J > 0$*,*

$$\begin{aligned} u^k &\to u && \text{in } L^2((-J, J) \times (0, T)),\\ v^k &\to v && \text{in } L^2((-J, J) \times (0, T)),\\ u^k - \tilde{u} &\rightharpoonup u - \tilde{u} && \text{in } L^2(0, T; H^1(\mathbb{R})), \end{aligned}$$

as $k \to \infty$*, where* $\tilde{u} \in C^\infty(\mathbb{R})$ *is a smooth function such that* $\tilde{u}(x) = u_0^\infty(x)$ *for all* $|x| \ge 1$*.*

The following segregation result is a key to the characterisation of the limits u, v in Lemma 2.19.

Lemma 2.20. *Let* $d_u > 0$*,* $d_v \ge 0$ *and* (u, v) *be as in Lemma 2.19. Then*

$$(2.41)\qquad uv = 0 \quad \text{a.e. in } Q_T.$$

Proof. It follows from Lemmas 2.12 and 2.19 that $F(u, v) = 0$ almost everywhere in Q_T, from which (2.41) follows since $F(u, v) = 0$ if and only if $u = 0$ or $v = 0$. □

To derive the limit problem, set

$$(2.42)\qquad w^k := u^k - v^k, \quad w := u - v.$$

Then it follows from Lemmas 2.19 and 2.20 that as a sequence $k_n \to \infty$,

$$w^{k_n} \to w \ \text{ in } \ L^2((-J,J)\times(0,T)) \ \text{ for all} : J > 0 \ \text{ and } a.e. \text{ in } Q_T,$$

and that

$$u = w^+ \ \text{ and } \ v = -w^-,$$

where $s^+ = \max\{0, s\}$ and $s^- = \min\{0, s\}$. Next note the following equality.

Lemma 2.21. *Let $d_u > 0$, $d_v \geq 0$ and (u, v) be as in Lemma 2.19. Then*

$$-\iint_{Q_T} (u - v)\psi_t \; dxdt - \int_{\mathbb{R}} (u_0^\infty - v_0^\infty)\, \psi(x, 0) \; dx = \iint_{Q_T} (d_u u - d_v v)\; \psi_{xx} \; dxdt, \tag{2.43}$$

for all $\psi \in \mathcal{F}_T = \{\psi \in C^{2,1}(Q_T) : \psi(\cdot, T) = 0$ and $\operatorname{supp}\psi \subset [-J, J] \times [0, T]$ for some $J > 0\}$.

Proof. Multiplying the difference between the equations for u^k and v^k by $\psi \in \mathcal{F}_T$ and integrating over Q_T yields

$$-\iint_{Q_T} (u^k - v^k)\psi_t \; dxdt - \int_{\mathbb{R}} (u_0^k - v_0^k)\, \psi(x, 0) \; dx = \iint_{Q_T} (d_u u^k - d_v v^k)\; \psi_{xx} \; dxdt,$$

from which (2.43) follows using Lemma 2.19 and the fact that $u_0^k \to u_0^\infty$ and $v_0^k \to v_0^\infty$ in $L^1(\mathbb{R})$ as $k \to \infty$. □

Now define

$$\mathcal{D}(s) := \begin{cases} d_u s & \text{if } s \geq 0, \\ d_v s & \text{if } s < 0, \end{cases} \tag{2.44}$$

and the limit problem

$$(P_1^{limit}) \begin{cases} w_t & = \mathcal{D}(w)_{xx}, & \text{in } \mathbb{R} \times [0, \infty), \\ w(x, 0) & = w_0(x) := \begin{cases} U_0 & \text{if } x < 0, \\ -V_0, & \text{if } x > 0. \end{cases} \end{cases}$$

Definition 2.22. *A function w is a weak solution of Problem (P_1^{limit}) if*

(i) $w \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$,

(ii) *for all $T > 0$,*

$$\iint_{Q_T} (w\psi_t + \mathcal{D}(w)\psi_{xx}) \; dxdt = -\int_{\mathbb{R}} w_0(x)\psi(x, 0) \; dx,$$

for all $\psi \in \mathcal{F}_T$

$= \{\psi \in C^{2,1}(Q_T) : \psi(\cdot, T) = 0$ and $\operatorname{supp}\psi \subset [-J, J] \times [0, T]$ for some $J > 0\}$.

Lemma 2.23. *The function w defined in equation (2.42) is the unique weak solution of Problem (P_1^{limit}) and the whole sequence (u^k, v^k) in Lemma 2.19 converges to $(w^+, -w^-)$.*

Proof. That w is a weak solution of (P_1^{limit}) follows immediately from Lemma 2.21 and the definition of $\mathcal{D}$. The uniqueness is a consequence of [2, Appendix, Proposition A], which extends the method of [1, Proposition 9] to unbounded domains, via exactly the argument used to establish uniqueness for a similar problem in [2, Appendix, Proof of Theorem C]. Note that although it is assumed throughout [2, Appendix] that the initial data of the problems considered is continuous, it is straightforward to verify that this is not in fact necessary for the proofs. □

We next identify the limit w as a certain self-similar solution of Problem (P_1^{limit}), the precise form of which depends on whether $d_v > 0$ or $d_v = 0$. To this end, we first state a free-boundary problem, including interface conditions, that is satisfied by the solution w of (P_1^{limit}) under some regularity assumptions and conditions on the form of the free boundary. The proof follows immediately from that of [10, Theorem 5] and we omit it.

Theorem 2.24. *Let w be the unique weak solution of Problem (P_1^{limit}). Suppose that there exists a function $\xi : [0, T] \to \mathbb{R}$ such that for each $t \in [0, T]$,*

$$w(x,t) > 0 \quad \textit{if} \quad x < \xi(t) \quad \textit{and} \quad w(x,t) < 0 \quad \textit{if} \quad x > \xi(t).$$

Then if $t \mapsto \xi(t)$ is sufficiently smooth and the functions $u := w^+$ and $v := -w^-$ are smooth up to $\xi(t)$, the functions u and v satisfy

$$(P_1^{limit}) \begin{cases} u_t = d_u u_{xx}, & \textit{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v_t = d_v v_{xx}, & \textit{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ [u] = d_v\,[v] = 0, & \textit{on } \Gamma_T := \{(x,t) \in Q_T : x = \xi(t)\}, \\ [v]\,\xi'(t) = [d_u u_x - d_v v_x]\,, & \textit{on } \Gamma_T := \{(x,t) \in Q_T : x = \xi(t)\}, \\ u(\cdot,0) = u_0^\infty(\cdot), & \textit{in } \mathbb{R}, \\ v(\cdot,0) = v_0^\infty(\cdot), & \textit{in } \mathbb{R}, \end{cases}$$

where $[\cdot]$ denotes the jump across $\xi(t)$ from $\{x < \xi(t)\}$ to $\{x > \xi(t)\}$, that is, $[a] := \lim_{x\downarrow\xi(t)} a(x,t) - \lim_{x\uparrow\xi(t)} a(x,t)$, $\xi'(t)$ denotes the speed of propagation of the free boundary $\xi(t)$.

Interpreting the interface conditions on Γ_T then yields the following two limit problems.

Corollary 2.25. *Let w and $\xi : [0, T] \to \mathbb{R}$ satisfy the hypotheses of Theorem 2.24. Then the functions $u := w^+$ and $v = -w^-$ satisfy one of two limit problems, depending on whether $d_v > 0$ or $d_v = 0$. If $d_v > 0$, then*

$$(P_{1,d_v>0}^{limit}) \begin{cases} u_t = d_u u_{xx}, & \textit{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = 0, & \textit{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v_t = d_v v_{xx}, & \textit{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ u = 0, & \textit{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ \lim_{x\uparrow\xi(t)} u(x,t) = 0 = \lim_{x\downarrow\xi(t)} v(x,t), & \textit{for each } t \in [0,T], \\ d_u \lim_{x\uparrow\xi(t)} u_x(x,t) = -d_v \lim_{x\downarrow\xi(t)} v_x(x,t), & \textit{for each } t \in [0,T], \\ u(\cdot,0) = u_0^\infty(\cdot), & \textit{in } \mathbb{R}, \\ v(\cdot,0) = v_0^\infty(\cdot), & \textit{in } \mathbb{R}, \end{cases}$$

whereas if $d_v = 0$ and we suppose additionally that $\xi(0) = 0$ and $t \mapsto \xi(t)$ is a non-decreasing function, then

$$(P_{1,d_v=0}^{limit}) \begin{cases} u_t = d_u u_{xx}, & \textit{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = 0, & \textit{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = V_0, & \textit{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ u = 0, & \textit{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ \lim_{x\uparrow\xi(t)} u(x,t) = 0, & \textit{for each } t \in [0,T], \\ V_0\ \xi'(t) = -d_u \lim_{x\uparrow\xi(t)} u_x(x,t), & \textit{for each } t \in [0,T], \\ u(\cdot,0) = u_0^\infty(\cdot), & \textit{in } \mathbb{R}, \\ v(\cdot,0) = v_0^\infty(\cdot), & \textit{in } \mathbb{R}, \end{cases}$$

where $\xi'(t)$ denotes the speed of propagation of the free boundary $\xi(t)$.

Proof. We interpret the meaning of the interface conditions in Theorem 2.24, depending on whether $d_v > 0$ or $d_v = 0$. The condition

$$[u] = d_v\,[v] = 0 \quad \text{on} \quad \Gamma_T := \{(x,t) \in Q_T : x = \xi(t)\},$$

implies that $u(\cdot,t)$ is continuous across $\xi(t)$, so that

$$\lim_{x\uparrow\xi(t)} u(x,t) = \lim_{x\downarrow\xi(t)} u(x,t) = 0.$$

Moreover, if $d_v > 0$, then $v(\cdot,t)$ is also continuous across $\xi(t)$, and so

$$\lim_{x\downarrow\xi(t)} v(x,t) = \lim_{x\uparrow\xi(t)} v(x,t) = 0,$$

whereas if $d_v = 0$, $v(\cdot,t)$ may jump across $\xi(t)$. Indeed, since $\xi(0) = 0$ and $t \mapsto \xi(t)$ is a non-decreasing function, it follows from the fact that $v_t = 0$ in $\{(x,t) \in Q_T : x > \xi(t)\}$ if $d_v = 0$, together with the initial condition that $v_0(x) = V_0$ if $x > 0$, that $v(x,t) \equiv V_0$ for all $x \geq \xi(t)$, and thus

$$[v] = V_0 - 0 = V_0 \quad \text{for all} \;\; t \in [0,T].$$

The normal derivative condition

$$[v]\,\xi'(t) = [d_u u_x - d_v v_x] \quad \text{on} \;\; \Gamma_T := \{(x,t) \in Q_T : x = \xi(t)\},$$

implies that if $d_v > 0$, then $0 = [d_u u_x - d_v v_x]$, which says that

$$d_u \lim_{x\uparrow\xi(t)} u_x(x,t) = -d_v \lim_{x\downarrow\xi(t)} v_x(x,t),$$

or equivalently,

$$d_u \lim_{x\uparrow\xi(t)} w_x^+(x,t) = d_v \lim_{x\downarrow\xi(t)} w_x^-(x,t).$$

On the other hand, if $d_v = 0$, then

$$\lim_{x\downarrow\xi(t)} v(x,t)\;\; \xi'(t) = -d_u \lim_{x\uparrow\xi(t)} u_x(x,t),$$

which in the case that $\xi(0) = 0$ and $t \mapsto \xi(t)$ is a non-decreasing function gives

$$V_0\,\xi'(t) = -d_u \lim_{x\uparrow\xi(t)} u_x(x,t).$$

□

It is then easy to show that the limit problems in Corollary 2.25 admit self-similar solutions.

Theorem 2.26. *The unique weak solution w of Problem (P_1^{limit}) has a self-similar form. There exists a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $a \in \mathbb{R}$ such that*

$$w(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \quad (x,t) \in Q_T, \quad \text{and} \quad \xi(t) = a\sqrt{t}, \quad t \in [0,T]. \tag{2.45}$$

If $d_v > 0$, then $a \in \mathbb{R}$ is the unique root of the equation

$$d_u U_0 \int_a^{\infty} e^{\frac{a^2-s^2}{4d_v}}\,ds = d_v V_0 \int_{-\infty}^{a} e^{\frac{a^2-s^2}{4d_u}}\,ds,$$

and

$$f(\eta) = \begin{cases} U_0\left(1 - \dfrac{\int_{-\infty}^{\eta} e^{-\frac{s^2}{4d_u}}\,ds}{\int_{-\infty}^{a} e^{-\frac{s^2}{4d_u}}\,ds}\right), & \text{if } \eta \leq a, \\ -V_0\left(1 - \dfrac{\int_{\eta}^{\infty} e^{-\frac{s^2}{4d_v}}\,ds}{\int_{a}^{\infty} e^{-\frac{s^2}{4d_v}}\,ds}\right), & \text{if } \eta > a. \end{cases} \tag{2.46}$$

On the other hand, if $d_v = 0$, then $a > 0$ is the unique root of the equation

$$U_0 = \frac{V_0 a}{2d_u} \int_{-\infty}^{a} e^{\frac{a^2-s^2}{4d_u}}\, ds,$$

and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_{-\infty}^{\eta} e^{-\frac{s^2}{4d_u}}\, ds}{\int_{-\infty}^{a} e^{-\frac{s^2}{4d_u}}\, ds} \right), & \text{if } \eta \le a, \\ -V_0, & \text{if } \eta > a. \end{cases} \tag{2.47}$$

Proof. Straightforward verification shows that the functions w defined in (2.45) and (2.46) or (2.47) satisfy $(P^{limit}_{1,d_v>0})$ or $(P^{limit}_{1,d_v=0})$ when $d_v > 0$ or $d_v = 0$ respectively, and hence give a solution of the problem (P^{limit}), which must therefore be the unique solution. □

As already mentioned in the Introduction, the constant a is not necessarily positive in the case when $d_v > 0$. Some sufficient conditions ensuring the sign of a are as follows.

Proposition 2.27. *Suppose that $d_u, d_v, U_0, V_0 \in \mathbb{R}$ are all strictly positive, and let $a \in \mathbb{R}$ be the unique root of the equation*

$$d_u U_0 \int_{a}^{\infty} e^{\frac{a^2-s^2}{4d_v}}\, ds = d_v V_0 \int_{-\infty}^{a} e^{\frac{a^2-s^2}{4d_u}}\, ds. \tag{2.48}$$

Then

(i) *if $d_u = d_v$ and $U_0 = V_0$, then $a = 0$;*
(ii) *if $d_u \le d_v$ and $\sqrt{d_u} U_0 \le \sqrt{d_v} V_0$, then $a < 0$;*
(iii) *if $d_u \ge d_v$ and $\sqrt{d_u} U_0 \ge \sqrt{d_v} V_0$, then $a > 0$.*

Proof. A straightforward rearrangement of (2.48) gives

$$\frac{\sqrt{d_u}\, U_0}{\sqrt{d_v}\, V_0}\, e^{\frac{a^2}{2}\left(\frac{1}{d_v} - \frac{1}{d_u}\right)} \int_{\frac{a}{2\sqrt{d_v}}}^{\infty} e^{-t^2}\, dt \;=\; \int_{-\infty}^{\frac{a}{2\sqrt{d_u}}} e^{-t^2}\, dt,$$

from which the result is clear. □

3. The half-line case: problem (P_2^k)

3.1. **Existence and uniqueness of solutions of (P_2^k) when $d_u > 0$ and $d_v > 0$.** Suppose that $d_u > 0$ and $d_v > 0$. Similarly to the whole-line case, we can use an approximate problem to establish existence of solutions of (P_2^k). Choose $M \ge \max\{U_0, V_0\}$ and for each $R > 1$, let $(P_2^{R,\mu})$ denote the problem

$$(P_2^{R,\mu}) \begin{cases} u_t &= d_u u_{xx} - kF_\mu(u,v) & \text{in } (0,R)\times(0,T), \\ v_t &= d_v v_{xx} - kF_\mu(u,v) & \text{in } (0,R)\times(0,T), \\ u(0,t) &= U_0 & \text{for } \ t \in (0,T), \\ u_x(R,t) &= 0 & \text{for } \ t \in (0,T), \\ v_x(0,t) &= 0 & \text{for } \ t \in (0,T), \\ v_x(R,t) &= 0 & \text{for } \ t \in (0,T), \\ u(x,0) &= u^k_{0,R}(x), \quad v(x,0) = v^k_{0,R}(x) & \text{for } \ x \in (0,R), \end{cases}$$

where $u_{0,R}^k, v_{0,R}^k \in C^2(\mathbb{R}^+)$ are such that $0 \le u_{0,R}^k \le M$, $0 \le v_{0,R}^k \le M$ and

$$u_{0,R}^k(x) = 0 \text{ for } x > \left(1 - \frac{1}{R}\right) R, \qquad v_{0,R}^k(x) = V_0 \text{ for } x > \left(1 - \frac{1}{R}\right) R, \tag{3.1}$$

which defines the functions $u_{0,R}^k$, $v_{0,R}^k$ on the half-line $(0, \infty)$. The regularisation F_μ is as defined in section 2.1.

Arguments analogous to those used in Section 2.1 yield existence of solutions to problem (P_2^k) by passing to the limits $R \to \infty$ and $\mu \to 0$ in problem $(P_2^{R,\mu})$. We omit repetition of the details the proof and simply state the result.

Theorem 3.1. *Suppose that $d_u > 0$ and $d_v > 0$. Then given $k > 0$, there exists a classical solution (u^k, v^k) of (P_2^k), such that for each $\delta > 0$, $J > 0$ and $p \ge 1$,*

$$u^k, v^k \in C^{2,1}(\mathbb{R}^+ \times [\delta, T]) \cap C^0(\mathbb{R}^+ \times [0, T]) \cap W_p^{2,1}((0, J) \times (0, T)), \tag{3.2}$$

and

$$0 \le u^k \le M, \quad 0 \le v^k \le M \quad on \quad \mathbb{R}^+ \times (0, T). \tag{3.3}$$

Uniqueness is again a consequence of a comparison theorem.

Lemma 3.2. *Suppose that $d_u > 0$, $d_v \ge 0$, and let $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ be such that for each $J > 0$ and $p \ge 1$, $\overline{u}, \underline{u} \in L^\infty(S_T) \cap W_p^{2,1}((0, J) \times (0, T))$, $\overline{v}, \underline{v} \in L^\infty(S_T) \cap W_p^{2,1}((0, J) \times (0, T))$ if $d_v > 0$, $\overline{v}, \underline{v} \in L^\infty(S_T) \cap W^{1,\infty}(0, T; L^\infty((-J, J)))$ if $d_v = 0$, and $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v})$ satisfy*

$$\begin{array}{ll} \overline{u}_t \ge d_u \overline{u}_{xx} - kF(\overline{u}, \overline{v}), & \underline{u}_t \le d_u \underline{u}_{xx} - kF(\underline{u}, \underline{v}), \\ \overline{v}_t \le d_v \overline{v}_{xx} - kF(\overline{u}, \overline{v}), & \underline{v}_t \ge d_v \underline{v}_{xx} - kF(\underline{u}, \underline{v}), \end{array} \quad in \ \ S_T,$$

$$\overline{u}(0, \cdot) \ge \underline{u}(0, \cdot), \qquad d_v \overline{v}_x(0, \cdot) \ge d_v \underline{v}_x(0, \cdot) \ \ on \ \ (0, T],$$

and

$$\overline{u}(\cdot, 0) \ge \underline{u}(\cdot, 0), \qquad \overline{v}(\cdot, 0) \le \underline{v}(\cdot, 0) \ \ on \ \mathbb{R}^+.$$

Then

$$\overline{u} \ge \underline{u} \quad and \quad \overline{v} \le \underline{v} \quad in \ \ S_T.$$

Proof. This follows from the same form of argument used to show Lemma 2.10, replacing Q_T with S_T, integrals over $\mathbb{R}$ with integrals over $\mathbb{R}^+$, and the cut-off function ψ^L by $\psi_+^L := \psi^L|_{\mathbb{R}^+}$. We omit most of the details and only note two key calculations involving the boundary $\{0\} \times (0, T)$. Taking $u := \underline{u} - \overline{u}$ and $v := \underline{v} - \overline{v}$, we have

$$u(0, \cdot) \le 0 \quad \text{and} \quad d_v v_x(0, \cdot) \le 0 \ \text{ on } \ (0, T].$$

Thus $(m_\alpha^+)'(u(0, \cdot)) = 0$, so that integrating over $\mathbb{R}^+ \times (0, t_0)$, $t_0 \in (0, T]$, gives

$$\int_0^{t_0} \int_{\mathbb{R}^+} u_{xx} (m_\alpha^+)'(u) \psi_+^L \, dxdt =$$
$$- \int_0^{t_0} \int_{\mathbb{R}^+} u_x \left[(m_\alpha^+)''(u) u_x \psi_+^L + (m_\alpha^+)'(u) \left(\psi_+^L\right)_x \right] \, dxdt \; \le \; \int_0^{t_0} \int_{\mathbb{R}^+} m_\alpha^+(u) \left(\psi_+^L\right)_{xx} \, dxdt,$$

whereas if $d_v > 0$, then

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^+} v_{xx}(m_\alpha^+)'(v)\psi_+^L\,dxdt \\
= &\int_0^{t_0} v_x(0,t)(m_\alpha^+)'(v(0,t))\,dt - \int_0^{t_0}\int_{\mathbb{R}^+} v_x\left[(m_\alpha^+)''(v)v_x\psi_+^L + (m_\alpha^+)'(v)\left(\psi_+^L\right)_x\right]\,dxdt \\
= &\int_0^{t_0} v_x(0,t)(m_\alpha^+)'(v(0,t))\,dt - \int_0^{t_0}\int_{\mathbb{R}^+}(m_\alpha^+)''(v)(v_x)^2\psi_+^L\,dxdt + \int_0^{t_0}\int_{\mathbb{R}^+} m_\alpha^+(v)\left(\psi_+^L\right)_{xx}\,dxdt \\
\leq &\int_0^{t_0}\int_{\mathbb{R}^+} m_\alpha^+(v)\left(\psi_+^L\right)_{xx}\,dxdt.
\end{aligned}$$

We refer the reader to Lemma 2.10 for the remainder of the proof. □

Corollary 3.3. *Suppose $d_u > 0$ and $d_v > 0$. Then given $k > 0$, there is at most one solution (u^k, v^k) of (P_2^k) in $L^\infty(S_T) \cap W_p^{2,1}((0,J)\times(0,T))$ for each $J > 0$, $p \geq 1$.*

3.2. **Existence and uniqueness of solutions for (P_2^k) when $d_u > 0$ and $d_v = 0$.** Again we begin with some preliminary estimates, counterparts of results in section 2.2. Here some different arguments are needed because of the boundary condition at $x = 0$.

The following key bound is the half-line counterpart of Lemma 2.12.

Lemma 3.4. *There exists a constant $C > 0$, independent of $d_v \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (P_2^k) satisfying (3.3), we have*

$$\int_0^T\int_0^\infty kF(u^k, v^k)\,dx\,dt \leq C.$$

Proof. Define a cut-off function $\beta \in C^\infty(\mathbb{R}^+)$ such that $0 \leq \beta(x) \leq 1$ for all $x \in \mathbb{R}^+$, $\beta^L(0) = \beta_x^L(0) = 0$, $\beta(x) = 1$ for all $x \in [1,2]$, and $\beta(x) = 0$ for $x \geq 3$. Then given $L \geq 2$, define the family of cut-off functions $\beta^L \in C^\infty(\mathbb{R})$ by $\beta^L(x) = \beta(x)$ when $x \in [0,1]$, $\beta^L(x) = 1$ when $x \in [1, L]$, and $\beta^L(x) = \beta(x - L + 2)$ when $x \geq L$. Note that $0 \leq \beta^L \leq 1$ for all L, and β_x^L, β_{xx}^L are bounded in both $L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)$ independently of L. Also let $\hat\beta \in C^\infty(\mathbb{R}^+)$ be such that $0 \leq \hat\beta \leq 1$, $\hat\beta(x) = 1$ for all $x \in [0,1]$ and $\hat\beta(x) = 0$ for all $x \geq 2$.

Then multiplying the equation for u^k by β^L and integrating over $\mathbb{R}^+ \times (0, t_0)$, $t_0 \in (0, T]$, gives that

$$\begin{aligned}
&\int_{\mathbb{R}^+}\beta^L(x)u^k(x,t_0)\,dx + \int_0^{t_0}\int_{\mathbb{R}^+}\beta^L(x)kF(u^k,v^k)\,dxdt = \\
&\qquad d_u\int_0^{t_0}\int_{\mathbb{R}^+}\beta_{xx}^L(x)u^k(x,t)\,dxdt + \int_{\mathbb{R}^+}\beta^L(x)u_0^k(x)\,dx,
\end{aligned} \tag{3.4}$$

from which, since $u^k \geq 0$, it follows that

$$k\int_0^T\int_1^L F(u^k, v^k)\,dxdt \tag{3.5}$$

is bounded independently of L, $k > 0$, since (3.3), the definition of β^L, and the fact that $\|u_0^k\|_{L^1(\mathbb{R}^+)}$ is bounded independently of k imply that the right-hand side of (3.4) is bounded independently of k and L. On the other hand, multiplying the equation for v^k by $\hat\beta$ and integrating over $\mathbb{R}^+ \times (0, t_0)$ yields

$$k\int_0^{t_0}\int_{\mathbb{R}^+}\hat\beta F(u^k, v^k)\,dxdt = d_v\int_0^{t_0}\int_{\mathbb{R}^+}\hat\beta_{xx}v^k\,dxdt - \int_0^2\hat\beta[v^k(x,T) - v_0^k(x)]\,dx,$$

which, together with (3.3), implies that

$$k \int_0^T \int_0^1 F(u^k, v^k)\ dxdt, \tag{3.6}$$

is bounded independently of $k > 0$ and of $d_v \geq 0$ sufficiently small. The result then follows from (3.5), (3.6), and Lebesgue's monotone convergence theorem. ☐

Lemma 3.5. *There exists a constant $C > 0$, independent of $d_v \geq 0$ and $k > 0$, such that for any solution (u^k, v^k) of (P_2^k) satisfying (3.3), we have*

$$\int_0^\infty u^k(x, t_0)\, dx \ \leq\ C \ \ \textit{and} \ \ \int_0^\infty |V_0 - v^k(x, t_0)|\ dx \leq C \quad \textit{for all}\ \ t_0 \in [0, T]. \tag{3.7}$$

Proof. The estimate for u^k is immediate from (3.3), (3.4), and Lebesgue's monotone convergence theorem. Then arguments similar to those used in the proof of Lemma 2.13 yield the estimate for v^k, since multiplying the equation satisfied by $\hat{v}^k := V_0 - v^k$ by $m'_\alpha(\hat{v}^k)\psi_+^L$, where ψ_+^L and m_α are as defined in the proofs of Lemmas 3.2 and 2.13 respectively, integrating over $\mathbb{R}^+ \times (0, t_0)$, and noting that, if $d_v > 0$, the boundary condition at $x = 0$ yields

$$\begin{aligned}
\int_0^\infty \psi_+^L m'_\alpha(\hat{v}^k)\hat{v}^k_{xx}\ dx \ &= \ -\int_0^\infty (\psi_+^L)_x m'_\alpha(\hat{v}^k)\hat{v}^k_x + \psi_+^L m''_\alpha(\hat{v}^k)(\hat{v}^k_x)^2\ dx \\
&\leq \ -\int_0^\infty (\psi_+^L)_x (m_\alpha(\hat{v}^k))_x\ dx = \int_0^\infty (\psi_+^L)_{xx} m_\alpha(\hat{v}^k)\ dx,
\end{aligned}$$

together gives, after letting $\alpha \to 0$, that for each $t_0 \in (0, T)$,

$$\begin{aligned}
\int_0^\infty \psi_+^L |\hat{v}^k(x, t_0)| dx \leq \int_0^\infty \psi_+^L |\hat{v}^k(x, 0)| dx \\
+ d_v \int_0^{t_0} \int_0^\infty |\hat{v}^k| \left(\psi_+^L\right)_{xx} dxdt + \int_0^{t_0} kF(u^k, V_0 - \hat{v}^k)\mathrm{sgn}(\hat{v}^k)\psi_+^L dxdt.
\end{aligned}$$

The result then follows using (3.3), Lemma 3.4, Lebesgue's monotone convergence theorem, and the fact that $\|V_0 - v_0^k\|_{L^1(\mathbb{R}^+)}$ is bounded independently of k. ☐

Next we prove the half-line analogue of Lemma 2.14.

Lemma 3.6. *Suppose that $d_u > 0$ and $d_v \geq 0$. Then there exists $C > 0$, independent of d_v and $k > 0$, such that for any solution (u^k, v^k) of (P_2^k) satisfying (3.3),*

$$d_u \int_0^T \int_0^\infty (u_x^k)^2(x, t)\ dxdt \leq C \quad \textit{and} \quad d_v \int_0^T \int_0^\infty (v_x^k)^2(x, t)\ dxdt \leq C. \tag{3.8}$$

Proof. Let $\hat{u} \in C^\infty(\mathbb{R}^+)$ be a fixed function such that $\hat{u}(0) = U_0$ and $\hat{u}(x) = 0$ when $x \geq 1$. Define $y^k := u^k - \hat{u}$. Then y^k satisfies

$$\begin{cases}
y_t^k &= d_u y_{xx}^k + d_u \hat{u}_{xx} - kF(y^k + \hat{u}, v^k) & \text{in } S_T, \\
y^k(0, t) &= 0 & \text{for } t \in [0, T], \\
y^k(x, 0) &= u_0^k(x) - \hat{u}(x), & \text{for } \ x \in \mathbb{R}^+,
\end{cases}$$

and multiplying the equation for y^k by $y^k\psi_+^L$ and integrating over S_T gives that

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \psi_+^L (y^k)^2(x, T) dx + d_u \int_0^T \int_0^\infty \psi_+^L (y_x^k)^2 dxdt = \frac{1}{2}\int_0^\infty \psi_+^L (u_0^k - \hat{u})^2(x) dx \\
+ \ \frac{d_u}{2}\int_0^T \int_0^\infty (y^k)^2 \left(\psi_+^L\right)_{xx} dxdt + d_u \int_0^T \int_0^\infty \psi_+^L \hat{u}_{xx} y^k dxdt - k\int_0^T \int_0^\infty \psi_+^L y^k F(u^k, v^k) dxdt.
\end{aligned}$$

Since $y^k_x = u^k_x - \hat{u}_x$, the first estimate in (3.8) then follows using (3.3), Lemma 3.4 and the fact that $\|u^k_0\|_{L^1(\mathbb{R}^+)}$ is bounded independently of k. A similar argument yields the estimate for v^k_x, using the equation for $\hat{v}^k := V_0 - v^k$ multiplied by $\psi^L_+ \hat{v}^k$ and the fact that $\hat{v}^k_x(0,t) = 0$ for all $t \in (0,T)$. □

Recall the notation for space and time translates introduced in (2.26).

Lemma 3.7. *Suppose that $d_u > 0$ and $d_v \geq 0$, and let (u^k, v^k) be a solution of (P^k_2) satisfying (3.3). Then for each $r \in (0,1)$, there exists a function $G_r \geq 0$, independent of $d_v \geq 0$ and $k > 0$, such that $G_r(\xi) \to 0$ as $|\xi| \to 0$, and for all $|\xi| \leq \frac{r}{4}$ and $t \in (0,T)$,*

$$\int_r^\infty |u^k(x,t) - S_\xi u^k(x,t)| + |v^k(x,t) - S_\xi v^k(x,t)| \; dx \;\leq\; G_r(\xi). \tag{3.9}$$

Proof. Let u, v, u_0 and v_0 be as defined in (2.28) and define a cut-off function $\gamma^1_r \in C^\infty(\mathbb{R}^+)$ such that $0 \leq \gamma^1_r \leq 1$, $\gamma^1_r(x) = 0$ when $x \in [0, \frac{r}{2}]$, $\gamma^1_r(x) = 1$ when $x \in [r,1]$, and $\gamma^1_r(x) = 0$ when $x \geq 2$. Then given $L \geq 1$, define the family of cut-off functions $\gamma^L_r \in C^\infty(\mathbb{R}^+)$ by $\gamma^L_r(x) = \gamma^1_r(x)$ when $x \in [0,r]$, $\gamma^L_r(x) = 1$ when $x \in [r,L]$, and $\gamma^L_r(x) = \gamma^1_r(x+1-L)$ when $x \geq L$. Note that $0 \leq \gamma^L_r \leq 1$ for all L, and $\left(\gamma^L_r\right)_x$, $\left(\gamma^L_r\right)_{xx}$ are bounded in both $L^\infty(\mathbb{R}^+)$ and $L^1(\mathbb{R}^+)$ independently of L. Then

$$\begin{aligned}
u_t &= d_u u_{xx} - k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} && \text{in } (\tfrac{r}{4},\infty)\times(0,T),\\
v_t &= d_v v_{xx} - k\{F(u^k,v^k) - F(S_\xi u^k, S_\xi v^k)\} && \text{in } (\tfrac{r}{4},\infty)\times(0,T),\\
u(x,0) &= u^k_0(x) - u^k_0(x+\xi), \quad v(x,0) = v^k_0(x) - v^k_0(x+\xi) && \text{for } \; x \in (\tfrac{r}{4},\infty),
\end{aligned}$$

so that arguing as in the proof of Lemma 2.15 yields that for each $t_0 \in (0,T)$,

$$\begin{aligned}
\int_{\frac{r}{2}}^\infty \gamma^L_r(x)\{|u(x,t_0)| + |v(x,t_0)|\} \; dx \leq& \int_{\frac{r}{2}}^\infty \gamma^L_r(x)\{|u_0(x)| + |v_0(x)|\} \; dx\\
&+ \int_0^{t_0}\int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x)\{d_u|u(x,t)| + d_v|v(x,t)|\} \; dxdt.
\end{aligned} \tag{3.10}$$

Now, by the definition of u,

$$\begin{aligned}
\int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x) d_u |u(x,t)| dx &= \int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x) d_u |u^k(x,t) - u^k(x+\xi,t)| dx\\
&= \int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x) d_u \left|\int_0^1 u^k_x(x+\theta\xi,t)\xi d\theta\right| dx\\
&\leq |\xi| d_u \int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x) \int_0^1 |u^k_x(x+\theta\xi,t)| d\theta dx,
\end{aligned}$$

thus

$$\begin{aligned}
&\int_0^{t_0}\int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)_{xx}(x) d_u |u(x,t)| dxdt\\
&\leq |\xi|\sqrt{d_u}\int_0^1 \left(\int_0^{t_0}\int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)^2_{xx} dxdt\right)^{\frac{1}{2}} \left(\int_0^{t_0}\int_{\frac{r}{2}}^\infty d_u |u^k_x(x+\theta\xi,t)|^2 dxdt\right)^{\frac{1}{2}} d\theta\\
&\leq |\xi|\sqrt{d_u}\left(\int_0^{t_0}\int_{\frac{r}{2}}^\infty \left(\gamma^L_r\right)^2_{xx} dxdt\right)^{\frac{1}{2}} \left(\int_0^{t_0}\int_{\frac{r}{4}}^\infty d_u |u^k_x(x,t)|^2 dxdt\right)^{\frac{1}{2}},
\end{aligned}$$

and hence applying Lemma 3.6 shows that

$$\int_0^{t_0}\int_{\frac{r}{2}}^{\infty}\left(\gamma_r^L\right)_{xx}(x)d_u|u(x,t)|dxdt \le K_r|\xi|, \tag{3.11}$$

for some constant K_r. The result then follows from (3.10) using (3.11), a similar estimate for $\int_0^{t_0}\int_{\frac{r}{2}}^{\infty}\left(\gamma_r^L\right)_{xx} d_v|v|dxdt$, the fact that $\|u_0^k(\cdot+\xi)-u_0^k(\cdot)\|_{L^1((r,\infty))}+\|v_0^k(\cdot+\xi)-v_0^k(\cdot)\|_{L^1((r,\infty))} \le \omega_r(|\xi|)$ where $\omega_r(|\xi|)\to 0$ as $\xi\to 0$, and Lebesgue's monotone convergence theorem. □

Lemma 3.8. *Suppose that $d_u>0$, $d_v\ge 0$ and let (u^k,v^k) be a solution of (P_2^k) satisfying (3.3). Then there exists $C>0$, independent of d_v and k, such that for any $\tau\in(0,T)$,*

$$\begin{aligned}\int_0^{T-\tau}\int_0^{\infty}|T_\tau u^k(x,t)-u^k(x,t)|^2\,dxdt &\le \tau C,\\ \int_0^{T-\tau}\int_0^{\infty}|T_\tau v^k(x,t)-v^k(x,t)|^2\,dxdt &\le \tau C.\end{aligned}$$

Proof. This follows from arguments analogous to those used in the proof of Lemma 2.16, replacing ψ^L by $\psi_+^L:=\psi^L|_{\mathbb{R}^+}$ and integrals over $\mathbb{R}$ by integrals over $\mathbb{R}^+$, noting that $u^k(0,t+\tau)-u^k(0,t)=0$ and $v_x^k(0,t+\tau)-v_x^k(0,t)=0$ for all $t\in(0,T-\tau)$ and using the bounds in Lemmas 3.4 and 3.6. □

Lemma 3.9. *Let $k>0$ and $d_u>0$ be fixed and $(u_{d_v}^k,v_{d_v}^k)$ be solutions of (P_2^k) satisfying (3.3) with $d_v>0$. Then there exists $(u_*^k,v_*^k)\in(L^\infty(S_T))^2$ such that up to a subsequence, for each $J>0$,*

$$\begin{aligned}u_{d_v}^k &\to u_*^k && \text{in } L^2((0,J)\times(0,T)),\\ v_{d_v}^k &\to v_*^k && \text{in } L^2((0,J)\times(0,T)),\\ u_{d_v}^k-\hat{u} &\rightharpoonup u_*^k-\hat{u} && \text{in } L^2(0,T;H_0^1(\mathbb{R}^+)),\end{aligned}$$

as $d_v\to 0$, where $\hat{u}\in C^\infty(\mathbb{R}^+)$ is a smooth function such that $\hat{u}(0)=U_0$ and $\hat{u}(x)=u_0^\infty(x)$ for all $x\ge 1$.

Proof. It follows from (3.3) and Lemma 3.5 that $\|u_{d_v}^k-u_0^\infty\|_{L^2(S_T)}$ and $\|v_{d_v}^k-v_0^\infty\|_{L^2(S_T)}$ are bounded independently of d_v. So Lemmas 3.7, 3.8 and the Riesz-Fréchet-Kolmogorov Theorem [3, Theorem 4.26] yield that the sets $\{u_{d_v}^k-u_0^\infty\}_{d_v>0}$ and $\{v_{d_v}^k-v_0^\infty\}_{d_v>0}$ are each relatively compact in $L^2((0,J)\times(0,T))$ for each $J>0$. The weak convergence of $u_{d_v}^k-\hat{u}$ in $L^2(0,T;H_0^1(\mathbb{R}))$ follows from the fact that $\|u_{d_v}^k\|_{L^2(S_T)}$ is bounded independently of d_v together with the proof of Lemma 3.6. □

Lemma 3.9 and Corollary 3.3 enable the following result to be established using arguments similar to those that yield Theorem 2.18. We omit details of the proof.

Theorem 3.10. *Let $d_v=0$ and $k>0$. Then problem (P_2^k) has a unique weak solution*

$$(u^k,v^k)\in W_p^{2,1}((0,J)\times(0,T))\times W^{1,\infty}(0,T;L^\infty((0,J)))\ \text{ for each } J>0,\ \ p\ge 1,$$

where (u^k,v^k) is a weak solution in the sense that

$$\begin{aligned}\iint_{S_T}u^k\psi_t\,dxdt+\iint_{S_T}\{d_u u^k\psi_{xx}-kF(u^k,v^k)\psi\}\,dxdt &= -\int_0^\infty u_0^k\psi(\cdot,0)\,dx\\ &\quad -d_uU_0\int_0^T\psi_x(0,t)\,dt,\end{aligned} \tag{3.12}$$

$$\iint_{Q_T}v^k\psi_t\,dxdt-\iint_{Q_T}kF(u^k,v^k)\psi\,dxdt = -\int_{\mathbb{R}}v_0^k\psi(\cdot,0)\,dx, \tag{3.13}$$

for all $\psi \in \hat{\mathcal{F}}_T = \{\psi \in C^{2,1}(S_T) : \psi(\cdot, T) = 0,\ \psi(0,t) = 0$ *for* $t \in (0,T)$ *and* $supp\, \psi \subset [0,J] \times [0,T]$ *for some* $J > 0\}$, *and also satisfies* $0 \le u^k, v^k \le M$.

3.3. **The limit problem for** (P_2^k) **as** $k \to \infty$**.** The next result follows directly from arguments similar to those used in section 2.3, exploiting the half-line estimates established in section 3.2.

Lemma 3.11. *Let* $d_u > 0$ *and* $d_v \ge 0$ *be fixed and* (u^k, v^k) *be solutions of* (P_2^k) *satisfying (3.3) with* $k > 0$. *Then there exists* $(u,v) \in (L^\infty(S_T))^2$ *such that up to a subsequence, for each* $J > 0$,

$$\begin{aligned} u^k &\to u && \text{in } L^2((0,J)\times(0,T)),\\ v^k &\to v && \text{in } L^2((0,J)\times(0,T)),\\ u^k - \hat{u} &\rightharpoonup u - \hat{u} && \text{in } L^2(0,T;H_0^1(\mathbb{R})), \end{aligned}$$

as $k \to \infty$, *where* $\hat{u} \in C^\infty(\mathbb{R}^+)$ *is a smooth function such that* $\hat{u}(0) = U_0$ *and* $\hat{u}(x) = 0$ *for all* $x \ge 1$. *Moreover,*

$$uv = 0 \quad \text{a.e. in } S_T. \tag{3.14}$$

Taking w^k and w as in (2.42), we clearly again have that as a sequence $k_n \to \infty$, $w^{k_n} \to w$ in $L^2(S_T)$ and almost everywhere in S_T, and that $u = w^+$ and $v = -w^-$.

The next result is our half-line counterpart of Lemma 2.21, with the boundary at $x = 0$ clearly now playing a rôle. Note that here, similarly to [10], the limit function w satisfies a Dirichlet boundary condition both when $d_v = 0$ and when $d_v > 0$.

Lemma 3.12. *Let* $d_u > 0$, $d_v \ge 0$ *and* (u,v) *be as in Lemma 3.11. Then*

$$-\iint_{S_T} (u-v)\psi_t\, dxdt - \int_0^\infty (u_0^\infty - v_0^\infty)\,\psi(x,0)\, dx = d_u U_0 \int_0^T \psi_x(0,t)dt + \iint_{S_T} (d_u u - d_v v)\,\psi_{xx}\, dxdt, \tag{3.15}$$

for all $\psi \in \hat{\mathcal{F}}_T = \{\psi \in C^{2,1}(S_T) : \psi(\cdot,T) = 0,\ \psi(0,t) = 0$ *for* $t \in (0,T)$ *and* $supp\,\psi \subset [0,J] \times [0,T]$ *for some* $J > 0\}$.

Proof. Multiplying the difference between the equations for u^k and v^k by $\psi \in \hat{\mathcal{F}}_T$ and integrating over S_T gives

$$\begin{aligned} -\iint_{S_T} (u^k - v^k)\psi_t\, dxdt &- \int_0^\infty (u_0^k - v_0^k)\,\psi(x,0)\, dx \\ &= \int_0^T \{d_u u^k(0,t) - d_v v^k(0,t)\}\psi_x(0,t)dt + \iint_{S_T} (d_u u^k - d_v v^k)\psi_{xx}\, dxdt. \end{aligned} \tag{3.16}$$

Now it follows exactly as argued in the proof of [10, Prop. 8] that if $d_v > 0$, then the segregation property (3.14) yields that as $k \to \infty$,

$$\gamma(d_u u^k - d_v v^k) \rightharpoonup d_u U_0 \quad \text{in } L^2(\{0\}\times(0,T)),$$

where γ denotes the trace on the boundary $\{0\} \times (0,T)$. So (3.15) follows by letting $k \to \infty$ in (3.16). □

Now recall the definition of $\mathcal{D}$ from (2.44) and define the limit problem

$$(P_2^{limit}) \begin{cases} w_t &= \mathcal{D}(w)_{xx}, & \text{in } [0,\infty)\times(0,\infty),\\ w(x,0) &= w_0(x) := -V_0, & \text{if } x < 0,\\ w(0,t) &= U_0, & \text{for } t \in (0,\infty). \end{cases}$$

Definition 3.13. *A function w is a weak solution of Problem (P_2^{limit}) if*

(i) $w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$,

(ii) *for all $T > 0$,*

$$\iint_{S_T} (w\psi_t + \mathcal{D}(w)\psi_{xx})\, dxdt = -d_u U_0 \int_0^T \psi_x(0,t)dt - \int_0^\infty w_0(x)\psi(x,0)\, dx,$$

for all $\psi \in \hat{\mathcal{F}}_T = \{\psi \in C^{2,1}(S_T) : \psi(\cdot,T) = 0,\ \psi(0,t) = 0$ for $t \in (0,T)$ and $\operatorname{supp}\psi \subset [0,J] \times [0,T]$ for some $J > 0\}$.

Lemma 3.14. *With (u,v) from Lemma 3.11, the function $w : u - v$ is the unique weak solution of Problem (P_2^{limit}) and the whole sequence (u^k, v^k) in Lemma 3.11 converges to $(w^+, -w^-)$.*

Proof. That w is a weak solution of (P_2^{limit}) follows immediately from Lemma 3.12 and the definition of $\mathcal{D}$. Minor modifications in the arguments used to establish [2, Appendix, Proposition A] and [1, Proposition 9] yield a corresponding estimate with the domain $\mathbb{R}$ in [2, Appendix, Proposition A] replaced by $\mathbb{R}^+$, from which uniqueness again follows via the reasoning in the proof of [2, Appendix, Proof of Theorem C]. □

As in the whole-line case, we can identify the limit w as a certain self-similar solution both when $d_v > 0$ and when $d_v = 0$. We first state the analogue of Corollary 2.25.

Proposition 3.15. *Let w be the unique weak solution of Problem (P_2^{limit}). Suppose that there exists a function $\xi : [0,T] \to \mathbb{R}^+$ such that for each $t \in [0,T]$,*

$$w(x,t) > 0 \quad \text{if} \quad x < \xi(t) \quad \text{and} \quad w(x,t) < 0 \quad \text{if} \quad x > \xi(t).$$

Then if $t \mapsto \xi(t)$ is sufficiently smooth and the functions $u := w^+$ and $v := -w^-$ are smooth up to $\xi(t)$, the functions u and v satisfy one of two limit problems, depending on whether $d_v > 0$ or $d_v = 0$. If $d_v > 0$, then

$$(P_{2,d_v>0}^{limit}) \begin{cases} u_t = d_u u_{xx}, & \text{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = 0, & \text{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v_t = d_v v_{xx}, & \text{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ u = 0, & \text{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ \lim_{x\uparrow\xi(t)} u(x,t) = 0 = \lim_{x\downarrow\xi(t)} v(x,t), & \text{for each } t \in [0,T], \\ d_u \lim_{x\uparrow\xi(t)} u_x(x,t) = -d_v \lim_{x\downarrow\xi(t)} v_x(x,t), & \text{for each } t \in [0,T], \\ u = U_0, & \text{on } \{0\} \times [0,T], \\ u(\cdot,0) = 0, & \text{in } (0,\infty), \\ v(\cdot,0) = V_0, & \text{in } (0,\infty), \end{cases}$$

whereas if $d_v = 0$ and we suppose additionally that $\xi(0) = 0$ and $t \mapsto \xi(t)$ is a non-decreasing function, then

$$(P_{2,d_v=0}^{limit}) \begin{cases} u_t = d_u u_{xx}, & \text{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = 0, & \text{in } \{(x,t) \in Q_T : x < \xi(t)\}, \\ v = V_0, & \text{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ u = 0, & \text{in } \{(x,t) \in Q_T : x > \xi(t)\}, \\ \lim_{x\uparrow\xi(t)} u(x,t) = 0, & \text{for each } t \in [0,T], \\ V_0\, \xi'(t) = -d_u \lim_{x\uparrow\xi(t)} u_x(x,t), & \text{for each } t \in [0,T], \\ u = U_0, & \text{on } \{0\} \times [0,T], \\ u(\cdot,0) = 0, & \text{in } (0,\infty), \\ v(\cdot,0) = V_0, & \text{in } (0,\infty), \end{cases}$$

where $\xi'(t)$ denotes the speed of propagation of the free boundary $\xi(t)$.

Theorem 3.16. *The unique weak solution w of Problem (P_2^{limit}) has a self-similar form. There exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ and a positive constant $a > 0$ such that*

$$w(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \quad (x,t) \in S_T, \quad \text{and} \quad \xi(t) = a\sqrt{t}, \quad t \in [0,T]. \tag{3.17}$$

If $d_v > 0$, then $a > 0$ is the unique root of the equation

$$d_u U_0 \int_a^\infty e^{\frac{a^2-s^2}{4d_v}} ds = d_v V_0 \int_0^a e^{\frac{a^2-s^2}{4d_u}},$$

and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_0^\eta e^{-\frac{s^2}{4d_u}} \, ds}{\int_0^a e^{-\frac{s^2}{4d_u}} \, ds}\right), & \text{if } \eta \le a, \\ -V_0 \left(1 - \dfrac{\int_\eta^\infty e^{-\frac{s^2}{4d_v}} \, ds}{\int_a^\infty e^{-\frac{s^2}{4d_v}} \, ds}\right), & \text{if } \eta > a. \end{cases} \tag{3.18}$$

On the other hand, if $d_v = 0$, then $a > 0$ is the unique root of the equation

$$U_0 = \frac{V_0 a}{2d_u} \int_0^a e^{\frac{a^2-s^2}{4d_u}} \, ds,$$

and

$$f(\eta) = \begin{cases} U_0 \left(1 - \dfrac{\int_0^\eta e^{-\frac{s^2}{4d_u}} \, ds}{\int_0^a e^{-\frac{s^2}{4d_u}} \, ds}\right), & \text{if } \eta \le a, \\ -V_0, & \text{if } \eta > a. \end{cases} \tag{3.19}$$

4. Long-time behaviour for (P_1^k) and (P_2^k) when k is fixed

We conclude our study by exploiting a scaling argument, due first to Kamin [8] and used also in [5, 6], to infer the self-similar $t \to \infty$ limits of solutions (u^k, v^k) of (P_1^k) or (P_2^k) from the $k \to \infty$ limits discussed in Sections 2.3 and 3.3. As mentioned in the Introduction, this enables us, in particular, to give rigorous justification to the long-time asymptotics of reaction fronts discussed by Trevelyan et al [14]. Note that in [5, 6], uniform convergence results as $k \to \infty$ implied pointwise convergence results as $t \to \infty$ for solutions of a problem with k fixed. Here, however, we simply use the L^2-convergence from Lemmas 2.19 and 3.11 to deduce convergence in a certain average sense of solutions of (P_1^k) and (P_2^k) as $t \to \infty$ along a subsequence.

Theorem 4.1. *Let (u^k, v^k) be the solution of problem (P_1^k) with initial data u_0^k, $v_0^k \in C^2(\mathbb{R})$ such that*

$$\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R})} < \infty, \quad \|v_0^k - v_0^\infty\|_{L^1(\mathbb{R})} < \infty, \tag{4.1}$$

and

$$u_0^k(x) \to U_0, 0 \;\; \text{as} \;\; x \to -\infty, \infty \;\; \text{and} \;\; v_0^k(x) \to 0, V_0 \;\; \text{as} \;\; x \to -\infty, \infty. \tag{4.2}$$

Then for each $J > 0$, there exists a sequence $t_n \to \infty$ such that

$$\frac{1}{\sqrt{t_n}} \int_{-J\sqrt{t_n}}^{J\sqrt{t_n}} \left| u^k(y,t_n) - f^+\left(\frac{y}{\sqrt{t_n}}\right)\right|^2 dy \;\; \to \;\; 0 \;\; \text{as} \;\; t_n \to \infty, \tag{4.3}$$

and

$$\text{(4.4)} \qquad \frac{1}{\sqrt{t_n}} \int_{-J\sqrt{t_n}}^{J\sqrt{t_n}} \left| v^k(y,t_n) + f^-\left(\frac{y}{\sqrt{t_n}}\right)\right|^2 \, dy \;\; \to \;\; 0 \;\; as \;\; t_n \to \infty.$$

where f is the self-similar profile given by (2.46) if $d_v > 0$, and by (2.47) if $d_v = 0$. Here, as usual, $f^+ := \max\{f, 0\}$, $f^- := \min\{f, 0\}$.

Proof. For each $l > 0$, the scaled functions

$$u_l^k(x,t) := u^k(lx, l^2 t), \quad v_l^k(x,t) := v_l^k(x,t) = v^k(lx, l^2 t),$$

satisfy the system

$$(P1_l^k) \begin{cases} u_t &= d_u u_{xx} - kl^2 F(u,v) & \text{in } Q_{l^2 T}, \\ v_t &= d_v v_{xx} - kl^2 F(u,v) & \text{in } Q_{l^2 T}, \\ u(x,0) &= u_0^k(lx), \quad v(x,0) = v_0^k(lx) & \text{for} \quad x \in \mathbb{R}. \end{cases}$$

Moreover, it follows from (4.1) that as $l \to \infty$,

$$\|u_l^k(\cdot,0) - u_0^\infty\|_{L^1(\mathbb{R})} \to 0, \quad \|v_l^k(\cdot,0) - v_0^\infty\|_{L^1(\mathbb{R})} \to 0.$$

So Lemma 2.19, Lemma 2.23 and Theorem 2.26 together imply that for each $T > 0$ and each $J > 0$,
(4.5)

$$\int_0^T \int_{-J}^J \left| u_l^k(x,t) - f^+\left(\frac{x}{\sqrt{t}}\right)\right|^2 \, dxdt \to 0, \quad \int_0^T \int_{-J}^J \left| v_l^k(x,t) + f^-\left(\frac{x}{\sqrt{t}}\right)\right|^2 \, dxdt \to 0 \;\text{ as }\; l \to \infty,$$

where f is as in the statement of the theorem. Now take $T \geq 1$ and fix $J > 0$. Then it follows from (4.5) that there exists $t_0 \in (\frac{1}{2}, 1)$ and a sequence $l_n \to \infty$ such that as $l_n \to \infty$,

$$\int_{-J}^J \left| u_{l_n}^k(x,t_0) - f^+\left(\frac{x}{\sqrt{t_0}}\right)\right|^2 \, dx \to 0, \quad \int_{-J}^J \left| v_{l_n}^k(x,t_0) + f^-\left(\frac{x}{\sqrt{t_0}}\right)\right|^2 \, dx \to 0,$$

or equivalently

$$\int_{-J}^J \left| u^k(l_n x, l_n^2 t_0) - f^+\left(\frac{x}{\sqrt{t_0}}\right)\right|^2 \, dx \to 0, \quad \int_{-J}^J \left| v^k(l_n x, l_n^2 t_0) + f^-\left(\frac{x}{\sqrt{t_0}}\right)\right|^2 \, dx \to 0,$$

which yields immediately that as $l_n \to \infty$,
(4.6)

$$\frac{1}{l_n}\int_{-l_n J}^{l_n J} \left| u^k(y, l_n^2 t_0) - f^+\left(\frac{y}{l_n\sqrt{t_0}}\right)\right|^2 \, dy \to 0, \quad \frac{1}{l_n}\int_{-l_n J}^{l_n J} \left| v^k(y, l_n^2 t_0) + f^-\left(\frac{y}{l_n\sqrt{t_0}}\right)\right|^2 \, dy \to 0,$$

Taking $s_n := l_n^2 t_0$ in (4.6) then gives that

$$\sqrt{\frac{t_0}{s_n}} \int_{-J\sqrt{\frac{s_n}{t_0}}}^{J\sqrt{\frac{s_n}{t_0}}} \left| u^k(y, s_n) - f^+\left(\frac{y}{\sqrt{s_n}}\right)\right|^2 \, dy \to 0, \quad \sqrt{\frac{t_0}{s_n}} \int_{-J\sqrt{\frac{s_n}{t_0}}}^{J\sqrt{\frac{s_n}{t_0}}} \left| v^k(y, s_n) + f^-\left(\frac{y}{\sqrt{s_n}}\right)\right|^2 \, dy \to 0,$$

as $s_n \to \infty$, from which the result follows. □

Minor modifications in the arguments above show the following, corresponding result for the half-line problem (P_2^k). We leave the details of the proof to the reader.

Theorem 4.2. *Let (u^k, v^k) be the solution of problem (P_2^k) with initial data u_0^k, $v_0^k \in C^2(\mathbb{R}^+)$ such that*

$$\|u_0^k - u_0^\infty\|_{L^1(\mathbb{R}^+)} < \infty, \quad \|v_0^k - v_0^\infty\|_{L^1(\mathbb{R}^+)} < \infty, \tag{4.7}$$

and

$$u_0^k(x) \to 0 \ \text{ as } \ x \to \infty \ \text{ and } \ v_0^k(x) \to V_0 \ \text{ as } \ x \to \infty. \tag{4.8}$$

Then for each $J > 0$, there exists a sequence $t_n \to \infty$ such that

$$\frac{1}{\sqrt{t_n}} \int_0^{J\sqrt{t_n}} \left| u^k(y, t_n) - f^+\left(\frac{y}{\sqrt{t_n}}\right)\right|^2 dy \quad \to \quad 0 \ \text{ as } \ t_n \to \infty, \tag{4.9}$$

and

$$\frac{1}{\sqrt{t_n}} \int_0^{J\sqrt{t_n}} \left| v^k(y, t_n) + f^-\left(\frac{y}{\sqrt{t_n}}\right)\right|^2 dy \quad \to \quad 0 \ \text{ as } \ t_n \to \infty. \tag{4.10}$$

where f is the self-similar profile given by (3.18) if $d_v > 0$, and by (3.19) if $d_v = 0$.

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Department of Mathematics, College of Science, Swansea University, Swansea SA2 8PP, U.K.

CNRS and Laboratoire de Mathématiques, Université de Paris-Sud 11, F-91405 Orsay Cedex, France