Cronfa - Swansea University Open Access Repository

This is an author produced version of a paper published in :
Physics Letters B

Cronfa URL for this paper:
http://cronfa.swan.ac.uk/Record/cronfa25961

Paper:

Hands, S. (2016). From domain wall to overlap in 2+1d. *Physics Letters B, 754*, 264-269.

http://dx.doi.org/10.1016/j.physletb.2016.01.037

This article is brought to you by Swansea University. Any person downloading material is agreeing to abide by the terms of the repository licence. Authors are personally responsible for adhering to publisher restrictions or conditions. When uploading content they are required to comply with their publisher agreement and the SHERPA RoMEO database to judge whether or not it is copyright safe to add this version of the paper to this repository.
http://www.swansea.ac.uk/iss/researchsupport/cronfa-support/

Physics Letters B 754 (2016) 264–269

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

From domain wall to overlap in $2+1d$

Simon Hands

Department of Physics, College of Science, Swansea University, Singleton Park, Swansea SA2 8PP, United Kingdom

ARTICLE INFO

Article history:
Received 27 December 2015
Received in revised form 12 January 2016
Accepted 20 January 2016
Available online 22 January 2016
Editor: J.-P. Blaizot

Keywords:
Lattice gauge field theories
Field theories in lower dimensions
Global symmetries

ABSTRACT

The equivalence of domain wall and overlap fermion formulations is demonstrated for lattice gauge theories in $2+1$ spacetime dimensions with parity-invariant mass terms. Even though the domain wall approach distinguishes propagation along a third direction with projectors $\frac{1}{2}(1\pm\gamma_3)$, the truncated overlap operator obtained for finite wall separation L_s is invariant under interchange of γ_3 and γ_5. In the limit $L_s\to\infty$ the resulting Ginsparg–Wilson relations recover the expected $\mathrm{U}(2N_f)$ global symmetry up to $\mathrm{O}(a)$ corrections. Finally it is shown that finite-L_s corrections to bilinear condensates associated with dynamical mass generation are characterised by whether even powers of the symmetry-breaking mass are present; such terms are absent for antihermitian bilinears such as $i\bar\psi\gamma_3\psi$, markedly improving the approach to the large-L_s limit.

© 2016 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP^3.

1. Introduction

Relativistic fermions moving in 2 spatial dimensions are the focus of much attention, in part due to the stability of Dirac points in graphene and surface states of topological band insulators when the underlying Hamiltonian is symmetric under time reversal and spatial inversion (see, e.g. [1]). Even in this case a gap may develop at the Dirac points in the presence of interactions. The corresponding issue in quantum field theory is the stability of the vacuum with respect to spontaneous generation of a parity-invariant bilinear condensate of the form $\langle\bar\psi\Gamma_i\psi\rangle\neq0$. Since the transition to a gapped phase generically occurs for strong interactions, it defines a quantum critical point (QCP) [2]; the phase diagram for planar fermionic systems with various interactions and characterisation of possible QCPs as a function of the number of fermion species N_f remain open questions [3].

To date there have been many lattice field theory simulations probing QCPs using the staggered fermion formulation [4] (a notable recent exception employs the SLAC derivative [5]); N staggered fermions describe $N_f=2N$ continuum flavors each having 4 spinor components [6], with global symmetry group $\mathrm{U}(N)\otimes\mathrm{U}(N)$ spontaneously broken by a parity-invariant mass to $\mathrm{U}(N)$. However, because there are two matrices γ_3 and γ_5 which anticommute with the kinetic operator, the correct continuum symmetry breaking is $\mathrm{U}(2N_f)\to\mathrm{U}(N_f)\otimes\mathrm{U}(N_f)$. For the strongly-interacting continuum limit at a QCP, there is no reason *a priori* to expect the correct symmetry-breaking pattern to be recovered.

For this reason the properties of domain wall fermions, which purportedly more faithfully reproduce continuum symmetries, were explored for $2+1+1d$ in Ref. [7]. In particular bilinear condensates and meson correlators constructed from distinct spinor combinations, but which should yield identical results in a $\mathrm{U}(2)$-invariant theory, were investigated as a function of the extent L_s of the "third" direction separating the domain walls. Numerical results obtained in the context of quenched non-compact QED_3 with variable coupling strength support $\mathrm{U}(2)$ symmetry being restored as $L_s\to\infty$. In $2+1d$ the Ginsparg–Wilson relation specifying the optimal requirements for lattice fermions to avoid species doubling while retaining as much of the continuum global symmetry as possible [8] generalises to a set of three relations (since chiral rotations are now specified by an element of $\mathrm{U}(2)$ rather than $\mathrm{U}(1)$). These were set out in [7], along with the specification of an overlap Dirac operator D_{ov} [9] defined in $2+1d$ in which realises them. As it must, D_{ov} has equivalent properties under the $\mathrm{U}(2)$ rotations generated by γ_3 and γ_5. Overlap formulations of massless fermions in $2+1d$ and the relation with the parity anomaly occurring for an odd number of two-component spinor flavors have previously been discussed in [10].

In the domain wall approach, the $2+1d$ fields $\psi,\bar\psi$ are defined in terms of surface states of fields $\Psi_\pm,\bar\Psi_\pm$ which are approximately localised on the walls and are $\pm$ eigenstates of γ_3 [11]. Some questions which remain unanswered in [7] are: the extent to which the domain wall formulation, in which propagation along

---

E-mail address: s.hands@swan.ac.uk.

http://dx.doi.org/10.1016/j.physletb.2016.01.037

0370-2693/© 2016 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP^3.

the direction separating the walls is governed by γ_3, can maintain the equivalence between γ_3 and γ_5 rotations for finite L_s; the reason for $O(a)$ violations of U(2) symmetries even in the overlap limit $L_s \to \infty$; and a better understanding of why finite-L_s corrections are minimised by choosing $i\langle\bar{\psi}\gamma_3\psi\rangle$, rather than $\langle\bar{\psi}\psi\rangle$, as the bilinear condensate to focus on.

In this brief technical Letter I outline how the overlap operator is recovered in the $L_s \to \infty$ limit of the domain wall formulation using a by now familiar sequence of matrix algebra operations. In particular, it will prove possible to extend the key results on the equivalence of γ_3 and γ_5 to a truncated overlap operator defined by domain wall fermions with finite L_s. As well as providing a firm conceptual foundation for domain wall fermions and their symmetry properties in $2+1d$, the proof sheds light on each of these outstanding issues.

2. From domain wall to overlap

First we review the passage from the domain wall formulation of lattice fermions to the overlap operator. The corresponding treatment for $4d$ gauge theories is well-known [12]: here we follow closely the treatment of [13]. We begin from the $2+1d$ domain wall operator defined in [7], correcting an overall (unphysical) sign:

$$S^{dw} = \sum_{x,y}\sum_{s,r} \bar{\Psi}(x,s)D(x,s|y,r)\Psi(y,r). \tag{1}$$

The fields $\Psi, \bar{\Psi}$ are four-component spinors defined in $2+1+1$ dimensions, and

$$D(x,s|y,s') = \delta_{s,s'}(D_W(x|y) - M) + \delta_{x,y}D_3(s|s'), \tag{2}$$

where the first term is the $2+1d$ Wilson operator defined on spacetime volume V

$$\begin{aligned}(D_W - M)_{x,y} &= -\frac{1}{2}\sum_{\mu=0,1,2}\left[(1-\gamma_\mu)U_\mu(x)\delta_{x+\hat{\mu},y} + (1+\gamma_\mu)U^\dagger_\mu(y)\delta_{x-\hat{\mu},y}\right] \\ &\quad + (3-M)\delta_{x,y},\end{aligned} \tag{3}$$

and D_3 controls hopping along the dimension separating the domain walls at $s=1$ and $s=L_s$, which we will refer to as the third direction:

$$D_{3\,s,s'} = -\left[P_-\delta_{s+1,s'}(1-\delta_{s',L_s}) + P_+\delta_{s-1,s'}(1-\delta_{s',1})\right] + \delta_{s,s'}, \tag{4}$$

where the projectors $P_\pm \equiv \frac{1}{2}(1\pm\gamma_3)$. Following convention, in (3) we include interaction with a SU(N_c) valued gauge connection field $U_\mu(x)$ located on the lattice links, noting in passing that some models relevant for $2+1d$ QCPs share the global U($2N_f$) symmetries of gauge theories.

Initially we supplement (1) with a hermitian mass term coupling fields on opposite walls:

$$m_hS_h = m_h\sum_x \bar{\Psi}(x,L_s)P_-\Psi(x,1) + \bar{\Psi}(x,1)P_+\Psi(x,L_s). \tag{5}$$

The operator $D_W - M + D_3 + m_hS_h$ can be represented as a $L_s \times L_s$ matrix consisting of $4VN_c \times 4VN_c$ blocks:

$$\begin{aligned} D(m_h) = &\begin{bmatrix} D_W-M+1 & 0 & \cdots & & +m_h \\ -1 & D_W-M+1 & 0 & & \\ 0 & -1 & \ddots & & \\ \vdots & & & & \\ 0 & & & -1 & D_W-M+1 \end{bmatrix} P_+ \\ + &\begin{bmatrix} D_W-M+1 & -1 & \cdots & 0 \\ 0 & D_W-M+1 & -1 & \\ \vdots & 0 & \ddots & \\ & & & -1 \\ +m_h & & & D_W-M+1 \end{bmatrix} P_-. \end{aligned} \tag{6}$$

Now define the cyclical shift operator $\mathcal{P}_{s,s'} \equiv [\delta_{s-1,s'}(1-\delta_{s,1}) + \delta_{s,1}\delta_{s',L_s}]P_- + \delta_{s,s'}P_+$ so that

$$D\mathcal{P} = \begin{bmatrix} Q_+ & 0 & \cdots & & Q_-C_- \\ Q_- & Q_+ & 0 & & 0 \\ 0 & Q_- & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ 0 & & & Q_- & Q_+C_+ \end{bmatrix} \tag{7}$$

with

$$Q_\pm = (D_W - M + 1)P_\pm - P_\mp; \tag{8}$$

$$C_\pm(m_h) = \frac{1}{2}(1-m_h) \pm \frac{1}{2}(1+m_h)\gamma_3 = P_\pm - m_hP_\mp. \tag{9}$$

Now define the block diagonal matrix $\mathcal{Q} = Q_+\mathbb{1}_{4VN_c\times 4VN_c}$; it is important to note that $Q_\pm \neq Q_\pm(m_h)$, $\mathcal{Q} \neq \mathcal{Q}(m_h)$, $\mathcal{P} \neq \mathcal{P}(m_h)$. With $\tilde{D} \equiv \mathcal{Q}^{-1}D\mathcal{P}$, we deduce

$$\det[\tilde{D}(1)^{-1}\tilde{D}(m_h)] \equiv \det[D(1)^{-1}D(m_h)], \tag{10}$$

where

$$\tilde{D} = \begin{bmatrix} 1 & 0 & \cdots & & -T^{-1}C_- \\ -T^{-1} & 1 & 0 & & 0 \\ 0 & -T^{-1} & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & -T^{-1} & C_+ \end{bmatrix}, \tag{11}$$

with $T = -Q_-^{-1}Q_+$.

In more detail,

$$\begin{aligned} T &= -[(D_W-M+1)P_- - P_+]^{-1}[(D_W-M+1)P_+ - P_-] \\ &= \left[1-\gamma_3\frac{(D_W-M)}{2+(D_W-M)}\right]^{-1}\left[1+\gamma_3\frac{(D_W-M)}{2+(D_W-M)}\right] \\ &= \frac{1-H}{1+H} \end{aligned} \tag{12}$$

where the hermitian $4VN_c \times 4VN_c$ matrix H is defined

$$H = -\gamma_3[2+(D_W-M)]^{-1}[D_W-M] \equiv -\gamma_3 A. \tag{13}$$

Hermiticity of H requires $\gamma_3A\gamma_3 = A^\dagger$, which is the case for A defined by (3). Up to an unphysical sign and with γ_3 assuming the role played by γ_5 in $4d$ gauge theories, H is identical with the Shamir kernel [14].

Next observe that in the form (11), $\tilde{D} = L\mathcal{D}U$ with

$$L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -T^1 & 1 & 0 & & \vdots \\ 0 & -T^{-1} & \ddots & & \\ \vdots & & \ddots & & \\ 0 & & & -T^{-1} & 1 \end{bmatrix};$$

$$U = \begin{bmatrix} 1 & 0 & \cdots & & -T^{-1}C_- \\ 0 & 1 & 0 & & -(T^{-1})^2C_- \\ \vdots & & 1 & \ddots & -(T^{-1})^3C_- \\ & & & \ddots & \vdots \\ 0 & & & & 1 \end{bmatrix} \tag{14}$$

and

$$\mathcal{D}=\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & 1 & & \\ & & & \ddots & \\ 0 & & & & C_+-(T^{-1})^{L_s}C_- \end{bmatrix}. \tag{15}$$

Again, note $L \neq L(m_h)$, and $\det L = \det U = 1$. We conclude

$$\begin{aligned} &\det[D(1)^{-1}D(m_h)] \\ &\quad = \det[\tilde{D}(1)^{-1}\tilde{D}(m_h)] = \det[\mathcal{D}_{L_s,L_s}(1)^{-1}\mathcal{D}_{L_s,L_s}(m_h)], \end{aligned} \tag{16}$$

where the $4VN_c \times 4VN_c$ matrix $\mathcal{D}_{L_s,L_s}$ is the Schur complement of $\tilde{D}$:

$$\begin{aligned} \mathcal{D}_{L_s,L_s}(m_h) &= C_+ - (T^{-1})^{L_s}C_- \\ &= (1+\mathcal{T}^{-1})\gamma_3\frac{1}{2}\left[(1+m_h)-(1-m_h)\gamma_3\frac{1-\mathcal{T}}{1+\mathcal{T}}\right] \\ &= \mathcal{D}_{L_s,L_s}(1)\frac{1}{2}\left[(1+m_h)-(1-m_h)\gamma_3\frac{1-\mathcal{T}}{1+\mathcal{T}}\right], \end{aligned} \tag{17}$$

with $\mathcal{T} \equiv T^{L_s}$. We now multiply both sides of (17) by $D^{-1}_{L_s,L_s}(1)$ to find that the combination of domain wall fermion determinants $\det[D(1)^{-1}D(m_h)]$ is the same as the determinant of the *truncated overlap* operator

$$D_{Ls}[H] = \frac{1}{2}\left[(1+m_h)-(1-m_h)\gamma_3\frac{1-\left(\frac{1-H}{1+H}\right)^{L_s}}{1+\left(\frac{1-H}{1+H}\right)^{L_s}}\right] \tag{18}$$

$$\equiv \frac{1}{2}\left[(1+m_h)-(1-m_h)\gamma_3\tanh(L_s\tanh^{-1}H)\right]. \tag{19}$$

In order for the tanh function to be defined by a power series the second equality (19) requires H to be a bounded operator, namely $|H| < 1$. The factor $D(1)^{-1}$ can be thought of as modelling Pauli–Villars boson fields which cancel the contributions of the fermions from the $4d$ bulk. Now, $\tanh(L_s\tanh^{-1}(x))$ is an analytic approximation to the signum function $\mathrm{sgn}(x)$ which becomes exact in the limit $L_s \to \infty$. So long as H is hermitian and bounded, we therefore recover the overlap operator [9]:

$$\begin{aligned} &\lim_{L_s\to\infty} D_{L_s} = D_{ov} \\ &= \frac{1}{2}\left[(1+m_h)-(1-m_h)\gamma_3\mathrm{sgn}\left(-\gamma_3\frac{D_W-M}{2+(D_W-M)}\right)\right] \\ &= \frac{1}{2}\left[(1+m_h)+(1-m_h)\frac{A}{\sqrt{A^\dagger A}}\right], \end{aligned} \tag{20}$$

where the unphysical nature of the sign of γ_3 is manifest. For $m_h \to 0$ (20) coincides with the $2+1d$ overlap operator given in [7].

Next let's check the overlap operator (20) has the expected weak-coupling limit. For link fields $U_\mu = 1$, and with lattice spacing set to unity, in momentum space $D_W = i\sum_\mu \gamma_\mu \sin p_\mu + \sum_\mu(1-\cos p_\mu)$, implying propagator poles at $p_\mu \approx 0$ and near the Brillouin Zone corners $p_\mu \approx \pi$. At the origin $D_W \approx i\gamma_\mu p_\mu$ so

$$\mathrm{sgn}(H) = \frac{H}{\sqrt{H^2}} \approx -\gamma_3\frac{(i\not{p}-M)}{(2-M)}\frac{(2-M)}{M} = -\gamma_3\left[\frac{i\not{p}}{M}-1\right] \tag{21}$$

so that the overlap operator

$$D_{ov} \approx i\not{p}\frac{(1-m_h)}{2M} + m_h. \tag{22}$$

Taking into account a benign wavefunction renormalisation, this is the propagator for a continuum species with mass proportional to m_h. By contrast near a doubler pole $\tilde{p}_\mu = p_\mu - (i,j,k)\pi \approx 0$, $i,j,k \in \{+1,-1\}$,

$$\mathrm{sgn}(H) \approx -\gamma_3\frac{i\tilde{\not{p}}+(2n-M)}{(2n-M)} = -\gamma_3\left[\frac{i\tilde{\not{p}}}{(2n-M)}+1\right] \tag{23}$$

with $n = |i|+|j|+|k|$, so the overlap is

$$D_{ov} \approx 1 + \frac{(1-m_h)}{2(2n-M)}i\tilde{\not{p}}. \tag{24}$$

So long as $(2n-M)$ is not too small, the species has a mass of O(1) in cutoff units, and decouples from low-energy physics.

Since m_h and M have opposite signs, for strong enough coupling there is the possibility of the system entering a parity-breaking Aoki phase signalled by a bilinear condensate with the quantum numbers of an isotriplet pion. This was investigated in the context of a 3d Gross–Neveu model in [15], where it was found that the Aoki phase was manifest for $m_h < 0$ with the width of the parity-broken region vanishing exponentially as $L_s \to \infty$.

3. Equivalence of γ_3 and γ_5

Despite the manifest independence of the overlap operator D_{ov} (20) of which matrix γ_3 or γ_5 is used to define the hermitian argument H of the signum function, for finite L_s it remains unclear whether the distinction is important or not [7], since clearly the definition (4) of the domain wall operator D_3 distinguishes them. We can address this using the analytic approximation for signum (19).

First, the series expansion for $\tanh^{-1}H$ is well-defined since $H = \gamma_3 A$ is a bounded operator, i.e. $|H| = M/(2-M) < 1$ for $0 < M < 1$[1]:

$$\tanh^{-1}H = H + \frac{H^3}{3} + \frac{H^5}{5} + \cdots \tag{25}$$

Each term is an odd power, so can be reexpressed using $\gamma_3 A \gamma_3 = A^\dagger$:

$$H^{2n+1} = \gamma_3 A(A^\dagger A)^n. \tag{26}$$

The signum approximation is then

$$\tanh(L_s\gamma_3 A\sum_n b_n(A^\dagger A)^n) = \frac{\sinh(L_s\gamma_3 A\sum_n b_n(A^\dagger A)^n)}{\cosh(L_s\gamma_3 A\sum_n b_n(A^\dagger A)^n)} \tag{27}$$

with $b_n = (2n+1)^{-1}$. In the McLaurin series expansions of the hyperbolic functions on the RHS of (27), expansion of the argument yields a general term of the form

$$L_s^m\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_m}^{\infty}\right)\prod_{i=1}^{m}[b_{n_i}(\gamma_3 A)(A^\dagger A)^{n_i}] \tag{28}$$

For the sinh series, m is an odd integer so that the term in square brackets reads

$$\begin{aligned} &(\prod b_{n_i})(\gamma_3 A)(A^\dagger)^{n_1}(\gamma_3 A)(A^\dagger A)^{n_2}\ldots(\gamma_3 A)(A^\dagger A)^{n_m} \\ &\quad = (\prod b_{n_i})(\gamma_3 A)(A^\dagger)^{n_1}(A^\dagger A)^{n_2+1}(A^\dagger A)^{n_3}\ldots \\ &\qquad (A^\dagger A)^{n_{m-1}+1}(A^\dagger A)^{n_m} \\ &\quad = (\prod b_{n_i})(\gamma_3 A)(A^\dagger A)^{\sum_i n_i+(m-1)/2}. \end{aligned} \tag{29}$$

---

[1] For free fermions the most stringent limit on M comes from the origin of momentum space. In practice on any finite lattice with antiperiodic temporal boundary conditions $M = 1$ is safe since $|H| = 1/\sqrt{5-4\cos\frac{\pi}{L_t}} < 1$ for $L_t < \infty$.

For the cosh series m is even and a similar argument gives the general term $(\prod b_{n_i})(A^\dagger A)^{\sum_i n_i + m/2}$.

The final step is to observe that $[(\gamma_3 A)^{-1}, (A^\dagger A)^n] = 0$ for any n; the RHS of (27) can therefore be manipulated to bring $\gamma_3 A$ to the left of all terms in the expansion, whereupon the γ_3 cancels in the expression (19) for the truncated overlap. Now using the fact that γ_5 has identical properties with respect to commutation with A, we can reverse all the steps to rewrite the truncated overlap operator

$$D_{Ls}[H] = \frac{1}{2}\left[(1+m_h) + (1-m_h)\gamma_5 \tanh(L_s \tanh^{-1}\gamma_5 A)\right]. \quad (30)$$

This establishes that the truncated overlap operator is equally blind to the distinction between γ_3 and γ_5 as the overlap (20).

4. Introducing $m_3, m_5 \neq 0$

In [7] we exploited the possibility of U(2)-rotating the fields leaving the kinetic term unaltered while changing the form of the mass term. In terms of continuum fields defined in $2+1d$ the alternative but physically equivalent, antihermitian but parity-invariant mass terms are $im_3\bar\psi\gamma_3\psi$, $im_5\bar\psi\gamma_5\psi$. In the domain wall approach (5) is replaced by one of

$$\begin{aligned} m_3 S_3 = im_3 \sum_x & \bar\Psi(x, L_s)\gamma_3 P_- \Psi(x, 1) \\ & + \bar\Psi(x, 1)\gamma_3 P_+ \Psi(x, L_s); \quad (31) \\ m_5 S_5 = im_5 \sum_x & \bar\Psi(x, L_s)\gamma_5 P_+ \Psi(x, L_s) \\ & + \bar\Psi(x, 1)\gamma_5 P_- \Psi(x, 1). \quad (32) \end{aligned}$$

First consider a mass term $m_3 S_3$. The matrix manipulations outlined in Sec. 2 leading to eqn. (7) go through as before, but with (9) replaced by

$$C_{3\pm} = P_\pm \pm im_3 P_\mp. \quad (33)$$

The Schur complement of $\tilde D = \mathcal{Q}^{-1} D \mathcal{P}$ is then

$$\mathcal{D}_{L_s,L_s}(m_h = 1)\frac{1}{2}\left[(1 + im_3\gamma_3) - \gamma_3 \frac{1-\mathcal{T}}{1+\mathcal{T}}(1 - im_3\gamma_3)\right], \quad (34)$$

implying a truncated overlap

$$D_{L_s} = \frac{1}{2}\left[(1 + im_3\gamma_3) - \gamma_3 \tanh(L_s \tanh^{-1}(\gamma_3 A))(1 - im_3\gamma_3)\right], \quad (35)$$

with A still given by (13). An important technical point is that the passage from domain wall to overlap requires the Pauli–Villars matrix $\mathcal{D}_{L_s,L_s}(1) = (1+\mathcal{T}^{-1})\gamma_3$ to continue to be defined with the *hermitian* mass term $1 \times S_h$. The overlap operator found in the limit $L_s \to \infty$ is thus

$$D_{ov} = \frac{1}{2}\left[(1 + im_3\gamma_3) + \frac{A}{\sqrt{A^\dagger A}}(1 - im_3\gamma_3)\right] \quad (36)$$

with A defined in (13). In the weak coupling long wavelength limit

$$D_{ov} \approx i\not{p}\frac{(1 - im_3\gamma_3)}{2M} + im_3\gamma_3. \quad (37)$$

This time there is an $O(a)$ term proportional to $\not{p}\gamma_3$ not present in the continuum action, which cannot be absorbed by wavefunction rescaling. It seems highly plausible that this lies at the heart of the $O(a)$ departures from U(2) symmetry observed when rotating fermion bilinears according to the remnant symmetries derived from the 3d Ginsparg–Wilson (GW) relations in Sec. 3 of [7].

Next consider the mass term $m_5 S_5$. Even though this term differs from the other masses by coupling fields on the same domain wall, rather than on opposite ones, the matrix manipulations of Sec. 2 still arrive at (7), with this time

$$C_{5\pm} = P_\pm - im_5\gamma_5 P_\pm = P_\pm - im_5 P_\mp \gamma_5, \quad (38)$$

where the second step is crucial. The truncated overlap in this case is

$$D_{L_s}[H] = \frac{1}{2}\left[(1 + im_5\gamma_5) - \gamma_3 \tanh(L_s \tanh^{-1} H)(1 - im_5\gamma_5)\right]; \quad (39)$$

however the considerations of Sec. 3 permit this to be rewritten

$$D_{L_s} = \frac{1}{2}\left[(1 + im_5\gamma_5) - \gamma_5 \tanh(L_s \tanh^{-1}(\gamma_5 A))(1 - im_5\gamma_5)\right]. \quad (40)$$

The complete equivalence between (40) and (35) is manifest.

5. Ginsparg–Wilson relations

Whilst the previous two sections have established the equivalence of the domain wall formulation with respect to a discrete interchange of the matrices γ_3 and γ_5, in order to study restoration of the full U($2N_f$) symmetry it is more convenient to examine the overlap operator. Following (20), (36), in the large-L_s limit we can write Lagrangian densities in terms of “Ginsparg–Wilson” fields $\Psi, \bar\Psi$:

$$\mathcal{L}_h = \bar\Psi[D^0_{ov} + m_h(1 - D^0_{ov})]\Psi; \quad (41)$$

$$\mathcal{L}_3 = \bar\Psi[D^0_{ov} + im_3(1 - D^0_{ov})\gamma_3]\Psi, \quad (42)$$

where D^0_{ov} is the overlap operator for massless fermions

$$D^0_{ov} = \frac{1}{2}\left[1 + \frac{A}{\sqrt{A^\dagger A}}\right]. \quad (43)$$

In both cases there is an $O(a)$ correction to the expected continuum form, but as noted above for the hermitian mass case (41) the correction can be absorbed into a harmless rescaling of the kinetic term. For the antihermitian case (42) by contrast the correction is not of the same form as a term in the continuum Lagrangian, as first noted in [7] (although (42) differs in detail from eq. (34) of that paper).

The reconciliation is made by first observing that the GW relation appropriate for the domain wall operator (2) is [8,13]

$$\gamma_3 D^0_{ov} + D^0_{ov}\gamma_3 = 2D^0_{ov}\gamma_3 D^0_{ov}. \quad (44)$$

As expected, there are further GW relations, first with γ_5 replacing γ_3 in (44), and also a rotation generated by $i\gamma_3\gamma_5$ which along with a simple global phase rotation completely specifies the U(2) [7]:

$$\gamma_5 D^0_{ov} + D^0_{ov}\gamma_5 = 2D^0_{ov}\gamma_5 D^0_{ov}; \quad \gamma_3\gamma_5 D^0_{ov} - D^0_{ov}\gamma_3\gamma_5 = 0. \quad (45)$$

We can further exploit $\gamma_3 D^0_{ov}\gamma_3 = \gamma_5 D^0_{ov}\gamma_5 = D^{0\dagger}_{ov}$ to express both non-trivial GW relations as $D^0_{ov} + D^{0\dagger}_{ov} = 2D^0_{ov}D^{0\dagger}_{ov}$ [10]. The associated symmetry in the massless limit is then [16,7]

$$\begin{aligned} \Psi &\mapsto e^{(i\alpha\gamma_3(1-D^0_{ov}))}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{(i\alpha(1-D^0_{ov})\gamma_3)} \\ \Psi &\mapsto e^{(i\alpha\gamma_5(1-D^0_{ov}))}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{(i\alpha(1-D^0_{ov})\gamma_5)} \\ \Psi &\mapsto e^{-\alpha\gamma_3\gamma_5}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{\alpha\gamma_3\gamma_5}. \quad (46) \end{aligned}$$

Strictly speaking, therefore, symmetry under global U(2) rotations of local fields is only recovered as $a \to 0$, under the assumption that the overlap operator D^0_{ov} is sufficiently localised in this limit.

Next, define projection operators as follows:

$$P_\pm = \frac{1}{2}(1 \pm \gamma_3); \quad \tilde{P}_\pm = P_\pm \mp D^0_{ov}\gamma_3 \tag{47}$$

with the property $\tilde{P}_\pm D^0_{ov} = D^0_{ov} P_\mp$ following from (44). With projected fields $\Psi_\pm = P_\pm \Psi$, $\bar{\Psi}_\pm = \bar{\Psi}\tilde{P}_\mp$, we can write

$$\mathcal{L}^0 = \bar{\Psi}_+ D^0_{ov}\Psi_+ + \bar{\Psi}_- D^0_{ov}\Psi_- = \bar{\Psi} D^0_{ov}\Psi; \tag{48}$$

$$m_h S_h^{GW} = m_h(\bar{\Psi}_-\Psi_+ + \bar{\Psi}_+\Psi_-) = m_h\bar{\Psi}(1 - D^0_{ov})\Psi; \tag{49}$$

$$m_3 S_3^{GW} = im_3(\bar{\Psi}_-\gamma_3\Psi_+ + \bar{\Psi}_+\gamma_3\Psi_-) = im_3\bar{\Psi}(1 - D^0_{ov})\gamma_3\Psi \tag{50}$$

consistent with (41), (42). The extension to the terms involving γ_5 is trivial [7].[2]

6. Bilinear condensates

The freedom to specify variants of the parity-invariant mass term can be exploited in the study of the corresponding bilinear condensates defined via

$$\langle\bar{\psi}\Gamma_i\psi\rangle = \frac{\partial \ln \mathcal{Z}}{\partial m_i} = \left\langle \mathrm{tr} M^{-1}\frac{\partial M}{\partial m_i}\right\rangle, \tag{51}$$

where $\det M$ is the part of the functional measure coming from the fermions. For a U(2)-invariant theory the condensates generated by the masses m_h, m_3, m_5 should all coincide, and indeed numerical evidence for this as $L_s \to \infty$ was presented for quenched non-compact QED$_3$ [7]. A particular useful result was that finite-L_s corrections are minimised by choosing the mass term antihermitian. We parametrise these in terms of residuals $\Delta_h, \epsilon_h, \epsilon_3, \epsilon_5$ which vanish exponentially as $L_s \to \infty$ by writing:

$$\begin{aligned}
\frac{1}{2}\langle\bar{\psi}\psi\rangle_{L_s} &= \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s\to\infty} + \Delta_h(L_s) + \epsilon_h(L_s);\\
\frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} &= \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s\to\infty} + \epsilon_3(L_s);\\
\frac{i}{2}\langle\bar{\psi}\gamma_5\psi\rangle_{L_s} &= \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s\to\infty} + \epsilon_5(L_s).
\end{aligned} \tag{52}$$

The numerically dominant residual is Δ_h, defined to be the imaginary component of $i\langle\bar{\psi}\gamma_3\psi\rangle$ evaluated on just the $+$ component of Ψ:

$$i\langle\bar{\Psi}(1)\gamma_3 P_+\Psi(L_s)\rangle = \frac{i}{2}\langle\bar{\psi}\gamma_3\psi\rangle_{L_s} + i\Delta_h(L_s). \tag{53}$$

The imaginary contribution from the Ψ_- component has opposite sign and hence cancels even for finite L_s.

In order to understand why Δ_h only contributes for the hermitian condensate, first consider the continuum case with $M = \not{D} + m_h$:

$$\begin{aligned}
\langle\bar{\psi}\psi\rangle = \mathrm{tr}(\not{D} + m_h)^{-1} &= \mathrm{tr}\frac{1}{\not{D}}\left[1 - \frac{m_h}{\not{D}} + \frac{m_h^2}{\not{D}^2} - \frac{m_h^3}{\not{D}^3} + \cdots\right]\\
&= -4\left[\frac{m_h}{D^2} + \frac{m_h^3}{D^4} + \cdots\right]
\end{aligned} \tag{54}$$

where we assume m_h is small enough to justify the binomial expansion. Since the trace over an odd number of gamma matrices is zero, all even powers of m_h vanish on taking the trace, which makes sense since $\langle\bar{\psi}\psi\rangle$ should be an odd function of m_h. The mass term $m_3 S_3$ yields the same series:

$$\begin{aligned}
i\langle\bar{\psi}\gamma_3\psi\rangle &= \mathrm{tr}(\not{D} + im_3\gamma_3)^{-1} i\gamma_3\\
&= \mathrm{tr}\left[1 - \frac{im_3}{\not{D}}\gamma_3 + \frac{m_3^2}{\not{D}^2} - \frac{im_h^3}{\not{D}^3}\gamma_3 + \cdots\right]\frac{1}{\not{D}}(i\gamma_3)\\
&= -4\left[\frac{m_3}{D^2} + \frac{m_3^3}{D^4} + \cdots\right]
\end{aligned} \tag{55}$$

where we have used $\gamma_3\not{D}\gamma_3 = -\not{D}$. This time the even powers vanish because they consist of products of an odd number of matrices γ_μ $(\mu = 0, 1, 2)$ with γ_3, so are proportional to either $\mathrm{tr}\gamma_\mu\gamma_3$ or $\mathrm{tr}\gamma_5$.

Now, for a theory with functional weight $\det D_{Ls}[H]$ the corresponding expression for $\langle\bar{\psi}\psi\rangle$ is

$$\begin{aligned}
\mathrm{tr}M^{-1}M' &= \mathrm{tr}[1 - \gamma_3\varepsilon_{L_s} + m_h(1 + \gamma_3\varepsilon_{L_s})]^{-1}[1 + \gamma_3\varepsilon_{L_s}]\\
&= \mathrm{tr}\frac{[1 + \gamma_3\varepsilon_{L_s}]}{[1 - \gamma_3\varepsilon_{L_s}]}\\
&\quad \times \left[1 - m_h\frac{[1 + \gamma_3\varepsilon_{L_s}]}{[1 - \gamma_3\varepsilon_{L_s}]} + m_h^2\frac{[1 + \gamma_3\varepsilon_{L_s}]^2}{[1 - \gamma_3\varepsilon_{L_s}]^2} + \cdots\right].
\end{aligned} \tag{56}$$

Here $\varepsilon_{L_s}[H] \equiv \tanh(L_s \tanh^{-1} H)$ is the finite-L_s approximation to the signum function. Now,

$$\begin{aligned}
\frac{1 + \gamma_3\varepsilon_{L_s}}{1 - \gamma_3\varepsilon_{L_s}} &= [1 - \gamma_3\varepsilon_{L_s}]^{-1}[1 + \varepsilon_{L_s}\gamma_3]^{-1}[1 + \varepsilon_{L_s}\gamma_3][1 + \gamma_3\varepsilon_{L_s}]\\
&= (1 - \varepsilon_{L_s}^2 - [\gamma_3, \varepsilon_{L_s}])^{-1}(1 + \varepsilon_{L_s}^2 + \{\gamma_3, \varepsilon_{L_s}\}).
\end{aligned} \tag{57}$$

In the limit $L_s \to \infty$, $\varepsilon_{L_s}^2 = 1$, and the long-wavelength weak coupling limit (21) gives

$$\lim_{L_s\to\infty}\{\gamma_3, \varepsilon_{L_s}\} = 2; \quad \lim_{L_s\to\infty}[\gamma_3, \varepsilon_{L_s}] = -\frac{2i\not{p}}{M}, \tag{58}$$

so (57) $\approx 2M/i\not{p}$ and we are on the right track. However, for finite L_s $1 - \varepsilon_{L_s}^2$ is a real quantity, and now there is no reason for the terms in (56) corresponding to even powers of m_h necessarily to vanish. Another way of saying this is that the form of A defining ε_{L_s} dictates that it is no longer the case that even powers of m_h are proportional to the trace over an odd number of gamma matrices. We conclude that the function $\langle\bar{\psi}\psi(m_h)\rangle$ in general contains an even component, labelled Δ_h in (52), weakly dependent on m_h as $m_h \to 0$ and only vanishing as $L_s \to \infty$.

Now repeat the exercise for the mass term $m_3 S_3$:

$$\begin{aligned}
\mathrm{tr}M^{-1}M' &= \mathrm{tr}[1 - \gamma_3\varepsilon_{L_s} + im_3\gamma_3(1 + \varepsilon_{L_s}\gamma_3)]^{-1} i\gamma_3[1 + \varepsilon_{L_s}\gamma_3]\\
&= \mathrm{tr}\left[1 + im_3\frac{[1 + \gamma_3\varepsilon_{L_s}]}{[1 - \gamma_3\varepsilon_{L_s}]}\gamma_3\right]^{-1}\frac{[1 + \gamma_3\varepsilon_{L_s}]}{[1 - \gamma_3\varepsilon_{L_s}]}(i\gamma_3)
\end{aligned} \tag{59}$$

Now, from (57) and the considerations of Sec. 3, all the terms in the binomial expansion of the first factor in (59) can only contain L_s dependence in terms of the form $(\gamma_3\varepsilon_{L_s})^p$, $\varepsilon_{L_s}^{2q}$ with p, q integer, which have the property that $\mathrm{tr}\gamma_3(\gamma_3\varepsilon_{L_s})^p = \mathrm{tr}\gamma_3(\varepsilon_{L_s})^{2q} = 0$. This implies that only odd powers of m_3 survive the trace. Hence $i\langle\bar{\psi}\gamma_3\psi(m_3)\rangle$ is an odd function of m_3, and the dominant residual Δ_h is necessarily absent. For finite L_s when the limiting forms (58) do not hold, we cannot exclude corrections which are odd functions of m_3, corresponding to the residual ϵ_3 in (52).

---

[2] Note that in order to recover the expressions for the antihermitian mass terms derived in [7] we should have chosen a matrix decomposition of $D(m_{3,5})$ with the projectors $P_\pm$ multiplying to the left rather than to the right as in (6).

Finally, the arguments of Sec. 3 then imply the identical property for the condensate $i\langle\bar\psi\gamma_5\psi\rangle$, consistent with the numerical results of [7].

7. Summary

In Sec. 2 we showed that the $2+1d$ domain wall fermion formulation introduced in [7] coincides with the overlap operator in the limit $L_s\to\infty$, and, importantly not simply in the continuum limit as suggested in the abstract of that paper. Whilst the Dirac matrices γ_3 and γ_5 enter the domain wall formulation (1) in very different ways, it was shown in Sec. 3 that the resulting $2+1d$ truncated overlap operator (19), (30) is blind to the distinction between them even for L_s finite. There seems to be no obstruction to modelling $\mathrm{U}(2N_f)\to\mathrm{U}(N_f)\otimes\mathrm{U}(N_f)$ symmetry breaking in lattice simulations of $2+1d$ fermions, so long as it is understood that the nature of the $a>0$ corrections to continuum symmetry operations, encapsulated in the GW relations (44), (45), and needed, say for identifying interpolating operators for Goldstone modes [7], is more complicated than for $4d$ gauge theories, as discussed in Sec. 5. In particular the antihermitian mass term (50) consistent with the GW relations contains an $\mathrm{O}(a)$ correction of a form not present in the continuum action. Ultimately, successful control of these corrections will depend on the locality properties of the overlap operator D_{ov} [17], which is a dynamical question.

On the other hand, the freedom to formulate alternative mass terms in $2+1d$ leads to a potentially important computational saving; as shown in Sec. 6, finite-L_s corrections to bilinear condensates may be classified by whether they are odd or even functions of the symmetry-breaking mass m_i, and the dominant even component Δ_h is absent for the antihermitian mass terms S_3, S_5, whose use in numerical simulations with finite L_s thus seems preferred, while recalling from Sec. 4 that the correct formulation of the Pauli–Villars bulk correction $\det D^{-1}_{L_s,L_s}(1)$ requires the hermitian mass $1\times S_h$.

Acknowledgements

This work was supported by a Royal Society Leverhulme Trust Senior Research Fellowship LT140052, and in part by STFC grant ST/L000369/1. I continue to benefit enormously from discussions with Tony Kennedy.

References

[1] B.A. Bernevig, T.L. Hughes, Topological Insulators and Topological Superconductors, Princeton University Press, 2013.
[2] D.T. Son, Phys. Rev. B 75 (2007) 235423.
[3] L. Janssen, H. Gies, Phys. Rev. D 86 (2012) 105007;
F. Gehring, H. Gies, L. Janssen, Phys. Rev. D 92 (8) (2015) 085046.
[4] C. Frick, J. Jersák, Phys. Rev. D 52 (1995) 340;
L. Del Debbio, S. Hands, J.C. Mehegan, Nucl. Phys. B 502 (1997) 269;
S. Christofi, S. Hands, C. Strouthos, Phys. Rev. D 75 (2007) 101701;
S. Chandrasekharan, A. Li, Phys. Rev. Lett. 108 (2012) 140404;
S. Chandrasekharan, A. Li, Phys. Rev. D 88 (2013) 021701;
V. Ayyar, S. Chandrasekharan, Phys. Rev. D 91 (6) (2015) 065035;
S. Catterall, arXiv:1510.04153 [hep-lat].
[5] D. Schmidt, B. Wellegehausen, A. Wipf, arXiv:1511.00522 [hep-lat].
[6] C. Burden, A.N. Burkitt, Europhys. Lett. 3 (1987) 545.
[7] S. Hands, J. High Energy Phys. 1509 (2015) 047.
[8] P.H. Ginsparg, K.G. Wilson, Phys. Rev. D 25 (1982) 2649.
[9] H. Neuberger, Phys. Lett. B 417 (1998) 141;
H. Neuberger, Phys. Lett. B 427 (1998) 353.
[10] R. Narayanan, J. Nishimura, Nucl. Phys. B 508 (1997) 371;
Y. Kikukawa, H. Neuberger, Nucl. Phys. B 513 (1998) 735;
W. Bietenholz, J. Nishimura, J. High Energy Phys. 0107 (2001) 015.
[11] D.B. Kaplan, Phys. Lett. B 288 (1992) 342.
[12] H. Neuberger, Phys. Rev. D 57 (1998) 5417.
[13] A.D. Kennedy, Algorithms for dynamical fermions, arXiv:hep-lat/0607038.
[14] Y. Shamir, Nucl. Phys. B 406 (1993) 90.
[15] P. Vranas, I. Tziligakis, J.B. Kogut, Phys. Rev. D 62 (2000) 054507.
[16] M. Lüscher, Phys. Lett. B 428 (1998) 342.
[17] P. Hernandez, K. Jansen, M. Lüscher, Nucl. Phys. B 552 (1999) 363.