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The $\mathcal{N}$=4 Supergravity NMHV six-point one-loop amplitude

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Abstract

We construct the six-point NMHV one-loop amplitude in $\mathcal{N} = 4$ supergravity using unitarity and recursion. The use of recursion requires the introduction of rational descendants of the cut-constructible pieces of the amplitude and the computation of the non-standard factorisation terms arising from the loop integrals.

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I. INTRODUCTION

Despite perturbative Quantum Gravity being a mature subject [1], it is a very challenging area computationally. Although great strides have been made in computing tree amplitudes [2], there remain a very limited number of loop calculations available to study. For the four and five point amplitudes all one-loop graviton scattering amplitudes have now been calculated [3–9]. For Maximal supergravity great progress has also be made at multi-loop level for the four point amplitude [10–12]. These computations are by necessity amplitudes which are "Maximally-Helicity-Violating" (MHV). MHV amplitudes are very special and have many features not shared by non-MHV amplitudes. In this article we compute the six graviton "Next-to-MHV" (NMHV) scattering amplitude for $\mathcal{N}=4$ supergravity. (The first NMHV amplitude appears at six-points.) The six-graviton scattering amplitude has been computed for $\mathcal{N}=8$ and $\mathcal{N}=6$ supergravity.

This amplitude has considerable algebraic complexity relative to the more supersymmetric cases including the appearance of rational terms. We construct the $M_6^{\mathcal{N}=4}(a^-,b^-,c^-,d^+,e^+,f^+)$[1] amplitude using unitarity and recursion augmented by limited off-shell behaviour.

One-loop amplitudes in a massless theory can be expressed as [13]

$$M_n^{\text{1-loop}} = \sum_{i\in\mathcal{C}} a_i\, I_4^i + \sum_{j\in\mathcal{D}} b_j\, I_3^j + \sum_{k\in\mathcal{E}} c_k\, I_2^k + R_n + O(\epsilon), \tag{1.1}$$

where the I_r^i are r-point scalar integral functions and the a_i etc. are rational coefficients. R_n is a purely rational term. The box, triangle and bubble coefficients can be determined via unitarity methods [13–17] using four dimensional on-shell tree amplitudes. These contributions are termed cut-constructible. Progress has been made both via the two-particle cuts [13, 15, 18] and using generalisations of unitarity [16] where, for example, triple [19–22] and quadruple cuts [17] are utilised to identify the triangle and box coefficients respectively. The remaining purely rational term, R_n, may in principle be obtained using unitarity but this requires a knowledge of $4-2\epsilon$ dimensional tree amplitudes [23]. In this paper a recursive approach is adopted that generates the rational term from four dimensional amplitudes.

An important technique for computing tree amplitudes is Britto-Cachazo-Feng-Witten (BCFW) [24] recursion which applies complex analysis to amplitudes. By shifting the momenta[2]

$$\bar\lambda_a \to \bar\lambda_a - z\bar\lambda_d \quad , \qquad \lambda_d \to \lambda_d + z\lambda_a \ , \tag{1.2}$$

the resultant amplitude $M_n(z)$ may be computed via Cauchy's theorem. Loop amplitudes, as functions of complexified momentum, contain both poles and cuts so BCFW does not immediately apply to these. However by defining

$$R_n = M_n - M_n^{cc} \ , \tag{1.3}$$

where M_n^{cc} is the cut-constructible part of the amplitude, we can compute the purely rational R_n from a knowledge of its singularities. R_n has singularities corresponding to the poles of

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[1] We use the normalisation for the full physical amplitudes $\mathcal{M}^{\text{tree}} = i(\kappa/2)^{n-2}M^{\text{tree}}$, $\mathcal{M}^{\text{1-loop}} = i(2\pi)^{-2}(\kappa/2)^n M^{\text{1-loop}}$.

[2] As usual, a null momentum is represented as a pair of two component spinors $p^\mu = \sigma^\mu_{\alpha\dot\alpha}\lambda^\alpha\bar\lambda^{\dot\alpha}$. For real momenta $\lambda = \pm\bar\lambda^*$ but for complex momenta λ and $\bar\lambda$ are independent [25]. We are using a spinor helicity formalism with the usual spinor products $\langle a\, b\rangle = \epsilon_{\alpha\beta}\lambda_a^\alpha\lambda_b^\beta$ and $[a\, b] = -\epsilon_{\dot\alpha\dot\beta}\bar\lambda_a^{\dot\alpha}\bar\lambda_b^{\dot\beta}$.

Helicity	2	3/2	1	1/2	0	−1/2	−1	−3/2	−2
graviton	1	4	6	4	2	4	6	4	1
matter	0	0	1	4	6	4	1	0	0

TABLE I: Particle content of the multiplets

the amplitude but also induced singularities arising because M_n^{cc} has singularities which are not present in the amplitude and which must be cancelled by equal and opposite singularities in R_n. We refer to these contributions as the *rational descendants* of M_n^{cc}.

The particle content of the $\mathcal{N}=4$ graviton and matter multiplets are shown in table I. For convenience, we will calculate the one-loop amplitude using the $\mathcal{N}=4$ matter multiplet, which is related to the amplitude with the graviton multiplet circulating in the loop by

$$M_n^{\mathcal{N}=4,\text{graviton}} = M_n^{\mathcal{N}=8} - 4M_n^{\mathcal{N}=6,\text{matter}} + 2M_n^{\mathcal{N}=4,\text{matter}}. \tag{1.4}$$

The $\mathcal{N}=8$ and $\mathcal{N}=6$ components are considerably simpler and given in [7, 26]. In particular $R_n=0$ for these two components.

II. CUT-CONSTRUCTIBLE PIECES

The cut-constructable pieces consist of box-functions, triangle functions and bubble integral functions. The analytic form of these depends upon how many of the external legs have non-null momentum: these are referred to as massive legs although non-null is more correct. For the one-mass box the fourth leg is conventionally the massive leg and the integral function depends upon $S \equiv (k_1+k_2)^2$, $T \equiv (k_2+k_3)^2$ and $M_4^2 \equiv K_4^2$. For the two-mass boxes there are two types: "two-mass-easy" ($2me$) where legs 2 and 4 are massive and "two-mass-hard" ($2mh$) where legs 3 and 4 are massive. For six-point amplitudes the three and four mass boxes do not appear and for the NMHV $M_6^{\mathcal{N}=4:\text{matter}}$ amplitude there are no two-mass-easy boxes.

The various box, triangle and bubble contributions present in the six-point NMHV are shown in fig. 1 together with the labelling of helicities which yield a non-zero coefficient. Permuting the positive and negative helicity legs separately gives eighteen one-mass boxes, thirty-six two-mass hard boxes, nine 2:4 bubbles, nine 3:3 bubbles and six three-mass triangles.

A. IR consistency and Choice of Integral Function Basis

For one-loop amplitudes Infra-Red (IR) consistency imposes a system of constraints on the rational coefficients of the integral functions. For the matter multiplets [27] there are in fact no IR singular terms in the amplitude, so the singular terms in the individual integral functions cancel. This gives enough information to fix the coefficients of the one- and two-mass triangles in terms of the box coefficients. The three-mass triangle is IR finite, so its coefficient is not determined by these constraints. It is convenient to combine the boxes and triangles in such a way that these infinities are manifestly absent. There are several ways to

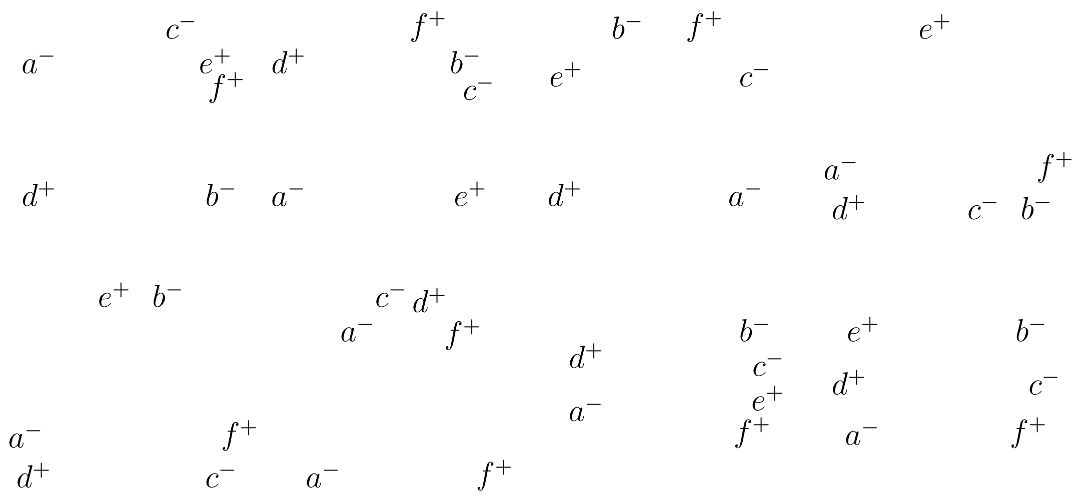

FIG. 1: The Integral functions appearing in the six-point NMHV amplitude

do this [15, 19, 28–30], here we choose to work with truncated box functions,

$$I_4^{\rm trunc} = I_4 - \sum_i \alpha_i \frac{(-K_i^2)^{-\epsilon}}{\epsilon^2}\,, \tag{2.1}$$

where the α_i and K_i^2 are chosen to make $I_4^{\rm trunc}$ IR finite. This effectively incorporates the one- and two-mass triangle contributions into the box contributions. Using these truncated boxes the amplitudes can be written as

$$M_n^{\text{1-loop}} = \sum_{i\in\mathcal{C}} a_i\, I_4^{i,\rm trunc} + \sum_{j\in\mathcal{D}'} b_j\, I_3^{j,\text{3-mass}} + \sum_{k\in\mathcal{E}} c_k\, I_2^k + R_n\,. \tag{2.2}$$

There is one remaining IR consistency constraint: $\sum c_k = 0$.

The one-mass and two-mass-hard truncated integral functions are

$$\begin{aligned}
I_4^{\rm 1m,trunc} &= -\frac{2r_\gamma}{ST}\left(\mathrm{Li}_2\left(1-\frac{K_4^2}{S}\right) + \mathrm{Li}_2\left(1-\frac{K_4^2}{T}\right) + \frac{1}{2}\ln^2\left(\frac{-S}{-T}\right) + \frac{\pi^2}{6}\right),\\
I_4^{\rm 2mh,trunc} &= -\frac{2r_\gamma}{ST}\Bigg(-\frac{1}{2}\ln\left(K_3^2/S\right)\ln\left(K_4^2/S\right) + \frac{1}{2}\ln^2\left(S/T\right)\\
&\qquad + \mathrm{Li}_2\left(1-\frac{K_3^2}{T}\right) + \mathrm{Li}_2\left(1-\frac{K_4^2}{T}\right)\Bigg),
\end{aligned} \tag{2.3}$$

where $S=(k_1+k_2)^2$ and $T=(k_1+K_4)^2$.

B. Boxes

The six-pt NMHV amplitude contains both one-mass and two-mass hard boxes. The coefficients of these boxes are readily obtained using quadruple cuts [17].

The coefficient of the first one-mass box in fig. 1 is

$$a^{\rm 1m[\mathcal{N}=4]}_{a,d,b,\{c,e,f\}} = \frac{1}{2}\frac{s_{ad}^2 s_{bd}^2 [d|K|c\rangle^4 [b|K|c\rangle [a|K|c\rangle}{[a\,b]^4\,\langle c\,e\rangle\, K^2 [b|K|f\rangle [a|K|f\rangle [b|K|e\rangle} \times \left(\frac{[cb][ef]}{\langle ef\rangle} - \frac{[eb][cf][a|K|c\rangle}{[a|K|e\rangle\langle cf\rangle}\right), \tag{2.4}$$

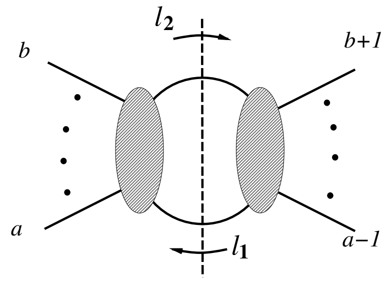

FIG. 2: A two-particle cut of a one-loop amplitude

where $K = K_{cef}$. The other one-mass box coefficients can be obtained from this by conjugation and relabelling. The coefficient of the two-mass-hard box is

$$a^{\text{2mh}[\mathcal{N}=4]}_{a,d,\{b,e\},\{c,f\}} = -\frac{1}{2}\frac{s_{ad}^2 K_{cfa}^2 [f|K_{cfa}|b\rangle^4}{[a|K_{cfa}|d\rangle^4}\frac{[be]\langle cf\rangle\langle bd\rangle[a|K_{cf}|b\rangle}{\langle be\rangle[cf]\langle ed\rangle[a|K_{cf}|e\rangle}\frac{[af][f|K_{cfa}|d\rangle}{[ca][c|K_{cfa}|d\rangle} \tag{2.5}$$

and the other coefficients are obtained by relabelling. The expressions for the box coefficients have the appropriate symmetries under exchange of external legs. Specifically the two-mass hard is invariant under the joint operation of $(a, b, c) \leftrightarrow (d, f, e)$ and conjugation. The one-mass coefficient is invariant under $a \leftrightarrow b$ although this is not manifest (and similarly under $e \leftrightarrow f$).

C. Canonical basis approach for triangle and bubble coefficients

The canonical basis approach [18] is a systematic method to determine the coefficients of triangle and bubble integral functions from the three and two-particle cuts. The two particle cut is shown in fig. 2 and is

$$C_2 \equiv i\int d^4\ell\,\delta(\ell_1^2)\delta(\ell_2^2)A^{\text{tree}}(-\ell_1, a, \cdots, b, \ell_2)\times A^{\text{tree}}(-\ell_2, \cdots, \ell_1)\,. \tag{2.6}$$

The product of tree amplitudes appearing in the two-particle cut can be decomposed in terms of canonical forms $\mathcal{F}_i$,

$$A^{\text{tree}}(-\ell_1, \cdots, \ell_2)\times A^{\text{tree}}(-\ell_2, \cdots, \ell_1) = \sum e_i\mathcal{F}_i(\ell_j), \tag{2.7}$$

where the e_i are coefficients independent of ℓ_j. We then use substitution rules to replace the $\mathcal{F}_i(\ell_j)$ by the rational functions $F_i(K)$ to obtain the coefficient of the bubble integral function as

$$\sum_i e_i F_i(K)\,. \tag{2.8}$$

Similarly, we can obtain the coefficient of the triangle functions from the triple cut [18, 20, 21] as shown in figure 3,

$$\begin{aligned} C_3 = -\int d^4\ell\,\delta(\ell_0^2)\delta(\ell_1^2)\delta(\ell_2^2)A^{\text{tree}}(-\ell_0, \cdots, \ell_1) \\ \times A^{\text{tree}}(-\ell_1, \cdots, \ell_2)\times A^{\text{tree}}(-\ell_2, \cdots, \ell_0)\,. \end{aligned} \tag{2.9}$$

The product of tree amplitudes can, again, be expressed in terms of standard forms of ℓ_i,

$$A^{\rm tree}(-\ell_0,\cdots,\ell_1)\times A^{\rm tree}(-\ell_1,\cdots,\ell_2)\times A^{\rm tree}(-\ell_2,\cdots,\ell_0)=\sum_i e_i\mathcal{E}_i(\ell_j) \tag{2.10}$$

and substitution rules used to replace the $\mathcal{E}_i(\ell_j)$ by the functions $E_i(K_j)$ to obtain the triangle coefficient as

$$\sum_i e_i E_i(K_j)\,. \tag{2.11}$$

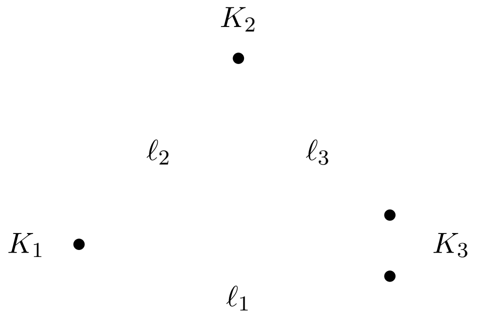

FIG. 3: Triple Cut

In general the integrands are rational functions of $\lambda(\ell_i)$ of degree $d = d_{num} - d_{denom}$. The simplest canonical forms have $d_{denom} = 1$ or 2. More complex denominators can be expressed in terms of the simplest forms by partial fractioning. Terms in the integrand with $d < 0$ only contribute to higher point integral functions. The degree generally decreases with increasing supersymmetry and for maximally supersymmetric Yang-Mills and supergravity there are no triangles. With increasing d the canonical forms become increasingly complex.

D. Triangles

Using the truncated box functions, we only need to compute the coefficient of the (IR finite) three-mass triangle: $b^{3m}_{\{a^-,d^+\}\{b^-,e^+\}\{c^-,f^+\}}$. Using the Kawai-Lewellen-Tye (KLT) relations [31], with a scalar circulating in the loop the cut integrand is

$$\begin{aligned}
C_3^0 = M(\ell_1,&a^-,d^+,-\ell_2)M(\ell_2,b^-,e^+,-\ell_3)M(\ell_3,c^-,f^+,-\ell_1)\\
&= s_{ad}A(\ell_1,a,d,\ell_2)A(\ell_1,d,a,\ell_2)s_{be}A(\ell_2,b,e,\ell_3)A(\ell_2,e,b,\ell_3)\\
&\quad\times s_{cf}A(\ell_3,c,f,\ell_1)A(\ell_3,f,c,\ell_1)\\
&= s_{ad}s_{be}s_{cf}C_3^{YM,0}(a,d;b,e;c,f)C_3^{YM,0}(d,a;e,b;f,c)\,.
\end{aligned} \tag{2.12}$$

The four-point Yang-Mills amplitudes above are simultaneously MHV and $\overline{\rm MHV}$ amplitudes, so there is a choice of two expressions for each. For algebraic convenience we take a mixed form for the first ordering:

$$C_3^{YM,0}(a,d;b,e;c,f)=\frac{\langle a\,l_1\rangle\,\langle a\,l_2\rangle^2}{\langle a\,d\rangle\,\langle d\,l_2\rangle\,\langle l_1\,l_2\rangle}\times\frac{[e\,l_3]\,[e\,l_2]^2}{[b\,e]\,[b\,l_2]\,[l_2\,l_3]}\times\frac{\langle c\,l_3\rangle\,\langle c\,l_1\rangle^2}{\langle c\,f\rangle\,\langle f\,l_1\rangle\,\langle l_3\,l_1\rangle} \tag{2.13}$$

and a purely MHV form for the second:

$$C_3^{YM,0}(d,a;e,b;f,c) = \frac{\langle a\, l_2\rangle\, \langle a\, l_1\rangle^2}{\langle a\, d\rangle\, \langle d\, l_1\rangle\, \langle l_1\, l_2\rangle} \times \frac{\langle b\, l_3\rangle\, \langle b\, l_2\rangle^2}{\langle b\, e\rangle\, \langle e\, l_2\rangle\, \langle l_2\, l_3\rangle} \times \frac{\langle c\, l_1\rangle\, \langle c\, l_3\rangle^2}{\langle c\, f\rangle\, \langle f\, l_3\rangle\, \langle l_3\, l_1\rangle}\,. \tag{2.14}$$

The contribution to the cut from a particle of helicity h circulating in the loop is given by

$$C_3^0 \times X^{2h}\,, \tag{2.15}$$

so that summing over the $\mathcal{N}=4$ matter multiplet gives

$$C_3^0 \times \left(X^{-2} - 4X^{-1} + 6 - 4X + X^2\right) = C_3^0 \times \frac{(1-X)^4}{X^2} = C_3^0 \times \rho^2\,. \tag{2.16}$$

The X-factor can be written in several ways. The natural choices given (2.13) and (2.14) are

$$X_1 = \frac{\langle a\ell_1\rangle}{\langle a\ell_2\rangle} \frac{\langle b\ell_2\rangle}{\langle b\ell_3\rangle} \frac{\langle c\ell_3\rangle}{\langle c\ell_1\rangle} \quad \text{and} \quad X_2 = \frac{\langle a\ell_1\rangle}{\langle a\ell_2\rangle} \frac{[d\,\ell_3]}{[d\,\ell_2]} \frac{\langle c\ell_3\rangle}{\langle c\ell_1\rangle}\,, \tag{2.17}$$

with the corresponding ρ-factors being

$$\rho_1 = \frac{\langle \ell_1\, Y\rangle^2\, \langle e\, b\rangle^2}{\langle a\ell_1\rangle\langle a\ell_2\rangle\langle b\ell_2\rangle\langle b\ell_3\rangle\langle c\ell_3\rangle\langle c\ell_1\rangle\, [\ell_2\, \ell_3]^2} \quad \text{and} \quad \rho_2 = \frac{\langle \ell_1\, Y\rangle^2}{\langle a\ell_1\rangle\langle a\ell_2\rangle\, [e\,\ell_3]\, [e\,\ell_2]\, \langle c\ell_3\rangle\langle c\ell_1\rangle}\,, \tag{2.18}$$

where

$$|Y\rangle = |a\rangle[e|f|c\rangle + |c\rangle[e|d|a\rangle\,. \tag{2.19}$$

The cut is then

$$C_3^{\mathcal{N}=4} = s_{ad}s_{be}s_{cf}C_3^{YM,0}(a,d;b,e;c,f) \times C_3^{YM,0}(d,a;e,b;f,c) \times \rho_1 \times \rho_2\,. \tag{2.20}$$

Using

$$\frac{\langle x\,\ell_2\rangle}{\langle y\,\ell_2\rangle} = \frac{\langle x\,\ell_2\rangle\,[\ell_2\,\ell_3]\,\langle \ell_3\,\ell_1\rangle}{\langle y\,\ell_2\rangle\,[\ell_2\,\ell_3]\,\langle \ell_3\,\ell_1\rangle} = \frac{\langle x|K_2K_3|\ell_1\rangle}{\langle y|K_2K_3|\ell_1\rangle} \quad , \quad \frac{\langle x\,\ell_3\rangle}{\langle y\,\ell_3\rangle} = \frac{\langle x\,\ell_3\rangle\,[\ell_3\,\ell_2]\,\langle \ell_2\,\ell_1\rangle}{\langle y\,\ell_3\rangle\,[\ell_3\,\ell_2]\,\langle \ell_2\,\ell_1\rangle} = \frac{\langle x|K_2K_1|\ell_1\rangle}{\langle y|K_2K_1|\ell_1\rangle} \tag{2.21}$$

and

$$\langle e\,\ell_3\rangle\,[b\,\ell_3]\,\langle \ell_1\,\ell_2\rangle\,[\ell_2\,\ell_3]\,\langle \ell_3\,\ell_1\rangle = [b|\ell_3|\ell_1\rangle\langle e|\ell_3\ell_2|\ell_1\rangle = [b|K_3|\ell_1\rangle\langle e|K_2K_1|\ell_1\rangle \tag{2.22}$$

we find

$$\begin{aligned} C_3^{\mathcal{N}=4} &= \frac{\langle b\,e\rangle\,[a\,d]\,[c\,f]}{[b\,e]\,\langle a\,d\rangle\,\langle c\,f\rangle} \frac{\langle 1\,\ell_1\rangle\,\langle 1|K_2K_3|\ell_1\rangle}{\langle d\,\ell_1\rangle\,\langle d|K_2K_3|\ell_1\rangle} \times \frac{[e\,\ell_2]\,\langle b\,\ell_2\rangle}{[b|K_3|\ell_1\rangle\langle e|K_2K_1|\ell_1\rangle} \times \frac{\langle c\,\ell_1\rangle\,\langle c|K_2K_1|\ell_1\rangle}{\langle f\,\ell_1\rangle\,\langle f|K_2K_1|\ell_1\rangle} \times \frac{\langle Y\,\ell_1\rangle^4}{\langle \ell_1|K_1K_2|\ell_1\rangle} \\ &= \frac{\langle b\,e\rangle\,[a\,d]\,[c\,f]}{[b\,e]\,\langle a\,d\rangle\,\langle c\,f\rangle} \prod_{i=1}^{6} \frac{\langle \alpha_i\,\ell_1\rangle}{\langle A_i\,\ell_i\rangle} \times \frac{\langle Y\,\ell_1\rangle^2}{\langle \ell_1|K_1K_2|\ell_1\rangle} \times \left([e\,\ell_1]\,\langle b\,\ell_1\rangle + [e|K_1|b\rangle\right)\,, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \{|\alpha_i\rangle\} &= \{|d\rangle, |f\rangle, K_3|b], K_1K_2|e\rangle, K_3K_2|d\rangle, K_1K_2|f\rangle\}\,, \\ \{|B_i\rangle\} &= \{|a\rangle, |c\rangle, |Y\rangle, |Y\rangle, K_1K_2|c\rangle, K_3K_2|a\rangle\}\,. \end{aligned} \tag{2.24}$$

Now we may *partial fraction* the expression, i.e. use the identity

$$\prod_{i=1}^{6} \frac{\langle \alpha_i\, \ell_1\rangle}{\langle A_i\, \ell_1\rangle} = \sum_{i=1}^{6} C_{A_i} \frac{\langle \alpha_6\, \ell_1\rangle}{\langle A_i\, \ell_1\rangle} \tag{2.25}$$

with

$$C_{A_i} = \frac{\prod_{j=1}^{5} \langle \alpha_j\, A_i\rangle}{\prod_{j\neq i} \langle A_j\, A_i\rangle}\,, \tag{2.26}$$

so that the cut is written as a sum of canonical forms:

$$\mathcal{J}_1^0[d;a,b,c;\ell] = \frac{\langle \ell\, a\rangle\, \langle \ell\, b\rangle\, \langle \ell\, c\rangle}{\langle \ell|K_1K_2|\ell\rangle\, \langle \ell\, d\rangle} \quad \text{and} \quad \mathcal{J}_1^1[f;b,c,d,e;B;\ell] = \frac{[B|\ell|b\rangle\, \langle \ell\, c\rangle\, \langle \ell\, d\rangle\, \langle \ell\, e\rangle}{\langle \ell|K_1K_2|\ell\rangle\, \langle \ell\, f\rangle}\,, \tag{2.27}$$

which have corresponding canonical functions

$$J_1^0[d;a,b,c;\{K_j\}] = \frac{\langle b|[K_1,K_2]|d\rangle\langle c|[K_1,K_2]|a\rangle + \Delta_3\, \langle b\, d\rangle\, \langle c\, a\rangle}{2\Delta_3\langle d|K_1K_2|d\rangle} - \frac{\langle d\, b\rangle\, \langle d\, c\rangle\, \langle a|[K_1,K_2]|d\rangle}{2\langle d|K_1K_2|d\rangle^2} \tag{2.28}$$

and

$$\begin{aligned} J_1^1[f;b,c,d,e;B;\{K_j\}] = & \sum_{P_{12}} \frac{[B|a_0|b\rangle\langle f|[K_1,K_2]|c\rangle\langle d|[K_1,K_2]|e\rangle}{4\Delta_3\langle f|K_1K_2|f\rangle} \\ & - \sum_{P_6} \frac{[B|a_0|f\rangle\langle b|[K_1,K_2]|c\rangle\langle d|[K_1,K_2]|e\rangle}{24\Delta_3\langle f|K_1K_2|f\rangle} \\ & - \sum_{P_6} \frac{[B|a_0|f\rangle\, \langle f\, b\rangle\, \langle f\, c\rangle\, \langle d|[K_1,K_2]|e\rangle}{12\langle f|K_1K_2|f\rangle^2} \\ & - \sum_{P_4} \frac{[B|a_0|b\rangle\langle f|[K_1,K_2]|c\rangle\langle f|[K_1,K_2]|d\rangle\langle f|[K_1,K_2]|e\rangle}{4\Delta_3\langle f|K_1K_2|f\rangle^2} \\ & - \sum_{P_6} \frac{[B|a_0|f\rangle\, \langle f\, b\rangle\, \langle f\, c\rangle\, \langle f|[K_1,K_2]|d\rangle\langle f|[K_1,K_2]|e\rangle}{12\langle f|K_1K_2|f\rangle^3}\,, \end{aligned} \tag{2.29}$$

where

$$\Delta_3 = 4(K_1{\cdot}K_2)^2 - 4K_1^2K_2^2 = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2(K_1^2K_2^2 + K_2^2K_3^2 + K_3^2K_1^2)\,, \tag{2.30}$$

$$a_0^\mu = \frac{-K_3^2(K_1^2+K_2^2-K_3^2)}{\Delta_3} K_1^\mu + \frac{K_1^2(K_3^2+K_2^2-K_1^2)}{\Delta_3} K_3^\mu \tag{2.31}$$

and P_n is the set of n permutations of $\{b,c,d,e\}$ necessary to generate symmetry in these variables.

We obtain the coefficient

$$\begin{aligned} b_{\{a,d\},\{b,e\},\{c,f\}} = & \frac{\langle b\, e\rangle\, [a\, d]\, [c\, f]}{[b\, e]\, \langle a\, d\rangle\, \langle c\, f\rangle} \times \sum_{i=1}^{6} C_{A_i} \Big(J_1^1[A_i;Y,Y,\alpha_6,b,e;\{K_i\}] \\ & - [e|K_1|b\rangle J_1^0[A_i;Y,Y,\alpha_6;\{K_i\}] \Big)\,. \end{aligned} \tag{2.32}$$

The corresponding three mass triangle integral function is [32, 33]

$$I_3^{3m} = \frac{i}{\sqrt{-\Delta_3}} \sum_{j=1}^{3} \left[\mathrm{Li}_2\left(-\left(\frac{1+i\delta_j}{1-i\delta_j} \right) \right) - \mathrm{Li}_2\left(-\left(\frac{1-i\delta_j}{1+i\delta_j} \right) \right) \right] + \mathcal{O}(\epsilon)\,, \tag{2.33}$$

where

$$\delta_1 = \frac{K_1^2 - K_2^2 - K_3^2}{\sqrt{-\Delta_3}}\,, \delta_2 = \frac{K_2^2 - K_1^2 - K_3^2}{\sqrt{-\Delta_3}} \quad \text{and} \quad \delta_3 = \frac{K_3^2 - K_1^2 - K_2^2}{\sqrt{-\Delta_3}}\,. \tag{2.34}$$

III. BUBBLES

The final cut constructible pieces are the bubbles integral functions $I_2(K^2)$. There are two distinct types of bubbles coefficients depending upon whether K is a two particle momentum, $K_{ij}^2 = (k_i + k_j)^2$, or three particle, $K_{ijk}^2 = (k_i + k_j + k_k)^2$. In the three particle case the cut only involves MHV tree amplitudes whereas the two particle case requires the NMHV tree amplitude.

For the K_{ade}^2 bubble the cut integrand with a scalar circulating in the loop is

$$\begin{aligned} C_2^0 &= M_5^{\text{tree}}(\ell_1, d^+, e^+, a^-, -\ell_2) \times M_5^{\text{tree}}(\ell_2, b^-, c^-, f^+, -\ell_1) \\ &= \Big(s_{\ell_1 d} s_{ea} A_5^{\text{tree}}(\ell_1, d, e, a, \ell_2) A_5^{\text{tree}}(d, \ell_1, a, e, \ell_2) + s_{\ell_1 e} s_{da} A_5^{\text{tree}}(\ell_1, e, d, a, \ell_2) A_5^{\text{tree}}(e, \ell_1, a, d, \ell_2) \Big) \\ &\times \Big(s_{\ell_2 b} s_{cf} A_5^{\text{tree}}(\ell_2, b, c, f, \ell_1) A_5^{\text{tree}}(b, \ell_2, f, c, \ell_1) + s_{\ell_2 c} s_{bf} A_5^{\text{tree}}(\ell_2, c, b, f, \ell_1) A_5^{\text{tree}}(c, \ell_2, f, b, \ell_1) \Big) \\ &= C_2^{p:0}(a,b,c,d,e,f) + C_2^{p:0}(a,b,c,e,d,f) + C_2^{p:0}(a,c,b,d,e,f) + C_2^{p:0}(a,c,b,e,d,f)\,, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} C_2^{p:0}(a,b,c,d,e,f) =& s_{\ell_1 d} s_{ea} A_5^{\text{tree}}(\ell_1, d, e, a, \ell_2) A_5^{\text{tree}}(d, \ell_1, a, e, \ell_2) \\ & \times s_{\ell_2 b} s_{cf} A_5^{\text{tree}}(\ell_2, b, c, f, \ell_1) A_5^{\text{tree}}(b, \ell_2, f, c, \ell_1) \\ =& \frac{\langle a\ell_1\rangle^3 \langle a\ell_2\rangle^3 [f\ell_1]^3 [f\ell_2]^3}{\langle de\rangle [bc] K^2} \times \left(\frac{[\ell_1 d][ea]}{\langle d\ell_1\rangle\langle ae\rangle\langle e\ell_2\rangle\langle \ell_2 d\rangle} \right) \times \left(\frac{\langle \ell_2 b\rangle\langle cf\rangle}{[b\ell_2][fc][c\ell_1][\ell_1 b]} \right). \end{aligned} \tag{3.2}$$

As in the three-mass triangle, the contribution from a particle of helicity h is $C_2^0 \times X^{2h}$. Summing over the multiplet has the effect of multiplying by ρ^2. For this cut

$$X = \frac{\langle a\ell_1\rangle}{\langle a\ell_2\rangle} \frac{[f\ell_1]}{[f\ell_2]} \quad \rightarrow \quad \rho = \frac{[f|K|a\rangle^2}{\langle a\ell_1\rangle\langle a\ell_2\rangle[f\ell_1][f\ell_2]}\,. \tag{3.3}$$

Multiplying $C_2^{p:0}$ by ρ^2 gives

$$\begin{aligned} C_2^{p:\mathcal{N}=4}(a,b,c,d,e,f) =& \frac{\langle a\ell_1\rangle\langle a\ell_2\rangle[f\ell_1][f\ell_2][f|K|a\rangle^4}{\langle de\rangle[bc]K^2} \\ & \times \left(\frac{[\ell_1 d][ea]}{\langle d\ell_1\rangle\langle ae\rangle\langle e\ell_2\rangle\langle \ell_2 d\rangle} \right) \left(\frac{\langle \ell_2 b\rangle\langle cf\rangle}{[b\ell_2][fc][c\ell_1][\ell_1 b]} \right). \end{aligned} \tag{3.4}$$

Using

$$\frac{\langle x\,\ell_2\rangle}{\langle y\,\ell_2\rangle}=\frac{\langle x|K|\ell_1]}{\langle y|K|\ell_1]}\quad\text{and}\quad\frac{[x\,\ell_2]}{[y\,\ell_2]}=\frac{[x|K|\ell_1\rangle}{[y|K|\ell_1\rangle}\tag{3.5}$$

we find

$$C_2^{p:\mathcal{N}=4}=\frac{[f|K|a\rangle^4\,[a\,e]\,\langle c\,f\rangle}{\langle de\rangle[bc]\,\langle a\,e\rangle\,[c\,f]\,K^2}\times\frac{\prod_{i=1}^{2}\langle\alpha_i\,\ell_1\rangle}{\prod_{i=1}^{2}\langle A_i\,\ell_1\rangle}\times\frac{\prod_{j=1}^{4}[\beta_j\,\ell_1]}{\prod_{j=1}^{4}[B_j\,\ell_1]}\,,\tag{3.6}$$

where

$$\begin{gathered}\{|\alpha_i\rangle\}=\{|a\rangle,K|f]\}\ \ ,\ \{|A_i\rangle\}=\{|d\rangle,K|b]\}\,,\\ \{|\beta_j]=|d],|f],|K|a\rangle,|K|b\rangle\}\ \ ,\ \{|B_j]\}=\{|b],|c],|K|d\rangle,|K|e\rangle\}\,.\end{gathered}\tag{3.7}$$

Partial fractioning on $|\ell_1\rangle$ and $|\ell_1]$ yields

$$C_2^{p:\mathcal{N}=4}=\frac{[f|K|a\rangle^4\,[a\,e]\,\langle c\,f\rangle}{\langle de\rangle[bc]\,\langle a\,e\rangle\,[c\,f]\,K^2}\times\sum_i\sum_j C_{i,j}\frac{[\beta_4\ell_1]}{[B_j\ell_1]}\frac{\langle\alpha_2\ell_1\rangle}{\langle A_i\ell_1\rangle}\,,\tag{3.8}$$

where

$$C_{i,j}=\frac{\langle\alpha_1\,A_i\rangle}{\prod_{k\neq i}\langle A_k\,A_i\rangle}\times\frac{\prod_{k=1}^{3}[\beta_k\,B_j]}{\prod_{k\neq j}[B_k\,B_j]}\,.\tag{3.9}$$

The cut is now expressed in terms of the canonical form

$$\mathcal{H}_0^d=\frac{\langle a\ell_1\rangle}{\langle A\ell_1\rangle}\frac{[b\ell_1]}{[B\ell_1]}\tag{3.10}$$

which has the corresponding canonical function

$$H_0^d\big[B,A,b,a\,;\,K\big]=\begin{cases}\dfrac{K^2[bB]\,\langle a\,B\rangle}{[B|K|A\rangle[B|K|B\rangle}+\dfrac{[b|K|A\rangle\langle a|K|A]}{[B|K|A\rangle[A|K|A\rangle} & [B|K|A\rangle\neq 0\\[2ex] \dfrac{[bA]\,\langle a\,B\rangle}{[BA]\,\langle A\,B\rangle} & [B|K|A\rangle=0\,.\end{cases}\tag{3.11}$$

The contribution of (3.8) to the bubble coefficient is then

$$c_p(a,b,c,d,e,f)=\frac{[f|K|a\rangle^4\,[a\,e]\,\langle c\,f\rangle}{\langle de\rangle[bc]\,\langle a\,e\rangle\,[c\,f]\,K^2}\times\sum_i\sum_j C_{i,j}H_0^d[B_j,A_i,\beta_4,\alpha_2]\,,\tag{3.12}$$

leading to the full bubble coefficient

$$c_{\{a,d,e\},\{b,c,f\}}=c_p(a,b,c,d,e,f)+c_p(a,b,c,e,d,f)+c_p(a,c,b,d,e,f)+c_p(a,c,b,e,d,f)\,.\tag{3.13}$$

The coefficient of the bubble $I_2(K_{cd}^2)$ is obtained from the cut

$$C_2=\sum_h M_4^{\rm tree}(-\ell_1^h,c^-,d^+,-\ell_2^{-h})\times M_6^{\rm tree}(\ell_2^h,a^-,b^-,e^+,f^+,\ell_1^{-h}).\tag{3.14}$$

This cut can also be decomposed in canonical forms. However the six-point NMHV tree amplitude in the cut is a sum of fourteen terms [34] which leads to a lengthy expression for this bubble coefficient. Full expressions for the bubble coefficients of this type are given in appendix B. An explicit form of the bubble coefficients is available in `Mathematica` format at `http://pyweb.swan.ac.uk/~dunbar/sixgraviton/R6.html`.

IV. CANCELLATION WEBS AND RATIONAL DESCENDANTS

Although we may split the amplitude into cut-constructible pieces and rational terms, when we examine the singularities in the amplitude there is a mixing between the two which is important when we reconstruct R_n from its singularities. This has proven useful in the context of QCD amplitudes [35–37]. The cut-constructible pieces of the amplitude introduce a number of singularities that cannot be present in the full amplitude. These can be spurious singularities that occur at kinematic points where the full amplitude should be finite or higher order singularities occurring at points where the amplitude has a simple pole. If these poles are of sufficiently high order, they generate rational descendants.

A. Higher Order Poles

As an example of a higher order pole consider the behaviour of the one-mass box contribution $a^{1m[\mathcal{N}=4]}_{a,d,b,\{c,e,f\}} I_4^{a,d,b,\{c,e,f\}}$ as $[ab] \to 0$. The box coefficient $a^{1m[\mathcal{N}=4]}_{a,d,b,\{c,e,f\}}$ in eq.(2.4) contains a factor of $[ab]^{-4}$ and the expansion of the box integral function around $U(\equiv s_{ab}) = 0$ is [8]

$$I^{\rm 1m,trunc}(S,T,U,M) = f_S \log(S) + f_T \log(T) + f_M \log(M^2) + f_R\,, \tag{4.1}$$

where

$$\begin{aligned}
f_S &= -2\frac{U}{ST^2} + \frac{U^2}{ST^3} - \frac{2}{3}\frac{U^3}{ST^4} + \cdots \\
f_T &= -2\frac{U}{S^2T} + \frac{U^2}{S^3T} - \frac{2}{3}\frac{U^3}{S^4T} + \cdots \\
f_M &= 2\frac{MU}{S^2T^2} - \frac{U^2M^2}{S^3T^3} + \frac{2}{3}\frac{U^3}{S^4T^4} + \cdots \\
f_R^{1m} &= -\frac{U^2}{S^2T^2} + \frac{1}{3}\frac{U^3M}{S^3T^3} + \cdots
\end{aligned} \tag{4.2}$$

As the cut constructible terms contain all of the logarithms and dilogarithms in the amplitude, the logarithmic pieces of this expansion must combine with the bubbles to give an effective coefficients that are linearly divergent as $U \to 0$. We have confirmed numerically that the relevant cancellations between the one-mass box contributions and the bubble contributions occur. The quadratic divergence in the rational descendant, $a^{1-m[\mathcal{N}=4]}_{a,d,b,\{c,e,f\}} f_R^{1m}$, must be cancelled by the rational piece of the amplitude. The full rational part of the amplitude will ultimately be obtained by recursion and one contribution to it will arise from this rational descendant if the shift *excites* the $[ab] = 0$ pole.

There are corresponding $\langle de\rangle^{-4}$ poles in $a^{1-m[\mathcal{N}=4]}_{d,a,e,\{b,c,f\}} I_4^{d,a,e,\{b,c,f\}}$ which are obtained from those above by conjugation.

B. $[a|K_{be}|d\rangle^{-4}$ Spurious Singularity

The coefficients of the two mass hard boxes have singularities of the form $[a|K_{be}|d\rangle^{-4}$. These singularities also occur in the three mass triangle contributions: three powers of the pole are explicit in the leading term of the canonical form and a fourth arises in the partial fractioning that splits the cut integrand into canonical forms.

In terms of kinematic variables, $[a|K_{be}|d\rangle \to 0$ corresponds to $UT - M_3^2M_4^2 \to 0$. The two mass hard integral functions depend on S, T, M_3^2 and M_4^2, while the three mass triangle depends on S, M_3^2 and M_4^2 and there is the kinematic constraint: $S+T+U = M_3^2+M_4^2$. For given S, M_3^2 and M_4^2 it is possible find T_0 and $U_0(= M_3^2+M_4^2-S-T_0)$ so that $U_0T_0 - M_3^2M_4^2 = 0$. Close to the pole the two mass hard box integral functions can be expanded as a series in terms of $(T-T_0)$. In this context it is convenient to work with dimensionless box integral functions $F^{2mh:trunc}$ defined by

$$I_4^{2mh:trunc} \equiv -2r_\gamma \frac{F^{2mh:trunc}}{ST}. \tag{4.3}$$

The expansion of this dimensionless box function is

$$\begin{aligned}F^{2mh:trunc}(S,T,M_3^2,M_4^2) =& F^{2mh:trunc}(S,T_0,M_3^2,M_4^2) + f_{M_3}^{2m}\log(M_3^2) + f_{M_4}^{2m}\log(M_4^2)\\ &+ f_S^{2m}\log(S) + f_T^{2m}\log(T) + f_R^{2m}\,,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
f_{M_3}^{2m} &= 2\Bigg[\frac{M_3^2}{T(T-M_3^2)}(T-T_0) + \Big(\frac{(M_3^2)^2}{T^2(T-M_3^2)^2} + 2\frac{M_3^2}{T^2(T-M_3^2)}\Big)\frac{(T-T_0)^2}{2}\\
&\quad + \Big(2\frac{(M_3^2)^3}{T^3(T-M_3^2)^3} + 6\frac{(M_3^2)^2}{T^3(T-M_3^2)^2} + 6\frac{M_3^2}{T^3(T-M_3^2)}\Big)\frac{(T-T_0)^3}{6} + \cdots\Bigg],\\
f_{M_4}^{2m} &= 2\Bigg[\frac{M_4^2}{T(T-M_4^2)}(T-T_0) + \Big(\frac{(M_4^2)^2}{T^2(T-M_4^2)^2} + 2\frac{M_4^2}{T^2(T-M_4^2)}\Big)\frac{(T-T_0)^2}{2}\\
&\quad + \Big(2\frac{(M_4^2)^3}{T^3(T-M_4^2)^3} + 6\frac{(M_4^2)^2}{T^3(T-M_4^2)^2} + 6\frac{M_4^2}{T^3(T-M_4^2)}\Big)\frac{(T-T_0)^3}{6} + \cdots\Bigg],\\
f_S^{2m} &= 2\Bigg[\frac{(T-T_0)}{T} + \frac{(T-T_0)^2}{2T^2} + \frac{(T-T_0)^3}{3T^3}\Bigg],\\
f_T^{2m} &= -f_S^{2m} - f_{M_3}^{2m} - f_{M_4}^{2m}
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
f_R^{2m} =& \Big(\frac{M_3^2}{T^2(T-M_3^2)} + \frac{M_4^2}{T^2(T-M_4^2)} + \frac{1}{T^2}\Big)\\
&\qquad \times \frac{(M_3^2+M_4^2-S-2T)^2\big(UT-M_3^2M_4^2\big)^2}{\Delta_t^2}\Big(1+6\frac{\big(UT-M_3^2M_4^2\big)}{\Delta_t}\Big)\\
&+\frac{1}{3}\Big(2\frac{(M_3^2)^2}{(T-M_3^2)^2T^3} + 5\frac{M_3^2}{T^3(T-M_3^2)} + 2\frac{(M_4^2)^2}{(T-M_4^2)^2T^3} + 5\frac{M_4^2}{T^3(T-M_4^2)} + \frac{3}{T^3}\Big)\\
&\qquad \times \frac{(M_3^2+M_4^2-S-2T)^3\big(UT-M_3^2M_4^2\big)^3}{\Delta_t^3},
\end{aligned} \tag{4.6}$$

with

$$\Delta_t = S^2 + M_3^2 + M_4^2 - 2\big(SM_3^2 + SM_4^2 + M_3^2M_4^2\big). \tag{4.7}$$

The same pole appears in the two boxes with integral function: $I_4^{2mh,a,d,\{b,e\},\{c,f\}}$ and $I_4^{2mh,a,d,\{c,f\},\{b,e\}}$. On the pole the coefficients of these two boxes are not equal and neither integral function vanishes. However, the sum of the dimensionless integral functions vanishes, i.e.

$$\left(F^{2mh}_{a,d,\{b,e\},\{c,f\}}+F^{2mh}_{a,d,\{c,f\},\{b,e\}}\right)\Big|_{[a|K_{be}|d\rangle=0} = 0. \tag{4.8}$$

On this pole the dilogarithms in the individual boxes and triangles survive, but cancel between them. Setting

$$F^{3mt} \equiv -i\sqrt{\Delta_3} I_3^{3m}\,, \tag{4.9}$$

the integral functions are related by

$$\left(F^{2mh}_{a,d,\{b,e\},\{c,f\}}-F^{2mh}_{a,d,\{c,f\},\{b,e\}}\right)\Big|_{[a|K_{be}|d\rangle=0} = \pm F^{3mt}_{\{a,d\},\{b,e\},\{c,f\}}\Big|_{[a|K_{be}|d\rangle=0} \tag{4.10}$$

where the sign ambiguity is associated with the choice of sign for $\sqrt{\Delta_3}$. Schematically, expressing the box coefficients in terms of their sum and difference, $a_{\text{box}\,1} = \mathcal{S}+\mathcal{D}$, $a_{\text{box}\,2} = \mathcal{S}-\mathcal{D}$, the box and triangle contributions are

$$(\mathcal{S}+\mathcal{D})\,F_{\text{box}\,1}+(\mathcal{S}-\mathcal{D})\,F_{\text{box}\,2}+\tilde{b}_{tri}F^{3mt} = \mathcal{S}\,(F_{\text{box}\,1}+F_{\text{box}\,2})+\mathcal{D}\,(F_{\text{box}\,1}-F_{\text{box}\,2})+\tilde{b}_{tri}F^{3mt}\,. \tag{4.11}$$

Expanding about $[a|K_{be}|d\rangle = 0$, thanks to (4.8) there is no dilogarithm component in the first term for any $\mathcal{S}$. However, we can only use (4.10) if $\mathcal{D}$ and $\tilde{b}_{tri}$ are equal. In fact

$$\mathcal{D} = \pm\tilde{b}_{tri}+\mathcal{O}([a|K_{be}|d\rangle^0)\,. \tag{4.12}$$

Hence the dilogarithms vanish from any term that is singular as $[a|K_{be}|d\rangle \to 0$, as required by the factorisation theorems [38].

As in the one-mass case, there are subleading singularities at cubic order multiplying logarithms. These combine with the bubble contributions and cancel up to and including order $[a|K_{be}|d\rangle^{-1}$, leaving no spurious singularity in the logarithms.

The rational descendant of this combination of boxes and triangle contain both $[a|K_{be}|d\rangle^{-2}$ and $[a|K_{be}|d\rangle^{-1}$ singularities. Both of these singularities must be cancelled by the rational piece of the amplitude R_n. As the expansion has been performed about a singularity specified in terms of S, M_3^2 and M_4^2, there is no need to expand the three mass triangle integral function when determining this rational descendant.

C. Δ_3 Spurious Singularity

The three mass triangle contributions have Δ_3^{-2} poles which can be seen explicitly in the canonical forms. Around the $\Delta_3 = 0$ pole the integral function has the expansion [37],

$$\begin{aligned} I_3(M_1^2,M_2^2,M_3^2) = \quad & \log(M_1^2)\Big(-\frac{2}{(M_1^2-M_2^2-M_3^2)}+\frac{2}{3}\frac{\Delta_3}{(M_1^2-M_2^2-M_3^2)^3}+\cdots\Big) \\ & +\log(M_2^2)\Big(-\frac{2}{(-M_1^2+M_2^2-M_3^2)}+\frac{2}{3}\frac{\Delta_3}{(-M_1^2+M_2^2-M_3^2)^3}+\cdots\Big) \\ & +\log(M_3^2)\Big(-\frac{2}{(-M_1^2-M_2^2+M_3^2)}+\frac{2}{3}\frac{\Delta_3}{(-M_1^2-M_2^2+M_3^2)^3}+\cdots\Big) \\ & +f_R^{3mt}\,, \end{aligned} \tag{4.13}$$

where

$$f_R^{3mt} = -\frac{4}{3}\frac{\Delta_3}{(M_1^2-M_2^2-M_3^2)(-M_1^2+M_2^2-M_3^2)(-M_1^2-M_2^2+M_3^2)} + \cdots . \tag{4.14}$$

The logarithmic terms in this expansion combine with the bubble contributions to yield a finite contribution on the pole. The rational piece in the expansion must cancel with the rational part of the amplitude.

V. OBTAINING R_6 BY RECURSION

BCFW [24] recursion applies complex analysis to amplitudes. Using Cauchy's theorem, if a complex function is analytic except at simple poles z_i (all non-zero) and $f(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$ then by considering the integral

$$\oint_C f(z)\frac{dz}{z} , \tag{5.1}$$

where the contour C is the circle at infinity, we obtain

$$f(0) = -\sum_i \frac{\text{Residue}(f, z_i)}{z_i} . \tag{5.2}$$

We wish to apply this with $f(z) = R_6(z)$, where R_6 has been complexified by a BCFW shift of momenta. Since

$$M_6 = \mathcal{C}_{\text{box}}+\mathcal{C}_{\text{tri}}+\mathcal{C}_{\text{bub}}+R_6 \rightarrow R_6 = M_6-\mathcal{C}_{\text{box}}-\mathcal{C}_{\text{tri}}-\mathcal{C}_{\text{bub}} , \tag{5.3}$$

the singularities and residues of R_6 are both those arising from the physical factorisations of M_6 and those induced by the necessity to cancel the spurious singularities of the cut-constructible pieces.

A. Choice of Shift

The rational part of an amplitude can be obtained recursively if the factorisation properties of the amplitude are understood at all of the relevant poles. There are three main obstacles to this: quadratic poles in the amplitude, non-standard factorisations for complex momenta and contributions for large shifted momenta. Quadratic poles in the amplitude lead to recursive contributions that depend on the off-shell behaviour of the factorised currents. This can be addressed using augmented recursion [39, 40]. For non-supersymmetric theories there are double poles of the form

$$V(a^+, b^+, K^+)\times\frac{1}{[a\,b]^2}\times M_{n-1}^{\text{tree}}(K^-, \cdots, n) . \tag{5.4}$$

For the six-point NMHV amplitude the tree amplitude vanishes since it has a single positive helicity leg. (This is no longer the case for seven and higher point NMHV amplitudes.)

Non-standard factorisations for complex momenta are unavoidable and are considered in detail below. The final obstacle is the possibility of contributions from asymptotically large shifted momenta. The amplitude doesn't factorise in this limit, so the residue is undetermined. This issue may be avoided if the shift employed causes the amplitude to vanish for asymptotically large shifted momenta. As the amplitude is as yet undetermined, its behaviour under any shift is unknown. However, if the cut constructible pieces don't vanish for asymptotically large shifted momenta there is little hope that the rational pieces would.

For example under a shift involving two negative helicity legs,

$$\bar{\lambda}_a \to \bar{\lambda}_{\hat{a}} = \bar{\lambda}_a + z\bar{\lambda}_b \qquad , \qquad \lambda_b \to \lambda_{\hat{b}} = \lambda_b - z\lambda_a \,, \tag{5.5}$$

the cut constructible pieces of the amplitude are divergent for large z.

However, for a shift involving one negative helicity leg and one positive helicity leg,

$$\bar{\lambda}_a \to \bar{\lambda}_{\hat{a}} = \bar{\lambda}_a + z\bar{\lambda}_d, \qquad , \qquad \lambda_d \to \lambda_{\hat{d}} = \lambda_d - z\lambda_a, \tag{5.6}$$

the cut-constructible pieces all vanish at large z, at least suggesting that the rational piece is also well behaved there. The shift (5.6) will be used to obtain R_6.

The contributions to R_6 can be grouped into three classes: standard factorisations, non-factorising contributions and rational descendants of the cut-constructible pieces :

$$R_6 = \mathcal{R}_6^{\rm SF} + \mathcal{R}_6^{\rm NF} + \mathcal{R}_6^{\rm RD} \,. \tag{5.7}$$

B. Standard Factorisations

The standard factorisations of a six-point one-loop amplitude have the forms:

$$M_3^{\rm tree}\frac{1}{K^2}M_5^{1-\rm loop} \qquad , \qquad M_4^{\rm tree}\frac{1}{K^2}M_4^{1-\rm loop} \qquad , \qquad M_5^{\rm tree}\frac{1}{K^2}M_3^{1-\rm loop} \,. \tag{5.8}$$

In a supersymmetric theory the 3-point loop amplitudes vanish and so the third class are absent in this case. With the shift (5.6) the factorisations of the first type are:

$$\begin{aligned}
&M_3^{\rm tree}(\hat{a}^-, m_1^-, \hat{K}^+)\frac{1}{K^2}M_5^{1-\rm loop}(-\hat{K}^-, m_2^-, \hat{d}^+, p_1^+, p_2^+),\\
&M_3^{\rm tree}(\hat{a}^-, \hat{K}^-, p_1^+)\frac{1}{K^2}M_5^{1-\rm loop}(m_1^-, m_2^-, p_2^+, \hat{d}^+, -\hat{K}^+),\\
&M_3^{\rm tree}(\hat{K}^-, \hat{d}^+, p_1^+)\frac{1}{K^2}M_5^{1-\rm loop}(\hat{a}^-, m_1^-, m_2^-, p_2^+, -\hat{K}^+),\\
&M_3^{\rm tree}(m_1^-, \hat{d}^+, \hat{K}^+)\frac{1}{K^2}M_5^{1-\rm loop}(\hat{a}^-, -\hat{K}^-, m_2^-, p_1^+, p_2^+).
\end{aligned} \tag{5.9}$$

While the factorisations of the second type are:

$$\begin{aligned}
&M_4^{\rm tree}(\hat{a}^-, \hat{K}^-, p_1^+, p_2^+)\frac{1}{K^2}M_4^{1-\rm loop}(m_1^-, m_2^-, \hat{d}^+, -\hat{K}^+),\\
&M_4^{\rm tree}(\hat{a}^-, m_1^-, p_1^+, \hat{K}^+)\frac{1}{K^2}M_4^{1-\rm loop}(-\hat{K}^-, m_2^-, \hat{d}^+, p_2^+),\\
&M_4^{1-\rm loop}(\hat{a}^-, \hat{K}^-, p_1^+, p_2^+)\frac{1}{K^2}M_4^{\rm tree}(m_1^-, m_2^-, \hat{d}^+, -\hat{K}^+),\\
&M_4^{1-\rm loop}(\hat{a}^-, m_1^-, p_1^+, K^+)\frac{1}{K^2}M_4^{\rm tree}(-\hat{K}^-, m_2^-, \hat{d}^+, p_2^+).
\end{aligned} \tag{5.10}$$

For generic six-point kinematics, the kinematic points at which the 4- and 5-point loop amplitudes appearing in these factorisations are evaluated are in no way special, hence the rational contribution to the residue comes solely from the rational part of the 4- and 5-point loop amplitudes. Each factorisation therefore gives a contribution to R_6 of

$$R_6^{SF:i} = \mathrm{Res}\left(\frac{\mathcal{M}_{8-n}^{\text{tree:i}}\,R_n^i}{zK_i^2}\right)\bigg|_{z\to z_i} = M_{8-n}^{\text{tree:i}}(z_i)\times\frac{1}{K_i^2}\times R_n^i(z_i)\,, \tag{5.11}$$

with [4, 7]

$$R_4(a^-,b^-,c^+,d^+) = \frac{1}{2}\frac{\langle a\,b\rangle^4\,[c\,d]^2}{\langle c\,d\rangle^2}$$

and

$$R_5(a^-,b^-,c^+,d^+,e^+] = \mathcal{R}_5^b(a^-,b^-,c^+,d^+,e^+)+\mathcal{R}_5^a(a^-,b^-,c^+,d^+,e^+)\,, \tag{5.12}$$

where

$$\begin{aligned}\mathcal{R}_5^a(a^-,b^-,c^+,d^+,e^+) &= -\frac{\langle a\,b\rangle^4}{2}\frac{[c\,d]^2}{\langle c\,d\rangle^2}\frac{[b\,e]}{\langle b\,e\rangle}\frac{\langle b\,c\rangle}{\langle c\,e\rangle}\frac{\langle b\,d\rangle}{\langle d\,e\rangle}+\mathcal{P}_6(\{a,b\},\{c,d,e\})\,,\\ \mathcal{R}_5^b(a^-,b^-,c^+,d^+,e^+) &= -\langle a\,b\rangle^4\frac{[c\,d]}{\langle c\,d\rangle}\frac{[c\,e]}{\langle c\,e\rangle}\frac{[d\,e]}{\langle d\,e\rangle}\end{aligned} \tag{5.13}$$

and $\mathcal{P}_6$ denotes a sum over the six distinct permutations of $\{a,b\}$ and $\{c,d,e\}$ noting the symmetry of $\mathcal{R}_5^a$ under $c\leftrightarrow d$. The full contribution of the standard factorisations is then

$$R_6^{SF} = \sum_i R_6^{SF:i}\,, \tag{5.14}$$

where the sum is over all of the standard factorisation channels given in (5.9) and (5.10).

C. Contribution Of Rational Descendants

As discussed above, higher order poles in the coefficients of the box and triangle contributions to the amplitude can generate rational descendants when those poles are excited. The shift (5.6) excites some poles of each type. Specifically we have the various singularities listed in table II (with $p_i \neq d$, $m_i \neq a$).

Denoting the rational descendant in each case by f_R^i, the corresponding coefficient by c_i and the value of z on the pole by z_i, the contribution on each of these poles is

$$\mathcal{R}_6^{\text{RD:i}} = -\mathrm{Res}\left(\frac{c_i(\hat{a},b,c,\hat{d},e,f)f_R^i(\hat{a},b,c,\hat{d},e,f)}{z}\right)\bigg|_{z\to z_i} \tag{5.15}$$

so that

$$\mathcal{R}_6^{\text{RD}} = \sum_i \mathcal{R}_6^{\text{RD:i}}\,, \tag{5.16}$$

where the sum is over all of the poles listed above.

Individual terms in the bubble coefficients contain a range of other higher order poles. In principle these could also generate rational descendants, however in the full bubble coefficients these are at most simple poles and so do not generate further rational descendants:

$$\mathcal{R}_6^{\text{RD:bub}} = 0\,. \tag{5.17}$$

$I_4^{1m}(\hat{a}^-, p_1^+, m_2^-, \{p_2^+, p_3^+, m_3^-\})$	$[a\, m_2]^{-4}$
$I_4^{1m}(\hat{d}^+, m_1^-, p_2^+, \{m_2^-, m_3^-, p_3^+\})$	$\langle d\, p_2 \rangle^{-4}$
$I_4^{2mh}(a^-, d^+, \{m_1, p_1\}, \{m_2, p_2\})$	$[\hat{a}\vert K_{p_i m_i}\vert\hat{d}\rangle^{-4}$
$I_4^{2mh}(a^-, p_2^+, \{m_1, d\}, \{m_2, p_2\})$	$[\hat{a}\vert K_{p_i m_i}\vert p_2\rangle^{-4}$
$I_4^{2mh}(m_1^-, d^+, \{a, p_1\}, \{m_2, p_2\})$	$[m_1\vert K_{p_2 m_2}\vert\hat{d}\rangle^{-4}$
$I_4^{2mh}(m_1^-, p_1^+, \{a, p_2\}, \{m_2, \hat{d}\})$	$[m_1\vert K_{\hat{d} m_2}\vert p_2\rangle^{-4}$
I_3^{3m}	Δ_3^{-2}

TABLE II: The various non-physical poles which induce terms in R_6

D. Non-standard Factorisations for Complex Momenta

Factorisations of the amplitude occur when propagators go on shell. The standard factorisation channels arise when the on-shell propagator is not in the loop and is explicit in, for example a Feynman diagram approach.

The loop momentum integral may also generate poles in the amplitude [38] particularly for complex momenta. Since we are computing the amplitude by recursion in complex momenta we must determine these complex factorisations.

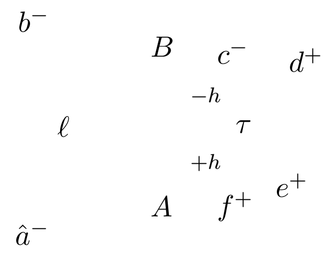

FIG. 4: Non-standard factorisations channel

Poles can arise when two adjacent massless legs on a loop became collinear as illustrated in fig. 4. This case has been discussed in the context of amplitudes with a single negative helicity leg [39, 40]. In the integration region $\ell \propto b$ the three propagators connected to a and b all become on shell when a and b are collinear. The diagrams of interest can be grouped together to form a one mass triangle in the integral reduction sense (i.e. the massive corner represents a sum of all possible tree diagrams). The integration region of interest has all three propagators on shell and so the pole may be determined by the triple cut of this triangle. This triple cut wouldn't normally exist, but *opens up* when a and b are collinear.

Using an axial gauge with reference spinor q [39–41], the contribution of fig. 4 with a scalar particle circulating in the loop is

$$\int d^D\ell \frac{1}{\ell^2 A^2 B^2}\left(\frac{[q|a\ell|q]}{[aq]^2}\right)^2\left(\frac{[q|b\ell|q]}{[bq]^2}\right)^2 \tau^{\rm tree}_{\rm MHV}(A,B,c^-,d^+,e^+,f^+)\,, \tag{5.18}$$

where $\tau^{\rm tree}_{\rm MHV}$ represents the sum of all possible tree diagrams. A and B are given by

$$A = \ell{+}a \;\;\text{and}\;\; B = -\ell{+}b \tag{5.19}$$

and satisfy $A+B = a+b$. The integrands for other particle types in the loop are related to the scalar contribution by an X-factor:

$$X = \frac{\langle \ell\, a\rangle}{\langle A\, a\rangle}\frac{\langle B\, b\rangle}{\langle \ell\, b\rangle}\frac{\langle A\, c\rangle}{\langle B\, c\rangle} = \frac{[q|A|c\rangle}{[q|B|c\rangle} \tag{5.20}$$

and the $\mathcal{N}=4$ contribution is obtained by multiplying the integrand by a factor of ρ^2 where,

$$\rho = \frac{[q|a{+}b|c\rangle^2}{[q|A|c\rangle[q|B|c\rangle}\,. \tag{5.21}$$

For $\ell \propto b$ the integrand of (5.18) contains $\langle a\, b\rangle$ factors in its numerator, leaving a pole in $[ab]$ in the integral. As $\mathcal{M}^{\rm tree}_{\rm MHV}$ is finite as $[ab]\to 0$, $\tau^{\rm tree}_{\rm MHV}\to\mathcal{M}^{\rm tree}_{\rm MHV}$ in the region of interest. For a scalar particle circulating in the loop the KLT relations [31] give

$$\begin{aligned}\mathcal{M}^{\rm tree:0}_{\rm MHV}(A,B,e,f,c,d) = \Big[&-is_{AB}s_{cf}A^{YM:0}(A,B,e,f,c,d)\big[s_{ec}A^{YM}(B,A,c,e,f,d)\\ &+(s_{ec}{+}s_{ef})A^{YM}(B,A,c,f,e,d)\big]\Big]+\mathcal{P}(B,e,f)\,.\end{aligned} \tag{5.22}$$

Of the six terms in the permutation sum in (5.22), the two which don't permute B can be neglected due to the explicit s_{AB} factor. The remaining four form two pairs with the members of each pair being related by interchange of legs e and f. The $\mathcal{N}=4$ contributions of one member of each of these pairs are

$$C_1 = \int d^D\ell \frac{1}{\ell^2 A^2 B^2}\frac{\langle A\, a\rangle^2\langle B\, b\rangle^2\langle A\, c\rangle\langle B\, c\rangle^2[Ae][Be]}{\langle A\, d\rangle\langle A\, e\rangle\langle B\, d\rangle\langle B\, e\rangle\langle B\, f\rangle}\frac{[cf][q|a{+}b|c\rangle^4}{[aq]^2[bq]^2\langle c\, d\rangle\langle c\, f\rangle\langle d\, f\rangle} \tag{5.23}$$

and

$$C_2 = -\int d^D\ell \frac{1}{\ell^2 A^2 B^2}\frac{\langle A\, a\rangle^2\langle B\, b\rangle^2\langle A\, c\rangle\langle B\, c\rangle[Ae][Bc][f|B{+}c|d\rangle}{\langle A\, d\rangle\langle A\, e\rangle\langle B\, d\rangle\langle B\, f\rangle}\frac{[cf][q|a{+}b|c\rangle^4}{[aq]^2[bq]^2\langle c\, d\rangle\langle d\, e\rangle\langle d\, f\rangle\langle e\, f\rangle}\,. \tag{5.24}$$

Partial fractioning the integrand of C_1 using the $\langle A\, c\rangle\langle B\, c\rangle^2$ factor in the numerator yields six terms whose integrands have loop momentum dependence

$$\frac{1}{\ell^2 A^2 B^2}\frac{\langle A\, a\rangle^2\langle B\, b\rangle^2[Ae][Be]}{\langle A\, x\rangle\langle B\, y\rangle}\,, \tag{5.25}$$

with $x\in\{d,e\}$ and $y\in\{d,e,f\}$. In the integration region of interest A^2 and B^2 are negligible allowing the integrands to be rewritten as quartic pentagon integrands

$$\frac{[e|A|a\rangle[e|B|b\rangle[x|A|a\rangle[y|B|b\rangle}{\ell^2A^2B^2(A{+}x)^2(B{-}y)^2}\,. \tag{5.26}$$

For $x \neq y$, using

$$\langle b|ByxA|a\rangle = 2B.y\langle b|xA|a\rangle - 2A.x\langle b|yB|a\rangle + \langle b|y(a+b)Ax|a\rangle \tag{5.27}$$

splits each of these quartic pentagons into a pair of cubic one-mass boxes and a cubic pentagon which can be neglected. As a box with two adjacent corners attached to single external legs of the same helicity has a vanishing quadruple cut, these cubic one-mass box integrals reduce to bubble and rational contributions only. The bubble coefficients can be evaluated by direct parametrisation. For example the box integral (5.28) which is illustrated in fig. 5 has bubbles associated with its $\{a, x\}$ and $\{b, a, x\}$ cuts.

$$\int d^D\ell \frac{[e|A|a\rangle[e|B|b\rangle[x|A|a\rangle}{\ell^2 A^2 B^2 (A+x)^2} \tag{5.28}$$

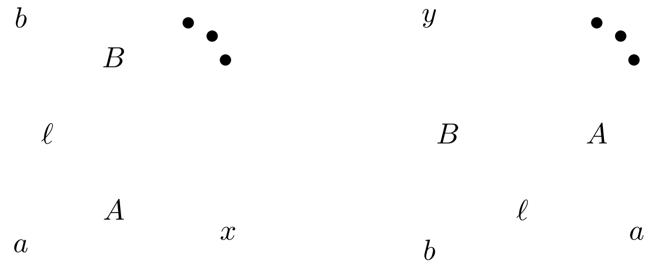

FIG. 5: The box integrals associated with (5.28)

The $\{a, x\}$ cut of (5.28) gives the bubble coefficient

$$C^{\rm bub}_{ax} = \frac{\langle a\, b\rangle^2}{[ab]} \frac{\langle x\, a\rangle\, [ea]^2}{\langle b\, x\rangle^2} + \mathcal{O}\big([ab]\big), \tag{5.29}$$

where terms of order $[ab]^1$ have been extracted from the leading term to simplify its denominator as far as possible. The remaining $\langle b\, x\rangle$ singularity in this bubble coefficient is spurious and must cancel with the $\{b, a, x\}$ bubble as $\langle b\, x\rangle \to 0$. So that this singularity is not present in the logarithmic part of the integral, the sum of the two bubble coefficients must be finite. The sum of the two bubble contributions then involves the singular parts of the $\{a, x\}$ bubble coefficient multiplied by the difference of the integral functions of the two bubbles. With s_{ab} and s_{bx} both being small, the rational descendant of the bubbles on the $\langle b\, x\rangle \to 0$ pole is

$$\frac{\langle a\, b\rangle^2}{[ab]} \frac{\langle a\, x\rangle\, [ea]^2}{\langle b\, x\rangle^2} \Big(\frac{s_{bx}}{s_{ax}} - \frac{1}{2} \frac{s_{bx}^2}{s_{ax}^2} + \cdots \Big) + \mathcal{O}\big([ab]^0\big). \tag{5.30}$$

The leading term of the rational descendant has a $\langle b\, x\rangle^{-1}$ spurious pole. This must be cancelled by the rational piece of the integral, allowing the rational term to be identified as,

$$\frac{\langle a\, b\rangle^2}{[ab]} \frac{[ae][be]}{\langle b\, x\rangle} + \mathcal{O}\big([ab]^0\big)\,. \tag{5.31}$$

For $x = y$ in (5.25) the reduction to boxes uses the identity

$$\frac{1}{(A+x)^2(B-x)^2} = \frac{1}{2P.x}\Big(\frac{1}{(A+x)^2}+\frac{1}{(B-x)^2}\Big)+\mathcal{O}\big(A^2, B^2\big) \tag{5.32}$$

which yields a pair of quartic box integrals whose rational pieces are evaluated using the approach described above. The full rational contribution of C_1 is

$$\begin{aligned} R^{C_1} = \frac{[f\,c]\,[q|P_{ab}|c\rangle^4}{[aq]^2[bq]^2\,\langle c\,d\rangle\,\langle c\,f\rangle\,\langle d\,e\rangle\,\langle d\,f\rangle}\Bigg(& \frac{\langle c\,d\rangle\,\langle c\,d\rangle^2}{\langle e\,d\rangle\,\langle f\,d\rangle}I^{5a}_{d,d,2}+\frac{\langle c\,d\rangle\,\langle c\,e\rangle^2}{\langle d\,e\rangle\,\langle f\,e\rangle}I^{5a}_{d,e,1} \\ & +\frac{\langle c\,d\rangle\,\langle c\,f\rangle^2}{\langle d\,f\rangle\,\langle e\,f\rangle}I^{5a}_{d,f,1}-\frac{\langle c\,e\rangle\,\langle c\,d\rangle^2}{\langle e\,d\rangle\,\langle f\,d\rangle}I^{5a}_{e,d,1} \\ & -\frac{\langle c\,e\rangle\,\langle c\,e\rangle^2}{\langle d\,e\rangle\,\langle f\,e\rangle}I^{5a}_{e,e,2}-\frac{\langle c\,e\rangle\,\langle c\,f\rangle^2}{\langle d\,f\rangle\,\langle e\,f\rangle}I^{5a}_{e,f,1}\Bigg), \end{aligned} \tag{5.33}$$

where

$$I^{5a}_{x,y,1} =-\frac{1}{4}\frac{\langle a\,b\rangle^2\,[ae][be]}{[ab]\,\langle x\,y\rangle} \quad \text{and} \quad I^{5a}_{x,x,2} = \frac{\langle a\,b\rangle^3}{12[ab][x|P_{ab}|x\rangle}\Big(-\frac{[eb]^2[ax]}{\langle a\,x\rangle}-\frac{[ea]^2[bx]}{\langle b\,x\rangle}\Big). \tag{5.34}$$

The C_2 contributions involve both quintic and quartic pentagon integrals, but their rational pieces can be obtained in a similar fashion to the C_1 contributions. Separating the quintic and quartic pentagon integrals,

$$R^{C_2} = R^{C_2}_{\text{quin}}+R^{C_2}_{\text{quar}}, \tag{5.35}$$

where

$$R^{C_2}_{\text{quin}} = \frac{[q|P_{ab}|c\rangle^4}{[aq]^2[bq]^2\,\langle c\,d\rangle\,\langle d\,e\rangle^2\,\langle e\,f\rangle\,\langle d\,f\rangle}\Big(\langle c\,d\rangle\,I^{5b}_d-\langle c\,e\rangle\,I^{5b}_e\Big) \tag{5.36}$$

with

$$\begin{aligned} I^{5b}_x = \frac{1}{\langle x\,f\rangle}\Bigg(& -\frac{1}{6}\,\langle a\,f\rangle\,[eb][cb]\,\langle a\,b\rangle^2\Big(\frac{\langle c\,b\rangle\,[af]}{\langle f\,a\rangle}+\frac{\langle f\,b\rangle\,\langle a\,c\rangle\,[af]}{\langle f\,a\rangle^2}-\frac{\langle a\,c\rangle\,[af]^2}{2\,\langle f\,a\rangle\,[bf]}\Big) \\ & +\frac{\langle a\,x\rangle}{4\,\langle x\,f\rangle\,\langle a\,f\rangle}\langle b|f\,P_{ab}|f\rangle[ea]\,\langle a\,c\rangle\,[cb]\,\langle a\,b\rangle \\ & -\langle a\,x\rangle\,[ea][fa][ca]\,\langle a\,x\rangle\,\langle c\,b\rangle\,\frac{\langle a\,b\rangle^2}{6\,\langle b\,x\rangle^2}\frac{s_{bx}}{s_{ax}} \\ & +\frac{\langle a\,x\rangle}{\langle x\,f\rangle}\langle b|f\,P_{ab}|x\rangle[ea][ca]\,\langle a\,x\rangle\,\langle c\,b\rangle\,\frac{\langle a\,b\rangle}{4\,\langle b\,x\rangle^2}\frac{s_{bx}}{s_{ax}}\Bigg) \end{aligned} \tag{5.37}$$

and

$$R^{C_2}_{\text{quar}} = \frac{[fc][q|P_{ab}|c\rangle^4}{[aq]^2[bq]^2\,\langle d\,e\rangle^2\,\langle d\,f\rangle^2\,\langle e\,f\rangle}\Bigg(\langle c\,d\rangle^2\,I^{5c}_{d,d,2}-\langle c\,e\rangle\,\langle c\,d\rangle\,I^{5c}_{e,d,1}-\langle c\,d\rangle\,\langle c\,f\rangle\,I^{5c}_{d,f,1}+\langle c\,e\rangle\,\langle c\,f\rangle\,I^{5c}_{e,f,1}\Bigg) \tag{5.38}$$

with

$$I^{5c}_{x,y,1} = -\frac{1}{4}\frac{\langle a\,b\rangle^2\,[ae][bc]}{[ab]\,\langle x\,y\rangle} \quad \text{and} \quad I^{5c}_{x,y,2} = \frac{\langle a\,b\rangle^3}{12[ab][x|P_{ab}|x\rangle}\Big(-\frac{[bc][be][ax]}{\langle a\,x\rangle}-\frac{[ac][ae][bx]}{\langle b\,x\rangle}\Big). \tag{5.39}$$

The contribution of these non-standard factorisations to the rational part of the 6-pt amplitude is obtained by recursion:

$$\mathcal{R}_6^{C_i}(a,b,c,d,e,f) = \text{Res}\left(\frac{R^{C_i}(\hat{a},b,c,\hat{d},e,f)}{z}\right)\Bigg|_{[\hat{a}b]\to 0} \tag{5.40}$$

The contributions arising from the conjugate poles, e.g. $\left\langle \hat{d}\,e\right\rangle \to 0$, can be obtained using the flip-conjugation symmetry of the amplitude. Defining

$$\mathcal{R}_6^{C}(a,b,c,d,e,f) = \mathcal{R}_6^{C_1}(a,b,c,d,e,f)+\mathcal{R}_6^{C_2}(a,b,c,d,e,f) \tag{5.41}$$

and

$$\mathcal{R}_6^{\tilde{C}}(a,b,c,d,e,f) = \mathcal{R}_6^{C}(d,e,f,a,b,c)\big|_{\langle x\,y\rangle\leftrightarrow[xy]}, \tag{5.42}$$

the full non-factorising contribution to $\mathcal{R}_6$ is

$$\begin{aligned}\mathcal{R}_6^{\text{n-f}}(a,b,c,d,e,f) = \; & \mathcal{R}_6^{C}(a,b,c,d,e,f)+\mathcal{R}_6^{C}(a,b,c,d,f,e)+\mathcal{R}_6^{C}(a,c,b,d,e,f)+\mathcal{R}_6^{C}(a,c,b,d,f,e)\\ & +\mathcal{R}_6^{\tilde{C}}(a,b,c,d,e,f)+\mathcal{R}_6^{\tilde{C}}(a,b,c,d,f,e)+\mathcal{R}_6^{\tilde{C}}(a,c,b,d,e,f)+\mathcal{R}_6^{\tilde{C}}(a,c,b,d,f,e)\,.\end{aligned} \tag{5.43}$$

We have computed R_6 systematically using its pole structure. Underlying this is the assumption that the amplitude vanishes for large shifts. This is difficult to justify a priori. However the expression obtained has the correct symmetries and collinear limits (checked numerically). Generically a BCFW recursion produces terms which are not manifestly symmetric and the restoration of symmetry is a good indicator that the amplitude has been correctly determined.

An explicit form of R_6 is available in `Mathematica` format at `http://pyweb.swan.ac.uk/~dunbar/sixgraviton/R6.html`.

VI. CONCLUSIONS

Graviton scattering amplitudes have a rich structure. In particular $\mathcal{N}=8$ supergravity has proven to have a much softer UV behaviour then previously expected with the underlying symmetry reason still unclear. It is important to understand which structures of $\mathcal{N}=8$ survive in theories with lower supersymmetry. It is also important to study amplitudes beyond MHV since this can often have a misleadingly simple structure. In this article we have constructed the six-point NMHV amplitude in $\mathcal{N}=4$ supergravity. Of particular interest is the rational term since in the MHV case a particularly simple and suggestive structure appears [42]. The rational terms in the NMHV case do not appear to have any such simple structure although this may be hiding given the algebraic complexity of the amplitude.

Computing the rational terms has required a blending of techniques including obtaining the rational descendants of the cut-constructible pieces. Amongst the cut-constructible pieces the coefficients of the bubble integral functions have been particularly cumbersome although, fortunately, these do not generate any rational descendants in this amplitude.

VII. ACKNOWLEDGEMENTS

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Appendix A: Six-Point Tree Amplitude Expression

The six-point tree amplitude needed for computing the bubble coefficients is

$$M((l_1)^-_h, a^-, b^-, e^+, f^+, (l_2)^+_h) = \sum_{i=1}^{14} T_i(h) = \sum_{i=1}^{14} A_i (X_i)^{2h}. \tag{A1}$$

The fourteen terms in (A1) are

$$
\begin{aligned}
T_1 &= -i\frac{\langle ab\rangle^7\langle el_2\rangle[bl_1][ef]^7}{\langle al_1\rangle\langle bl_1\rangle[e|K_{abl_1}|a\rangle[l_2|K_{abl_1}|a\rangle[f|K_{abl_1}|b\rangle[f|K_{abl_1}|l_1\rangle[el_2][fl_2]t_{abl_1}}\Big[\delta_{h,-2}\Big],\\
T_2 &= -i\frac{\langle bl_1\rangle[f|K_{bfl_1}|a\rangle^8[el_2]}{\langle ae\rangle\langle al_2\rangle\langle el_2\rangle[b|K_{bfl_1}|a\rangle[l_1|K_{bfl_1}|a\rangle[f|K_{bfl_1}|e\rangle[f|K_{bfl_1}|l_2\rangle[bf][bl_1][fl_1]t_{bfl_1}}\left[-i\frac{\langle al_2\rangle[fl_1]}{[f|K_{bfl_1}|a\rangle}\right]^A,\\
T_3 &= -i\frac{\langle ab\rangle^7\langle el_1\rangle[bl_2][ef]^7}{\langle al_2\rangle\langle bl_2\rangle[e|K_{efl_1}|a\rangle[l_1|K_{efl_1}|a\rangle[f|K_{efl_1}|b\rangle[f|K_{efl_1}|l_2\rangle[el_1][fl_1]t_{efl_1}}\Big[\delta_{h,2}\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_4 &= i\frac{\langle al_1\rangle^7\langle be\rangle[ef]^7[l_1l_2]}{\langle al_2\rangle\langle l_1l_2\rangle[b|K_{bef}|a\rangle[e|K_{bef}|a\rangle[f|K_{bef}|l_1\rangle[f|K_{bef}|l_2\rangle[be][bf]t_{bef}}\left[i\frac{\langle al_2\rangle}{\langle al_1\rangle}\right]^A,\\
T_5 &= i\frac{\langle a\,b\rangle^7\,\langle l_1\,l_2\rangle\,[b\,e]\,[f\,l_2]^7}{\langle a\,e\rangle\,\langle b\,e\rangle\,[l_1|K_{abe}|a\rangle[l_2|K_{abe}|a\rangle[f|K_{abe}|b\rangle[f|K_{abe}|e\rangle\,[f\,l_1]\,[l_1\,l_2]\,t_{abe}}\left[i\frac{[fl_1]}{[fl_2]}\right]^A,\\
T_6 &= -i\frac{\langle al_1\rangle^7\langle bl_2\rangle[el_1][fl_2]^7}{\langle ae\rangle\langle el_1\rangle[b|K_{ael_1}|a\rangle[l_2|K_{ael_1}|a\rangle[f|K_{ael_1}|e\rangle[f|K_{ael_1}|l_1\rangle[bf][bl_2]t_{ael_1}}\left[\frac{[f|K_{ael_1}|a\rangle}{\langle al_1\rangle[fl_2]}\right]^A,\\
T_7 &= i\frac{[ef][l_2|K_{aef}|a\rangle^7\big(\langle al_2\rangle\langle bl_1\rangle[l_1|K_{aef}|e\rangle[bl_2]-\langle al_1\rangle\langle bl_2\rangle[l_2|K_{aef}|e\rangle[bl_1]\big)}{\langle af\rangle^2\langle ef\rangle[b|K_{aef}|a\rangle[b|K_{aef}|e\rangle[l_1|K_{aef}|a\rangle[l_1|K_{aef}|e\rangle[l_2|K_{aef}|e\rangle[bl_1][bl_2][l_1l_2]t_{aef}}\left[i\frac{[l_1|K_{aef}|a\rangle}{[l_2|K_{aef}|a\rangle}\right]^A,\\
T_8 &= i\frac{[e|K_{bel_1}|a\rangle^7[fl_2](\langle ae\rangle\langle bl_1\rangle[l_1|K_{bel_1}|l_2\rangle[be]-\langle be\rangle\langle al_1\rangle[e|K_{bel_1}|l_2\rangle[bl_1])}{\langle af\rangle^2\langle fl_2\rangle[b|K_{bel_1}|a\rangle[l_1|K_{bel_1}|a\rangle[b|K_{bel_1}|l_2\rangle[e|K_{bel_1}|l_2\rangle[l_1|K_{bel_1}|l_2\rangle[be][bl_1][el_1]t_{bel_1}}\left[i\frac{\langle l_2a\rangle[el_1]}{[e|K_{bel_1}|a\rangle}\right]^A,\\
T_9 &= i\frac{\langle al_1\rangle^8[el_2]^7[fl_1]\big(\langle al_2\rangle\langle be\rangle[b|K_{afl_1}|l_1\rangle[el_2]-\langle ab\rangle\langle el_2\rangle[l_2|K_{afl_1}|l_1\rangle[be]\big)}{\langle af\rangle^2\langle fl_1\rangle[b|K_{afl_1}|a\rangle[e|K_{afl_1}|a\rangle[l_2|K_{afl_1}|a\rangle[b|K_{afl_1}|l_1\rangle[e|K_{afl_1}|l_1\rangle[l_2|K_{afl_1}|l_1\rangle[be][bl_2]t_{afl_1}}\Bigg[i\frac{[e|K_{afl_1}|a}{\langle l_1a\rangle[el_2}\\
T_{10} &= i\frac{\langle ab\rangle^8[bf][el_2]^7\big(\langle al_2\rangle\langle el_1\rangle[l_1|K_{abf}|b\rangle[el_2]-\langle al_1\rangle\langle el_2\rangle[l_2|K_{abf}|b\rangle[el_1]\big)}{\langle af\rangle^2\langle bf\rangle[e|K_{abf}|a\rangle[l_1|K_{abf}|a\rangle[l_2|K_{abf}|a\rangle[e|K_{abf}|b\rangle[l_1|K_{abf}|b\rangle[l_2|K_{abf}|b\rangle[el_1][l_1l_2]t_{abf}}\left[i\frac{[el_1]}{[el_2]}\right]^A,\\
T_{11} &= i\frac{\langle ae\rangle\langle bl_1\rangle^7[ef]^8\big(\langle bl_2\rangle[e|K_{aef}|l_1\rangle[bl_1][fl_2]-\langle bl_1\rangle[e|K_{aef}|l_2\rangle[bl_2][fl_1]\big)}{\langle bl_2\rangle\langle l_1l_2\rangle[e|K_{aef}|b\rangle[f|K_{aef}|b\rangle[e|K_{aef}|l_1\rangle[f|K_{aef}|l_1\rangle[e|K_{aef}|l_2\rangle[f|K_{aef}|l_2\rangle[ae][af]^2t_{aef}}\left[i\frac{\langle bl_2\rangle}{\langle bl_1\rangle}\right]^A,\\
T_{12} &= -i\frac{\langle al_2\rangle\langle bl_1\rangle^7[fl_2]^8\big(\langle be\rangle[l_2|K_{bel_1}|l_1\rangle[bl_1][ef]+\langle bl_1\rangle[l_2|K_{bel_1}|e\rangle[be][fl_1]\big)}{\langle be\rangle\langle el_1\rangle[f|K_{bel_1}|b\rangle[l_2|K_{bel_1}|b\rangle[f|K_{bel_1}|l_1\rangle[f|K_{bel_1}|e\rangle[l_2|K_{bel_1}|e\rangle[l_2|K_{bel_1}|l_1\rangle[al_2][af]^2t_{bel_1}}\Bigg[i\frac{[f|K_{bel_1}|b}{\langle l_1b\rangle[fl_2]}\\
T_{13} &= -i\frac{\langle al_1\rangle[f|K_{afl_1}|b\rangle^7(\langle be\rangle[l_1|K_{afl_1}|l_2\rangle[bf][el_2]+\langle el_2\rangle[l_1|K_{afl_1}|b\rangle[be][fl_2])}{\langle be\rangle\langle bl_2\rangle\langle el_2\rangle[l_1|K_{afl_1}|b\rangle[f|K_{afl_1}|e\rangle[l_1|K_{afl_1}|e\rangle[f|K_{afl_1}|l_2\rangle[l_1|K_{afl_1}|l_2\rangle[af]^2[al_1]t_{afl_1}}\left[i\frac{\langle l_2b\rangle[fl_1]}{[f|K_{afl_1}|b\rangle}\right]^A\\
T_{14} &= i\frac{\langle ab\rangle[f|K_{abf}|l_1\rangle^7\big(\langle el_2\rangle[b|K_{abf}|l_1\rangle[el_1][fl_2]-\langle el_1\rangle[b|K_{abf}|l_2\rangle[el_2][fl_1]\big)}{\langle el_1\rangle\langle el_2\rangle\langle l_1l_2\rangle[b|K_{abf}|e\rangle[f|K_{abf}|e\rangle[b|K_{abf}|l_1\rangle[b|K_{abf}|l_2\rangle[f|K_{abf}|l_2\rangle[ab][af]^2t_{abf}}\left[i\frac{[f|K_{abf}|l_2\rangle}{[f|K_{abf}|l_1\rangle}\right]^A,
\end{aligned}
\tag{A2}
$$

with $A=4-2h$.

Appendix B: Bubble coefficient

The 2:4 bubbles involve the 6-pt NMHV tree amplitude. This has fourteen terms and consequently, the bubble coefficient has fourteen sources. Each of these generates a collection of terms leading to an algebraically complicated expression,

$$c = \sum_i C_{Ti}\,. \tag{B1}$$

Of these fourteen terms two (T_1 and T_3) don't enter the $\mathcal{N}=4$ matter multiple calculation and the rest split evenly into *massless* and *massive* types. Massive terms involve a factor of $((\ell+Q)^2)^{-1}$ where $Q^2 \neq 0$. Terms T_4, T_5, T_7, T_{10}, T_{11} and T_{14} are of the massless type and their ℓ dependent factors in the denominator are of the from $\langle x\ell\rangle$ or $[x\ell]$. The bubble coefficients for these massless type terms can be evaluated using the H_0^d canonical forms presented above (see (3.11)). The overall result for these terms is then

$$C_{T4}+C_{T5}+C_{T7}+C_{T10}+C_{T11}+C_{T14} = \sum_{j=1}^{65} D_j H_0^d[B_j, A_j, b_j, a_j]\,. \tag{B2}$$

The explicit results are

$$C_{T4} = \frac{[c\,d]\,\langle a\,c\rangle^4\,\langle b\,e\rangle\,[e\,f]^7\,s_{cd}}{[b|K_{bef}|a\rangle[e|K_{bef}|a\rangle\,\langle c\,d\rangle\,[b\,e]\,[b\,f]\,K_{bef}^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_i\alpha_j H_0^d[B_i, A_j, b_2, a_2] \tag{B3}$$

$$\{|a_j\rangle\} = \{|c\rangle, |a\rangle\} \quad \{|A_j\rangle\} = \{|d\rangle, K_{be}|f]\} \tag{B4}$$
$$\{|b_i]\} = \{|d], K_{cd}|a\rangle\} \quad \{|B_i]\} = \{|c], K_{cd}K_{be}|f]\} \tag{B5}$$

$$\alpha_j = \frac{\prod_{i=1}^{n_A-1}\langle a_i\,A_j\rangle}{\prod_{k\neq j}\langle A_k\,A_j\rangle} \quad \beta_i = \frac{\prod_{i=1}^{n_B-1}[b_i\,B_i]}{\prod_{k\neq i}[B_k\,B_i]} \tag{B6}$$

For C_{T4}, $n_A = n_B = 2$ and the numerator products simplify to $\langle a_1\,A_j\rangle$ and $[b_1\,B_i]$.

$$C_{T5} = -\frac{[f|K_{cd}|c\rangle^4\,\langle a\,b\rangle^7\,[b\,e]}{\langle c\,d\rangle^2\,\langle a\,e\rangle\,\langle b\,e\rangle\,[f|K_{abe}|b\rangle[f|K_{abe}|e\rangle K_{abe}^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_i\alpha_j H_0^d[B_i, A_j, b_2, a_2] \tag{B7}$$

where

$$\{|a_j\rangle\} = \{|c\rangle, K_{cd}|f]\} \quad \{|A_j\rangle\} = \{|d\rangle, K_{cd}K_{abe}|a\rangle\} \tag{B8}$$
$$\{|b_i]\} = \{|f], |d], \} \quad \{|B_i]\} = \{|c], K_{be}|a\rangle\} \tag{B9}$$

and α_i and β_j are given by B6 with $n_A = n_B = 2$.

$$C_{T7} = \frac{[e\,f]\,\langle c\,d\rangle^2\,[d|K_{aef}|a\rangle^4}{\langle a\,f\rangle^2\,\langle e\,f\rangle\,[b|K_{aef}|a\rangle[b|K_{aef}|e\rangle K_{aef}^2;}\Bigg([b|K_{aef}|a\rangle\sum_{i=1}^{2}\sum_{j=1}^{3}\beta_i\alpha_j H_0^d[B_i, A_j, b_2, a_3] - K_{aef}^2\,\langle b\,e\rangle\sum_{i=1}^{2}\sum_{j=1}^{3}\delta_i\gamma_j H_0^d[D_i, C_j, d_2, c_3];\Bigg) \tag{B10}$$

$$\begin{aligned}
\{|a_j\rangle\} =& \{|b\rangle, |c\rangle, K_{cd}K_{aef}|a\rangle\} \quad \{|A_j\rangle\} = \{|d\rangle, K_{cd}|b], K_{cd}K_{aef}|e\rangle\} \\
\{|b_i]\} =& \{|d], K_{aef}|a\rangle, \} \quad \{|B_i]\} = \{|b], |c]\} \\
\{|c_j\rangle\} =& \{|a\rangle, |c\rangle, K_{cd}K_{aef}|a\rangle\} \quad \{|C_j\rangle = \{|d\rangle, K_{cd}|b], K_{cd}K_{aef}|e\rangle\} \\
\{|d_i]\} =& \{|f], K_{aef}|a\rangle\} \quad \{|D_i] = \{|c], K_{aef}|e]\}
\end{aligned} \tag{B11}$$

where α_j and β_i are given in eq.(B6) with $n_A = 3, n_B = 2$ and

$$\gamma_j = \frac{\prod_{k=1}^{n_C-1} \langle c_k\, C_j\rangle}{\prod_{k\neq j} \langle C_k\, C_j\rangle} \quad \delta_i = \frac{\prod_{k=1}^{n_D-1} [d_k\, D_i]}{\prod_{k\neq i} [D_k\, D_i]} \tag{B12}$$

with $n_C = 3, n_D = 2$.

$$\begin{aligned}
C_{T10} =& -\frac{\langle a\, b\rangle^8 [b\, f]\, [e|K_{cd}|c\rangle^4 [c\, d]}{\langle c\, d\rangle \langle a\, f\rangle^2 \langle b\, f\rangle\, [e|K_{abf}|a\rangle K_{abf}^2 [e|K_{abf}|b\rangle s_{cd}} \Bigg(\\
& -[e|K_{cd}|a\rangle \sum_{i=1}^{3}\sum_{j=1}^{3} \beta_i\alpha_j H_0^d[B_i, A_j, b_3, a_3] + \langle e|K_{cd}|K_{abf}|b\rangle \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_i\gamma_j H_0^d[D_i, C_j, d_3, c_3]; \Bigg)
\end{aligned} \tag{B13}$$

where

$$\begin{aligned}
\{|a_j\rangle\} =& \{|e\rangle, |c\rangle, K_{cd}|e]\} \quad \{|A_j\rangle\} = \{|d\rangle, K_{cd}K_{abf}|a\rangle, K_{cd}K_{abf}|b\rangle\} \\
\{|b_i]\} =& \{|d], |e], K_{abf}|b\rangle, \} \quad \{|B_i]\} = \{|c], K_{abf}|b\rangle, K_{abf}|a\rangle, \} \\
\{|c_j\rangle\} =& \{|a\rangle, |c\rangle, K_{cd}|e]\} \quad \{|C_j\rangle = \{|d\rangle, K_{cd}K_{abf}|a\rangle, K_{cd}K_{abf}|b\rangle\} \\
\{|d_i]\} =& \{|e], |e], |d]\} \quad \{|D_i] = \{|c], K_{abf}|a\rangle, K_{abf}|b\rangle\}
\end{aligned} \tag{B14}$$

and the $\alpha_i, \beta_j, \gamma_i, \delta_j$ are given in eqns.(B6) and (B12) with $n_A = n_B = n_C = n_D = 3$.

$$\begin{aligned}
C_{T14} =& \frac{\langle a\, b\rangle\, [f|K_{abf}|K_{cd}|d]^4}{[a\, b]\, [a\, f]^2\, K_{abf}^2 [b|K_{abf}|e\rangle [f|K_{abf}|e\rangle \langle c\, d\rangle^2} \Bigg(\\
& -[f|K_{cd}|e\rangle \sum_{i=1}^{3}\sum_{j=1}^{2} \beta_i\alpha_j H_0^d[B_i, A_j, b_3, a_2] - [b|K_{abf}|K_{c,d}|e] \sum_{i=1}^{3}\sum_{j=1}^{2} \delta_i\gamma_j H_0^d[D_i, C_j, d_3, c_3]; \Bigg)
\end{aligned} \tag{B15}$$

$$\begin{aligned}
\{|a_j\rangle\} =& \{|c\rangle, K_{abf}|f]\} \quad \{|A_j\rangle\} = \{|e\rangle, |d\rangle\} \\
\{|b_i]\} =& \{|d], |e], K_{cd}K_{abf}|f], \} \quad \{|B_i]\} = \{|c], K_{cd}|e\rangle, K_{cd}K_{abf}|b\rangle, \} \\
\{|c_j\rangle\} =& \{|c\rangle, K_{abf}|f]\} \quad \{|C_j\rangle = \{|d\rangle, K_{abf}|b]\} \\
\{|d_i]\} =& \{|d], |f], K_{cd}K_{abf}|f]\} \quad \{|D_i] = \{|c], K_{cd}|e\rangle, K_{cd}K_{abf}|b]\}
\end{aligned} \tag{B16}$$

and the $\alpha_i, \beta_j, \gamma_i, \delta_j$ are given in eqns.(B6) and (B12) with $n_A = 2, n_B = 3, n_C = 2, n_D = 3$.

$$C_{T11} = \frac{[c\, d]^2 \langle a\, e\rangle\, [e\, f]^8 \langle b\, c\rangle^4}{K_{aef}^2\, [a\, e]\, [a\, f]^2\, [e|K_{aef}|b\rangle [f|K_{aef}|b\rangle} \Bigg(\tag{B17}$$

$$[f|K_{abef}|b\rangle \sum_{i=1}^{3}\sum_{j=1}^{2} \beta_i\alpha_j H_0^d[B_i, A_j, b_3, a_2] - K_{aef}^2\, [b\, e] \sum_{i=1}^{3}\sum_{j=1}^{2} \delta_i\gamma_j H_0^d[D_i, C_j, d_3, c_3]; \Bigg) \tag{B18}$$

where

$$
\begin{aligned}
\{|a_j\rangle\} =& \{|b\rangle, |c\rangle\} \quad \{|A_j]\} = \{|d], K_{ae}|f\rangle\} \\
\{|b_i]\} =& \{|b], |d], K_{cd}|b\rangle\} \quad \{|B_i]\} = \{|c], K_{cd}K_{aef}|f], K_{cd}K_{aef}|f]\} \\
\{|c_j\rangle\} =& \{|b\rangle, |b\rangle, |c\rangle\} \quad \{|C_j\rangle = \{|d\rangle, K_{aef}|e], K_{aef}|f]\} \\
\{|d_i]\} =& \{|f], |d], K_{cd}|b\rangle\} \quad \{|D_i] = \{|c], K_{cd}K_{aef}|e], K_{cd}K_{aef}|e]\}
\end{aligned}
\tag{B19}
$$

and the $\alpha_i, \beta_j, \gamma_i, \delta_j$ are given in eqns.(B6) and (B12) with $n_A = n_B = n_C = n_D = 3$.

The remaining pieces come from T_2, T_6, T_8, T_9, T_{12} and T_{13} and all involve massive propagators. These terms generically take the form

$$
\sim \frac{f(\ell)}{(\ell+Q)^2}\,. \tag{B20}
$$

Other denominators such as $[\ell|Q|\ell\rangle^{-1}$ and $[\alpha|K_{\ell+Q}|\beta\rangle^{-1}$ appear but these can be manipulated into a common $[\ell|Q|\ell\rangle^{-1}$ form using

$$
[\alpha|(\ell+Q)|\beta\rangle = [\ell|\alpha]\langle\beta\ell\rangle + [\alpha|Q|\beta\rangle = [\ell|\left(\alpha]\langle\beta + \frac{[\alpha|Q|\beta\rangle}{P^2}P\right)\ell\rangle = \frac{[\alpha|Q|\beta\rangle}{P^2}[\ell|\tilde{Q}|\ell\rangle \tag{B21}
$$

with

$$
\tilde{Q} = P + \frac{P^2}{[\alpha|Q|\beta\rangle}\bar{\lambda}_\alpha\lambda_\beta \tag{B22}
$$

where we have used that, on the cut $(l_1-P)^2=0$, so that $[l_1|P|l_1\rangle = P^2$. Also

$$
(\ell+Q)^2 = [\ell|Q|\ell\rangle + Q^2 \equiv [\ell|Q|\ell\rangle + Q^2\frac{[\ell|P|\ell\rangle}{P^2} = [\ell|\tilde{Q}|\ell\rangle \tag{B23}
$$

where

$$
\tilde{Q}^\mu = Q^\mu + \frac{Q^2}{P^2}P^\mu\,. \tag{B24}
$$

The previous six terms were of overall order ℓ^4. When combined with the other tree amplitude and multiplying by the ρ-factor the resulting cut was of order ℓ^0. The ℓ count of these terms is +6 and there is no straightforward way of implementing a reduction. As in the massless case the ρ-factors lower the overall power count of the $\mathcal{N}=4$ contribution by 8, leading to an overall power count of +2. This significantly increases the complexity of the expressions. The leading large ℓ contributions cancel between the terms at large ℓ indicating that there is probably an underlying simpler version of the bubble coefficient. The form of the six-point NMHV amplitude was obtained using a BCFW shift on legs a^- and f^+. We have evaluated alternative forms using alternative shifts e.g. a^- and b^- but the resultant expressions still include terms which would be ℓ^6 or have equivalent problems.

Fortunately only three of these contributions are required since

$$
C_{T6} + C_{T9} + C_{T12} = C_{T2} + C_{T8} + C_{T13}\,. \tag{B25}
$$

We present the analysis of term C_{T2} in detail below and the results for C_{T8} and C_{T13} after.

Term T_2 is

$$
T_2 = -i\frac{\langle bl_1\rangle[f|P_{bfl_1}|a\rangle^8[el_2]}{\langle ae\rangle\langle al_2\rangle\langle el_2\rangle[b|P_{bfl_1}|a\rangle[l_1|P_{bfl_1}|a\rangle[f|P_{bfl_1}|e\rangle[f|P_{bfl_1}|l_2\rangle[bf][bl_1][fl_1]t_{bfl_1}}\left[-i\frac{\langle al_2\rangle[fl_1]}{[f|P_{bfl_1}|a\rangle}\right]^A, \tag{B26}
$$

where $A = 4 - 2h$.

The denominator of (B26) includes a products of three massive factors:

$$\frac{1}{[b|P_{bfl_1}|a\rangle[f|P_{bfl_1}|e\rangle t_{bfl_1}}\,. \tag{B27}$$

These denominator factors can be rewritten as

$$\begin{aligned}
[b|P_{bfl_1}|a\rangle &= [b|f|a\rangle + [l_1 b]\langle a l_1\rangle = [l_1\Bigg|\left(\frac{[b|f|a\rangle}{P^2}P + \bar{\lambda}_b\lambda_a\right)\Bigg|l_1\rangle \equiv \frac{[b|f|a\rangle}{P^2}[l_1|Q_{2:1}|l_2\rangle \\
[f|P_{bfl_1}|a\rangle &= [f|b|a\rangle + [l_1 f]\langle a l_1\rangle = [l_1\Bigg|\left(\frac{[f|b|a\rangle}{P^2}P + \bar{\lambda}_f\lambda_a\right)\Bigg|l_1\rangle \equiv \frac{[f|b|a\rangle}{P^2}[l_1|Q_{2:2}|l_1\rangle \\
t_{bfl_1} &= [l_1|P_{bf}|l_1\rangle + s_{bf} = [l_1\Bigg|\left(\frac{s_{bf}}{P^2}P + P_{bf}\right)\Bigg|l_1\rangle \equiv [l_1|Q_{2:3}|l_1\rangle
\end{aligned} \tag{B28}$$

Drawing in two l_1 dependent factors from the numerator, the massive factor can be separated using

$$\begin{aligned}
\frac{\langle x l_1\rangle\langle y l_1\rangle}{[l_1|Q_{2:1}|l_1\rangle[l_1|Q_{2:2}|l_1\rangle[l_1|Q_{2:3}|l_1\rangle} &= \frac{\langle y l_1\rangle}{[l_1|Q_{2:1}|l_1\rangle[l_1|Q_{2:2}Q_{2:3}|l_1]}\left(\frac{[l_1|Q_{2:2}|x\rangle}{[l_1|Q_{2:2}|l_1\rangle} - \frac{[l_1|Q_{2:3}|x\rangle}{[l_1|Q_{2:3}|l_1\rangle}\right) \\
&= \frac{1}{[l_1|Q_{2:2}Q_{2:3}|l_1]}\left(\frac{[l_1|Q_{2:2}|x\rangle\langle y l_1\rangle}{[l_1|Q_{2:1}Q_{2:2}|l_1]}\left(\frac{[l_1|Q_{2:1}|y\rangle}{[l_1|Q_{2:1}|l_1\rangle} - \frac{[l_1|Q_{2:2}|y\rangle}{[l_1|Q_{2:2}|l_1\rangle}\right)\right. \\
&\quad \left. - \frac{[l_1|Q_{2:3}|x\rangle}{[l_1|Q_{2:1}Q_{2:3}|l_1]}\left(\frac{[l_1|Q_{2:1}|y\rangle}{[l_1|Q_{2:1}|l_1\rangle} - \frac{[l_1|Q_{2:3}|y\rangle}{[l_1|Q_{2:3}|l_1\rangle}\right)\right)\,.
\end{aligned} \tag{B29}$$

The $[l_1|Q_{2:i}Q_{2:j}|l_1]$ factors can be split by defining $\hat{Q}_2^{ij} = Q_{2:i} - \alpha Q_{2:j}$ where $(\hat{Q}_2^{ij})^2 = 0$, so that,

$$[l_1|Q_{2:i}Q_{2:j}|l_1] = [l_1|(Q_{2:i} - \alpha Q_{2:j})Q_{2:j}|l_1] = [l_1|\hat{Q}_2^{ij}Q_{2:j}|l_1] = [l_1\hat{Q}_2^{ij}]\langle\hat{Q}_2^{ij}|Q_{2:j}|l_1]\,. \tag{B30}$$

These factors can then be treated in the same way as the massless factors. As the full denominator may contain factors of the form $\langle x\, l_1\rangle$, partial fractioning on both $|l_1\rangle$ and $|l_1]$ yields terms with loop momentum dependent factors of the form:

$$\frac{[A|l_1|B\rangle[C|l_1|D\rangle + \gamma[E|l_1|F\rangle}{[l_1|Q|l_1\rangle}\frac{[Y l_1]}{[X l_1]}\frac{\langle y l_1\rangle}{\langle x l_1\rangle}\,. \tag{B31}$$

The canonical form arising from the terms in (B31) are the G_{111} functions as defined in appendix C.

The cut momentum P_{cd} is specified by the sum of the two null momenta c and d, however it is often convenient to express P_{cd} as the sum of two alternative null momenta. For any null momentum x, setting

$$x^* = \frac{P_{cd}^2}{[x|P_{cd}|x\rangle}x, \tag{B32}$$

gives

$$P = P_x^\flat + x^* \tag{B33}$$

with $(P_x^\flat)^2 = 0$.

Defining

$$
\begin{aligned}
\lambda_X &= [f|P_{bc}|a\rangle\lambda_c + [f\,d]\,\langle c\,a\rangle\,\lambda_d \\
\lambda_{Q^x_{2:1}} &= \frac{s_{cd}}{[b|f|a\rangle}\lambda_a \qquad \bar\lambda_{Q^x_{2:1}} = \lambda^*_{Q^x_{2:1}} \\
\lambda_{Q^x_{2:2}} &= \frac{s_{cd}}{[f|b|e\rangle}\lambda_e \qquad \bar\lambda_{Q^x_{2:2}} = \lambda^*_{Q^x_{2:2}} \\
\{|b_i]\} &= \{\langle X_2|P_{cd}|,\ \langle X_2|P_{cd}|,\ \langle X_2|P_{cd}|,\ \langle X_2|P_{cd}|\} \\
\{|B_i]\} &= \{[b|,\ [c|,\ \langle e|P_{cd}|,\ \langle a|P_{bf}|,\ [f|P_{ae}P_{cd}|\} \\
\{|a^1_i\rangle\} &= \{\langle b|,\ \langle c|,\ [f|Q_{2:1}|,\ \langle a|P_{cd}Q_{2:1}|\} \\
\{|a^2_i\rangle\} &= \{\langle b|,\ \langle c|,\ [f|Q_{2:2}|,\ \langle a|P_{cd}Q_{2:2}|\} \\
\{|a^3_i\rangle\} &= \{\langle b|,\ \langle c|,\ [f|Q_{2:3}|,\ \langle a|P_{cd}Q_{2:3}|\} \\
\{|A^1_i\rangle\} &= \{\langle d|,\ \langle f|,\ \langle \hat Q^{1,2}_2|,\ [\hat Q^{1,2}_2|Q_{2:2}|,\ \langle a|P_{bf}Q_{2:3}|\} \\
\{|A^2_i\rangle\} &= \{\langle b|,\ \langle d|,\ \langle \hat Q^{1,2}_2|,\ [\hat Q^{1,2}_2|Q_{2:2}|,\ \langle e|P_{bf}Q_{2:3}|\} \\
\{|A^3_i\rangle\} &= \{\langle b|,\ \langle d|,\ \langle f|,\ \langle a|P_{bf}Q_{2:3}|,\ \langle e|P_{bf}Q_{2:3}|\}
\end{aligned} \tag{B34}
$$

$$
\alpha^k_j = \frac{\prod_{i=1}^4 \langle a^k_i\, A^k_j\rangle}{\prod_{j\neq i}\langle A^k_i\, A^k_j\rangle} \quad \beta_j = \frac{\prod_{i=1}^4 [b_i\, B_j]}{\prod_{i\neq j}^5 [B_i\, B_j]} \tag{B35}
$$

the contribution of term T_2 to the bubble coefficient is

$$
\begin{aligned}
C_{T2} = \frac{[c\,d]\,\langle f\,b\rangle}{\langle a\,e\rangle\,[b\,f]\,s_{cd}^2\,\langle c\,d\rangle} \sum_{j=1}^5\sum_{i=1}^5\sum_{k=1}^3 \rho^k\alpha^k_j\beta_i \\
\times\Big(\quad G^s_{111}\big[[e|P_{cd}|, A^k_i, [d|, B_j, [f|, \langle a|, Q_{2:1}, \{b^*, P^\flat_b\}, P_{cd}, Q^x_{2:1}\big] \\
-2[f|b|a\rangle G_{011}\big[[e|P_{cd}|, A^k_i, [d|, B_j, [f|, \langle a|, Q_{2:1}, \{b^*, P^\flat_b\}, P_{cd}, Q^x_{2:1}\big] \\
+[f|b|a\rangle^2 G_{s11}\big[[e|P_{cd}|, A^k_i, [d|, B_j, [f|, \langle a|, Q_{2:1}, \{b^*, P^\flat_b\}, P_{cd}, Q^x_{2:1}\big]\Big)
\end{aligned} \tag{B36}
$$

where

$$
\rho^1 = \frac{1}{[f|b|e\rangle},\quad \rho^2 = \frac{1}{[b|f|a\rangle},\quad \rho^3 = -\frac{\langle f\,b\rangle}{s_{cd}}\,. \tag{B37}
$$

In the above x and w represent arbitrary null momenta.

C_{T8} and C_{T13}
For C_{T8} we have

$$\begin{aligned}
C_{T8} =& \frac{\langle a\, e\rangle\, [c\, d]}{\langle a\, f\rangle^3 \langle c\, d\rangle\, s_{cd}^2\, [e\, b]} \sum_{i=1}^{6}\sum_{j=1}^{3} \beta_i \Bigg(\\
& \alpha_j^1 \begin{pmatrix} G_{111}^s[\langle c|, A_j, [d|, B_i, [e|, \langle a|, Q_{8:1}, \{b^*, P_b^\flat\}, P_{cd}, Q_{8:1}^x] \\ -[e|b|a\rangle G_{011}[\langle c|, A_j, [d|, B_i, [e|, \langle a|, Q_{8:1}, \{b^*, P_b^\flat\}, P_{cd}, Q_{8:1}^x] \end{pmatrix} \\
& -\alpha_j^2 \begin{pmatrix} G_{111}^s[\langle c|, A_j, [d|, B_i, [e|, \langle a|, Q_{8:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \\ -[e|b|a\rangle G_{011}[\langle c|, A_j, [d|, B_i, [e|, \langle a|, Q_{8:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \end{pmatrix} \Bigg) \\
& + \frac{\langle b\, e\rangle\, [cd]}{\langle a\, f\rangle^3 \langle c\, d\rangle\, s_{cd}^2 [be]^2} \sum_{i=1}^{6}\sum_{j=1}^{2} \gamma_i \Bigg(\\
& \delta_j^1 \begin{pmatrix} G_{111}^s[\langle c|, C_j, [d|, D_i, [e|, \langle a|, Q_{8:1}, \{b^*, P_b^\flat\}, P_{cd}, Q_{8:1}^x] \\ -[e|b|a\rangle G_{011}[\langle c|, C_j, [d|, D_i, [e|, \langle a|, Q_{8:1}, \{b^*, P_b^\flat\}, P_{cd}, Q_{8:1}^x] \end{pmatrix} \\
& -\delta_j^2 \begin{pmatrix} G_{111}^s[\langle c|, C_j, [d|, D_i, [e|, \langle a|, Q_{8:2}, \{x^*, P_x^\flat\}, w] \\ -[e|b|a\rangle G_{011}[\langle c|, C_j, [d|, D_i, [e|, \langle a|, Q_{8:2}, \{x^*, P_x^\flat\}, w] \end{pmatrix} \Bigg)
\end{aligned} \tag{B38}$$

where we define

$$\lambda_X = -\lambda_c [e|b|a\rangle + \lambda_c\, [e\, c]\, \langle a\, c\rangle + \lambda_d\, [e\, d]\, \langle a\, c\rangle \tag{B39}$$

$$\lambda_{Q_{8:1}^a} = \lambda_a \frac{s_{cd}}{[b|e|a\rangle}\ ,\ \bar{\lambda}_{Q_{8:1}^a} = \bar{\lambda}_b\ ,\ \lambda_{Q_{8:1}^x} = \lambda_{Q_{8:1}^a}\ ,\ \bar{\lambda}_{Q_{8:1}^x} = \lambda^*_{Q_{8:1}^a} \tag{B40}$$

$$Q_{8:1} = Q_{8:1}^a + k_c + k_d \tag{B41}$$

$$\lambda_{Q_{8:2}^a} = \lambda_c \frac{s_{be}}{s_{cd}}\ ,\ \bar{\lambda}_{Q_{8:2}^a} = \bar{\lambda}_c\ ,\ \lambda_{Q_{8:2}^b} = \lambda_d\ ,\ \bar{\lambda}_{Q_{8:2}^b} = \bar{\lambda}_d \frac{s_{be}}{s_{cd}} \tag{B42}$$

$$Q_{8:2} = Q_{8:2}^a + Q_{8:2}^b + k_b + k_e \tag{B43}$$

$$\hat{P}_8 = P_{cd} + \alpha P_{af}\ \text{s.t.}\ \hat{P}_8^2 = 0 \tag{B44}$$

$$\begin{aligned}
\{|A_j\rangle\} &= \{\langle a|,\ \langle d|,\ \langle e|\}\ ,\ \ \{|a_j^1\rangle\} = \{\langle b|,\ \langle a|P_{cd}Q_{8:1}|\}\ ,\ \ \{|a_j^2\rangle\} = \{\langle b|,\ \langle a|P_{cd}Q_{8:2}|\} \\
\{|B_i]\} &= \{[b|,\ \langle a|P_{be}|,\ [b|P_{af}P_{cd}|,\ [c|,\ \langle f|P_{cd}|,\ [e|P_{af}P_{cd}|\} \\
\{|b_i]\} &= \{\langle a|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|\} \\
\{|C_j\rangle\} &= \{\langle d|,\ \langle e|\}\ ,\ \ \{|c_j^1\rangle\} = \{\langle a|P_{cd}Q_{8:1}|\}\ ,\ \ \{|c_j^2\rangle\} = \{\langle a|P_{cd}Q_{8:2}|\} \\
\{|D_i]\} &= \{[c|,\ \langle a|P_{be}|,\ [b|P_{af}P_{cd}|,\ \langle f|P_{cd}|,\ [\hat{P}_8|,\ \langle \hat{P}_8|P_{af}|\} \\
\{|d_i]\} &= \{\langle a|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|\}
\end{aligned} \tag{B45}$$

$$\alpha_j^k = \frac{\prod_{i=1}^{2} \langle a_i^k \, A_j^k \rangle}{\prod_{i\neq j} \langle A_i^k \, A_j^k \rangle} \ , \ \beta_j = \frac{\prod_{i=1}^{5} [b_i \, B_j]}{\prod_{i\neq i} [B_i \, B_j]} \tag{B46}$$

$$\gamma_j^k = \frac{\langle c_1^k \, C_j^k \rangle}{\prod_{i\neq j} \langle C_i^k \, C_j^k \rangle} \ , \ \delta_j = \frac{\prod_{i=1}^{5} [d_i \, D_j]}{\prod_{i\neq j} [D_i \, D_j]} \tag{B47}$$

Finally,

$$\begin{aligned}
C_{T13} = & -\frac{[f b][c d] \, [e \, b]}{\langle c \, d\rangle \, \langle e \, b\rangle^2 \, [f \, a]^3 \, s_{cd}^2} \sum_{i=1}^{6} \sum_{j=1}^{3} \beta_i \Bigg(\\
& \alpha_j^1 \begin{pmatrix} G_{111}^s[\langle c|, A_j, [d|, B_i, [f|, \langle b|, Q_{13:1}, \{f^*, P_f^\flat\}, P_{cd}, Q_{13:1}^x] \\ -[f|a|b\rangle G_{011}[\langle c|, A_j, [d|, B_i, [f|, \langle b|, Q_{13:1}, \{f^*, P_f^\flat\}, P_{cd}, Q_{13:1}^x] \end{pmatrix} \\
& -\alpha_j^2 \begin{pmatrix} G_{111}^s[\langle c|, A_j, [d|, B_i, [f|, \langle b|, Q_{13:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \\ -[f|a|b\rangle G_{011}[\langle c|, A_j, [d|, B_i, [f|, \langle b|, Q_{13:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \end{pmatrix} \Bigg) \\
& +\frac{[c \, d] \, [e \, b]}{\langle c \, d\rangle \, \langle e \, b\rangle^2 \, [f \, a]^3 \, s_{cd}^2} \sum_{i=1}^{6} \sum_{j=1}^{3} \delta_i \Bigg(\\
& \gamma_j^1 \begin{pmatrix} G_{111}^s[\langle c|, D_j, [d|, C_i, [f|, \langle b|, Q_{13:1}, \{f^*, P_f^\flat\}, P_{cd}, Q_{13:1}^x] \\ -[f|a|b\rangle G_{011}[\langle c|, D_j, [d|, C_i, [f|, \langle b|, Q_{13:1}, \{f^*, P_f^\flat\}, P_{cd}, Q_{13:1}^x] \end{pmatrix} \\
& -\gamma_j^2 \begin{pmatrix} G_{111}^s[\langle c|, D_j, [d|, C_i, [f|, \langle b|, Q_{13:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \\ -[f|a|b\rangle G_{011}[\langle c|, D_j, [d|, C_i, [f|, \langle b|, Q_{13:2}, \{x^*, P_x^\flat\}, P_{cd}, w] \end{pmatrix} \Bigg)
\end{aligned} \tag{B48}$$

where

$$\lambda_X = \lambda_c[f|a|b\rangle + \lambda_c \, \langle c \, b\rangle \, [f c] + \lambda_d \, \langle c \, b\rangle \, [f d] \tag{B49}$$

$$\lambda_{Q_{13:1}^a} = \lambda_e \frac{s_{cd}}{[f|a|e\rangle} \ , \ \bar{\lambda}_{Q_{13:1}^a} = \bar{\lambda}_f \ , \ \lambda_{Q_{13:1}^x} = \lambda_{Q_{13:1}^a} \ , \ \bar{\lambda}_{Q_{13:1}^x} = \lambda_{Q_{13:1}^a}^* \tag{B50}$$

$$Q_{13:1} = Q_{13:1}^a + k_c + k_d \tag{B51}$$

$$\lambda_{Q_{13:2}^a} = \lambda_c \frac{[a|f|a\rangle}{s_{cd}} \ , \ \bar{\lambda}_{Q_{13:2}^a} = \bar{\lambda}_c \ , \ \lambda_{Q_{13:2}^b} = \lambda_d \ , \ \bar{\lambda}_{Q_{13:2}^b} = \bar{\lambda}_d \frac{[a|f|a\rangle}{s_{cd}} \tag{B52}$$

$$Q_{13:2} = Q_{13:2}^a + Q_{13:2}^b + k_a + k_f \tag{B53}$$

$$
\begin{aligned}
\{|A_j\rangle\} &= \{\langle b|,\ \langle d|,\ [b|P_{cd}|\}\ ,\quad \{|a_j^1\rangle\} = \{[e|P_{cd}|, [f|Q_{13:1}|\}\ ,\quad \{|a_j^2\rangle\} = \{[e|P_{cd}|, [f|Q_{13:2}|\} \\
\{|B_i]\} &= \{[a|, \langle e|P_{af}|, [f|P_{be}P_{cd}|, [c|, \langle b|P_{af}|, \langle e|P_{cd}|\} \\
\{|b_i]\} &= \{\langle b|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|\} \\
\{|C_j\rangle\} &= \{\langle b|,\ \langle d|,\ [b|P_{cd}|\}\ ,\quad \{|c_j^1\rangle\} = \{[f|P_{cd}|,\ [f|Q_{13:1}|\}\ ,\quad \{|c_j^2\rangle\} = \{[f|P_{cd}|,\ [f|Q_{13:2}|\} \\
\{|D_i]\} &= \{[a|,\ \langle e|P_{af}|,\ [f|P_{be}P_{cd}|,\ [c|,\ [\hat{P}_{13}|,\ \langle \hat{P}_{13}|P_{af}|\} \\
\{|d_i]\} &= \{\langle b|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|,\ \langle X|P_{cd}|\}
\end{aligned} \tag{B54}
$$

$$
\alpha_j^k = \frac{\prod_{i=1}^2 \langle a_i^k\, A_j^k\rangle}{\prod_{i\neq j} \langle A_i^k\, A_j^k\rangle}\ ,\ \beta_j = \frac{\prod_{i=1}^5 [b_i\, B_j]}{\prod_{i\neq i} [B_i\, B_j]} \tag{B55}
$$

$$
\gamma_j^k = \frac{\prod_{i=1}^2 \langle c_i^k\, C_j^k\rangle}{\prod_{i\neq j} \langle C_i^k\, C_j^k\rangle}\ ,\ \delta_j = \frac{\prod_{i=1}^5 [d_i\, D_j]}{\prod_{i\neq j} [D_i\, D_j]} \tag{B56}
$$

Appendix C: G_{111} Canonical Form

The massive canonical form required for the bubble coefficients is

$$\mathcal{G}_{111}[x,y,a,A,b,B,f,e,\tilde{Q},\ell] = [\ell x]\,\langle \ell\, y\rangle\,\frac{\langle \ell\, a\rangle}{\langle \ell\, A\rangle}\frac{[\ell b]}{[\ell B]}\frac{[\ell f]\,\langle \ell\, e\rangle}{[\ell|\tilde{Q}|\ell\rangle} \tag{C1}$$

with corresponding canonical function

$$G_{111}[x,y,a,A,b,B,f,e,\tilde{Q},c,d,P] \tag{C2}$$

Specifically the 2:4 bubble coefficients involve cut integrands of the form

$$\mathcal{G}^{s}_{111}[a,A,b,B,f,e,\tilde{Q},\ell] = \frac{\langle \ell\, a\rangle}{\langle \ell\, A\rangle}\frac{[\ell b]}{[\ell B]}\frac{[\ell f]^2\,\langle \ell\, e\rangle^2}{[\ell|\tilde{Q}|\ell\rangle} \tag{C3}$$

and

$$\mathcal{G}_{011}[a,A,b,B,f,e,\tilde{Q},\ell] = \frac{\langle \ell\, a\rangle}{\langle \ell\, A\rangle}\frac{[\ell b]}{[\ell B]}\frac{[\ell f]\,\langle \ell\, e\rangle}{[\ell|\tilde{Q}|\ell\rangle} \tag{C4}$$

which are special cases of $\mathcal{G}_{111}$ with corresponding functions given by

$$\begin{aligned} G^{s}_{111}[a,A,b,B,f,e,\tilde{Q},c,d,P] =& G_{111}[f,e,a,A,b,B,f,e,\tilde{Q},c,d,P] \\ G_{011}[a,A,b,B,f,e,\tilde{Q},c,d,P] =& -(G_{111}[c,c,a,A,b,B,f,e,\tilde{Q},c,d,P] \\ & +G_{111}[d,d,a,A,b,B,f,e,\tilde{Q},c,d,P])/s_{cd} \end{aligned} \tag{C5}$$

The function of G_{111} is derived below.

a. *Real* $\tilde{Q}$

For $\tilde{Q}$ real the canonical form G_{111} can be evaluated using a basis for the loop momentum based on any pair of real null momenta c and d since, if the momentum crossing the cut is the sum of two null momenta, $P = c + d$, the cut loop momentum can be parametrised by

$$\lambda_\ell = \cos\frac{\theta}{2}\lambda_c + \sin\frac{\theta}{2}\epsilon^{-i\phi}\lambda_d \quad , \qquad \bar{\lambda}_\ell = \cos\frac{\theta}{2}\bar{\lambda}_c + \sin\frac{\theta}{2}\epsilon^{i\phi}\bar{\lambda}_d\,. \tag{C6}$$

The expression for the canonical form has a range of special cases if certain combinations of A, B, P and $\tilde{Q}$ vanish:

$$G_{111} = \begin{cases} G^{g}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] \neq 0 \\ G^{x}_{111} & [B|P|A\rangle = 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] \neq 0 \\ G^{y1}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] \neq 0 \\ G^{y2}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] = 0 \\ G^{y12}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] = 0 \\ G^{xy}_{111} & [B|P|A\rangle = 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] = 0 \end{cases} \tag{C7}$$

The full canonical form can be split into terms involving just the $[\ell|\tilde{Q}|\ell\rangle^{-1}$ pole and terms involving one or both of $\langle \ell\, A\rangle^{-1}$ and $[\ell B]^{-1}$. The contributions from terms involving no extra pole (np), an extra angle pole (ap), an extra square pole (sp) and both extra poles (dp) are given explicitly below. The decomposition of each of the special cases for G_{111} into these pieces is:

$$G^{g}_{111} = (C_{np:0}+C_{np:\gamma R}+C_{np:\gamma I}+C_{ap:0}+C_{ap:a}+C_{ap:\gamma R}+C_{ap:\gamma I} \\ +C_{sp:0}+C_{sp:s}+C_{sp:\gamma R}+C_{sp:\gamma I}+C_{dp:0}+C_{dp:a}+C_{dp:s}+C_{dp:\gamma R}+C_{dp:\gamma I}) \tag{C8}$$

$$G^{x}_{111} = (C_{np:0}+C_{np:\gamma R}+C_{np:\gamma I}+C_{ap:0}+C_{ap:a}+C_{ap:\gamma R}+C_{ap:\gamma I} \\ +C_{sp:0}+C_{sp:s}+C_{sp:\gamma R}+C_{sp:\gamma I}+C_{dp:0}+C_{dp:ax}+C_{dp:sx}+C_{dp:\gamma Rx}+C_{dp:\gamma Ix}) \tag{C9}$$

$$G^{y1}_{111} = (C_{np:0}+C_{np:\gamma R}+C_{np:\gamma I}+C_{ap:0}+C_{ap:ax}+C_{ap:\gamma Rx}+C_{ap:\gamma Ix} \\ +C_{sp:0}+C_{sp:s}+C_{sp:\gamma R}+C_{sp:\gamma I}+C_{dp:0}+C_{dp:ay}+C_{dp:s}+C_{dp:\gamma Ry1}+C_{dp:\gamma Iy1}) \tag{C10}$$

$$G^{y2}_{111} = (C_{np:0}+C_{np:\gamma R}+C_{np:\gamma I}+C_{ap:0}+C_{ap:a}+C_{ap:\gamma R}+C_{ap:\gamma I} \\ +C_{sp:0}+C_{sp:sx}+C_{sp:\gamma Rx}+C_{sp:\gamma Ix}+C_{dp:0}+C_{dp:a}+C_{dp:sy}+C_{dp:\gamma Ry2}+C_{dp:\gamma Iy2}) \tag{C11}$$

$$G^{xy}_{111} = (C_{np:0}+C_{np:\gamma R}+C_{np:\gamma I}+C_{ap:0}+C_{ap:ax}+C_{ap:\gamma Rx}+C_{ap:\gamma Ix} \\ +C_{sp:0}+C_{sp:sx}+C_{sp:\gamma Rx}+C_{sp:\gamma Ix}+C_{dp:0}+C_{dp:axy}+C_{dp:sxy}+C_{dp:\gamma Rxy}+C_{dp:\gamma Ixy}) \tag{C12}$$

Using the definitions,

$$\begin{aligned}
\mathcal{A}_1 &= \frac{\langle d\,a\rangle\,\langle d\,e\rangle}{\langle d\,A\rangle^2}\left(\langle c\,d\rangle\left(\frac{\langle a\,A\rangle\,\langle d\,y\rangle}{\langle d\,a\rangle}+\frac{\langle e\,A\rangle\,\langle d\,y\rangle}{\langle d\,e\rangle}+\langle y\,A\rangle\right)+2\,\langle c\,A\rangle\,\langle d\,y\rangle\right) \\
\mathcal{A}_0 &= \frac{\langle d\,a\rangle\,\langle d\,e\rangle}{\langle d\,A\rangle^3}\Bigg(\langle c\,A\rangle\,\langle c\,d\rangle\left(\frac{\langle a\,A\rangle\,\langle d\,y\rangle}{\langle d\,a\rangle}+\frac{\langle e\,A\rangle\,\langle d\,y\rangle}{\langle d\,e\rangle}+\langle y\,A\rangle\right) \\
&\qquad +\langle c\,d\rangle^2\left(\frac{\langle a\,A\rangle\,\langle e\,A\rangle\,\langle d\,y\rangle}{\langle d\,a\rangle\,\langle d\,e\rangle}+\frac{\langle a\,A\rangle\,\langle y\,A\rangle}{\langle d\,a\rangle}+\frac{\langle e\,A\rangle\,\langle y\,A\rangle}{\langle d\,e\rangle}\right)+\langle c\,A\rangle^2\,\langle d\,y\rangle\Bigg) \\
\mathcal{A}_p &= \frac{\langle a\,A\rangle\,\langle e\,A\rangle\,\langle y\,A\rangle\,\langle c\,d\rangle^3}{\langle d\,A\rangle^4}
\end{aligned} \tag{C13}$$

$$\begin{aligned}
\mathcal{S}_1 &= \frac{[c\,b]\,[c\,f]}{[c\,B]^2}\left([d\,c]\left(\frac{[b\,B]\,[c\,x]}{[c\,b]}+\frac{[f\,B]\,[c\,x]}{[c\,f]}+[x\,B]\right)+2\,[d\,B]\,[c\,x]\right) \\
\mathcal{S}_0 &= \frac{[c\,b]\,[c\,f]}{[c\,B]^3}\Bigg([d\,B]\,[d\,c]\left(\frac{[b\,B]\,[c\,x]}{[c\,b]}+\frac{[f\,B]\,[c\,x]}{[c\,f]}+[x\,B]\right) \\
&\qquad +[d\,c]^2\left(\frac{[b\,B]\,[f\,B]\,[c\,x]}{[c\,b]\,[c\,f]}+\frac{[b\,B]\,[x\,B]}{[c\,b]}+\frac{[f\,B]\,[x\,B]}{[c\,f]}\right)+[d\,B]^2\,[c\,x]\Bigg) \\
\mathcal{S}_p &= \frac{[b\,B]\,[f\,B]\,[x\,B]\,[d\,c]^3}{[c\,B]^4}
\end{aligned} \tag{C14}$$

and

$$C_{pre} = \frac{\langle d\,a\rangle}{\langle d\,A\rangle}\frac{[c\,b]}{[c\,B]}\frac{\langle d\,e\rangle\,[c\,f]}{[c|\tilde{Q}|d\rangle}\,\langle d\,y\rangle\,[c\,x] \tag{C15}$$

the contributions of the no-extra-pole piece are

$$C_{np:0} = -\frac{C_{pre}}{2}[c|\tilde{Q}|d\rangle\left[\frac{\mathcal{A}_0\mathcal{S}_1+\mathcal{S}_0\mathcal{A}_1}{[d|\tilde{Q}|c\rangle}-\frac{\mathcal{A}_0\mathcal{S}_0\big([c|\tilde{Q}|c\rangle+[d|\tilde{Q}|d\rangle\big)}{[d|\tilde{Q}|c\rangle^2}\right], \tag{C16}$$

$$C_{np:\gamma R} = -\frac{C_{pre}}{4}\Big(-\mathcal{P}_Q(1+2r_Q)+\mathcal{A}_1+\mathcal{S}_1-\frac{1}{Q_C}\big(\mathcal{A}_0\mathcal{S}_1+\mathcal{S}_0\mathcal{A}_1\big)+\frac{\mathcal{P}_Q(1+2r_Q)}{Q_C^2}\mathcal{A}_0\mathcal{S}_0\Big), \tag{C17}$$

and

$$\begin{aligned} C_{np:\gamma I} = -\frac{C_{pre}}{16\mathcal{P}_\delta^2}\Bigg[&\mathcal{P}_Q^2\big[1+\frac{1}{Q_C^2}\mathcal{A}_0\mathcal{S}_0\big]\Bigg((2+8r_Q-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(8r_Q-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\\ &+\mathcal{P}_Q\Bigg[\big[\mathcal{A}_1+\frac{\mathcal{A}_0\mathcal{S}_1}{Q_C}\big]\Bigg((-2+4r_Q-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(-4+4r_Q-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\\ &-\big[\mathcal{S}_1+\frac{\mathcal{S}_0\mathcal{A}_1}{Q_C}\big]\Bigg((2+4r_Q-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(4r_Q-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\Bigg]\\ &-2\big[\mathcal{A}_1\mathcal{S}_1-Q_C-\frac{\mathcal{A}_0\mathcal{S}_0}{Q_C}\big]\Bigg((-2-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(-4-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\\ &+2\mathcal{A}_0\Bigg((-6-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(-8-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\\ &+2\mathcal{S}_0\Bigg((2-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Bigg)\Bigg] \end{aligned} \tag{C18}$$

with

$$\begin{aligned} \mathcal{P}_Q &= \Big(\frac{[d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\ ,\ r_Q = \frac{[c|\tilde{Q}|c\rangle}{[d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle}\ ,\ Q_C = \frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\\ \mathcal{P}_\delta &= \sqrt{\frac{\big([d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle\big)^2+4[d|\tilde{Q}|c\rangle[c|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle^2}}\\ d_\delta &= -2\left[\frac{[c|\tilde{Q}|c\rangle\big([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle\big)+2[c|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle^2}\right]\frac{1}{\mathcal{P}_\delta^2} \end{aligned} \tag{C19}$$

The contributions of the bonus angle pole pieces are

$$C_{ap:0} = -\frac{C_{pre}\mathcal{A}_p}{2}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{[c|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\left[\mathcal{S}_1-\mathcal{S}_0\Big(\frac{[c|\tilde{Q}|c\rangle}{[d|\tilde{Q}|c\rangle}+\frac{[d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}+\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big)\right] \tag{C20}$$

$$C_{ap:a} = -\frac{C_{pre}\mathcal{A}_p}{\mathcal{P}_{\mathcal{A}}}\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big[\Big(\frac{\langle c\,A\rangle^2}{\langle d\,A\rangle^2}+\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\mathcal{S}_1+\mathcal{S}_0\Big)\Big(\frac{[A|d|A\rangle^2}{2[A|P|A\rangle^2}-b_{\mathcal{A}}\frac{[A|d|A\rangle}{[A|P|A\rangle}\Big)$$
$$-\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\mathcal{S}_1+2\mathcal{S}_0\Big)\frac{[A|d|A\rangle}{[A|P|A\rangle}\Big] \tag{C21}$$

$$C_{ap:ax} = C_{pre}\mathcal{A}_p\frac{\langle d\,A\rangle^3\,[c|\tilde{Q}|d\rangle}{\langle c\,A\rangle^3\,[d|\tilde{Q}|A\rangle}$$
$$\times\Big[\frac{1}{3}\frac{[A|d|A\rangle^3}{[A|P|A\rangle^3}\Big(\frac{\langle c\,A\rangle^2}{\langle d\,A\rangle^2}+\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\mathcal{S}_1+\mathcal{S}_0\Big)-\frac{1}{2}\frac{[A|d|A\rangle^2}{[A|P|A\rangle^2}\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\mathcal{S}_1+2\mathcal{S}_0\Big)+\frac{[A|d|A\rangle}{[A|P|A\rangle}\mathcal{S}_0\Big] \tag{C22}$$

$$C_{ap:\gamma R} = -C_{pre}\mathcal{A}_p\Big[-\frac{1}{2\mathcal{P}_{\mathcal{A}}}\Big((\frac{3}{2}+b_1)+(\frac{1}{2}+b_1)\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}-(\frac{1}{2}-b_1)\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big)$$
$$-\frac{1}{4Q_C}\Big(\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}-\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big)+\frac{\mathcal{P}_Q(1+2r_Q)}{4Q_C^2}\mathcal{S}_0\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big] \tag{C23}$$

$$C_{ap:\gamma Rx} = -C_{pre}\mathcal{A}_p\Big[-\frac{1}{2}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{1}{\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q}\Big(\frac{1}{3}-\frac{1}{6}\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}+\frac{1}{3}\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big)$$
$$-\frac{1}{4Q_C}\Big(\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}-\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big)+\frac{\mathcal{P}_Q(1+2r_Q)}{4Q_C^2}\mathcal{S}_0\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big] \tag{C24}$$

$$C_{ap:\gamma I} = -\frac{C_{pre}\mathcal{A}_p}{4\mathcal{P}_\delta^2}\Bigg[-\frac{\mathcal{P}_{r1}}{\mathcal{P}_{\mathcal{A}}}\Big[1+\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}+\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big]\Big[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_1-4b_1-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\Big]$$
$$+4\frac{\mathcal{P}_{r1}}{\mathcal{P}_{\mathcal{A}}}\Big[2+\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big]\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)$$
$$+\frac{\mathcal{P}_Q}{2Q_C}\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_Q-4-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\Big]$$
$$+\frac{\mathcal{P}_Q}{2Q_C}\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\Big[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_Q-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\Big]$$
$$+\mathcal{S}_0\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{\mathcal{P}_Q^2}{2Q_C^2}\Big[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(8r_Q-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\Big]$$
$$+\frac{\mathcal{S}_0}{Q_C}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(-4-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\Big]\Bigg] \tag{C25}$$

$$
\begin{aligned}
C_{ap:\gamma Ix} = -\frac{C_{pre}\mathcal{A}_p}{4\mathcal{P}_\delta^2}\Bigg[&\frac{1}{6}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{\mathcal{P}_{r1}}{\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q}\left[1+\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}+\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\right]\\
&\times\left[(8+12r_1-10d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(15d_\delta^2-18d_\delta r_1-16c_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\\
&-\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{\mathcal{P}_{r1}}{\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q}\left[2+\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\right]\left[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_1-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\\
&+4\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{\mathcal{P}_{r1}}{\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q}\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\\
&+\frac{\mathcal{P}_Q}{2Q_C}\mathcal{S}_1\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\left[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_Q-4-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\\
&+\frac{\mathcal{P}_Q}{2Q_C}\mathcal{S}_0\frac{\langle d\,A\rangle^2}{\langle c\,A\rangle^2}\left[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(4r_Q-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\\
&+\mathcal{S}_0\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\frac{\mathcal{P}_Q^2}{2Q_C^2}\left[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(8r_Q-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\\
&+\frac{\mathcal{S}_0}{Q_C}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\left[2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}+(-4-3d_\delta)\Big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\right]\Bigg]
\end{aligned}
\tag{C26}
$$

with

$$
\begin{aligned}
\mathcal{P}_{\mathcal{A}} &= \frac{\langle c\,A\rangle^2}{\langle d\,A\rangle^2}-\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\Big(\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\Big)-\frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle} = -\frac{\langle c\,d\rangle\,\langle A|P\tilde{Q}|A\rangle}{\langle d\,A\rangle^2\,[c|\tilde{Q}|d\rangle},\\
b_{\mathcal{A}} &= -\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\frac{1}{\mathcal{P}_{\mathcal{A}}} = -\frac{\langle c\,d\rangle\,[d|\tilde{Q}|A\rangle}{\langle d\,A\rangle\,[c|\tilde{Q}|d\rangle}\frac{1}{\mathcal{P}_{\mathcal{A}}},
\end{aligned}
\tag{C27}
$$

$$
\begin{aligned}
c_\delta &= \frac{[c|\tilde{Q}|c\rangle^2}{4[c|\tilde{Q}|d\rangle^2}\frac{1}{\mathcal{P}_\delta^2}.\ ,\ b_1 = \Big(\mathcal{P}_Q r_Q\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\frac{\langle c\,A\rangle^2}{\langle d\,A\rangle^2}\Big)\frac{1}{\mathcal{P}_A}\\
\mathcal{P}_{r1} &= -\frac{\mathcal{P}_Q}{2}-\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\ ,\ r_1 = \Big(-\frac{\mathcal{P}_Q r_Q}{2}+\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\Big)\frac{1}{\mathcal{P}_{r1}}
\end{aligned}
\tag{C28}
$$

Similarly the bonus square pole pieces give

$$
C_{sp:0} = -\frac{C_{pre}\mathcal{S}_p}{2}\frac{[c|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\frac{[c\,B]}{[d\,B]}\left[\mathcal{A}_1-\mathcal{A}_0\Big(\frac{\big([c|\tilde{Q}|c\rangle+[d|\tilde{Q}|d\rangle\big)}{[d|\tilde{Q}|c\rangle}+\frac{[c\,B]}{[d\,B]}\Big)\right]
\tag{C29}
$$

$$
\begin{aligned}
C_{sp:s} =& -\frac{C_{pre}\mathcal{S}_p}{\mathcal{P}_{\mathcal{S}}}\frac{[c\,B]^2}{[d\,B]^2}\\
&\times\Big(\big(\frac{[d\,B]^2}{[c\,B]^2}+\frac{[d\,B]}{[c\,B]}\mathcal{A}_1+\mathcal{A}_0\big)\big[\frac{1}{2}\frac{[B|c|B\rangle^2}{[B|P|B\rangle^2}-b_{\mathcal{S}}\frac{[B|c|B\rangle}{[B|P|B\rangle}\big]-\big(\frac{[d\,B]}{[c\,B]}\mathcal{A}_1+2\mathcal{A}_0\big)\frac{[B|c|B\rangle}{[B|P|B\rangle}\Big)
\end{aligned}
\tag{C30}
$$

$$C_{sp:sx} =-C_{pre}\mathcal{S}_p\frac{[c\,B]^3\,[c|\tilde{Q}|d\rangle}{[d\,B]^2\,[c\,d]\,[B|\tilde{Q}|c\rangle}$$
$$\times\Bigg[\frac{1}{3}\frac{[B|c|B\rangle^3}{[B|P|B\rangle^3}\Big(\frac{[d\,B]^2}{[c\,B]^2}+\frac{[d\,B]}{[c\,B]}\mathcal{A}_1+\mathcal{A}_0\Big)-\frac{1}{2}\frac{[B|c|B\rangle^2}{[B|P|B\rangle^2}\Big(\frac{[d\,B]}{[c\,B]}\mathcal{A}_1+2\mathcal{A}_0\Big)+\frac{[B|c|B\rangle}{[B|P|B\rangle}\mathcal{A}_0\Bigg] \tag{C31}$$

$$C_{sp:\gamma R} = -C_{pre}\mathcal{S}_p\Bigg[-\frac{1}{2\mathcal{P}_{\mathcal{S}}}\Big(\frac{1}{2}-b_{\bar{2}}-(\frac{1}{2}+b_{\bar{2}})\mathcal{A}_1\frac{[c\,B]}{[d\,B]}-(\frac{3}{2}+b_{\bar{2}})\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Big)$$
$$-\frac{1}{4Q_C}\Big(\mathcal{A}_1\frac{[c\,B]}{[d\,B]}-\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Big)+\frac{\mathcal{P}_Q(1+2r_Q)}{4Q_C^2}\mathcal{A}_0\frac{[c\,B]}{[d\,B]}\Bigg] \tag{C32}$$

$$C_{sp:\gamma Rx} = -C_{pre}\mathcal{S}_p\Bigg[\frac{1}{2}\frac{1}{\mathcal{P}_Q r_Q\frac{[d\,B]}{[c\,B]}-Q_C}\Big(\frac{1}{3}-\frac{1}{6}\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+\frac{1}{3}\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Big)$$
$$-\frac{1}{4Q_C}\Big(\mathcal{A}_1\frac{[c\,B]}{[d\,B]}-\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Big)+\frac{\mathcal{P}_Q(1+2r_Q)}{4Q_C^2}\mathcal{A}_0\frac{[c\,B]}{[d\,B]}\Bigg] \tag{C33}$$

$$C_{sp:\gamma I} = \frac{C_{pre}\mathcal{S}_p}{\mathcal{P}_\delta^2}\Bigg[\frac{\mathcal{P}_{r2}}{4\mathcal{P}_{\mathcal{S}}}\Big(\big[1+\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big]\big[(2-4b_{\bar{2}}+4r_{\bar{2}}-3d_\delta)\frac{[d|\tilde{Q}|c\rangle}{[d|\tilde{Q}|d\rangle}-(-4b_{\bar{2}}+4r_{\bar{2}}-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big]$$
$$-4\big[\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+2\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big]\big[\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big]\Big)$$
$$+\frac{\mathcal{P}_Q}{8Q_C}\Big(\big[\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big]\big[(2+4r_Q-3d_\delta)\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-(4r_Q-3d_\delta)\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big]-4\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big[\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big]\Big)$$
$$-\mathcal{A}_0\frac{\mathcal{P}_Q^2\big[(2+8r_Q-3d_\delta)[d|\tilde{Q}|d\rangle-(8r_Q-3d_\delta)[c|\tilde{Q}|c\rangle\big]-2Q_C\big[(2+3d_\delta)[d|\tilde{Q}|d\rangle-(4+3d_\delta)[c|\tilde{Q}|c\rangle\big]}{8Q_C^2[c|\tilde{Q}|d\rangle}\frac{[c\,B]}{[d\,B]}\Bigg] \tag{C34}$$

$$C_{sp:\gamma Ix} = -\frac{C_{pre}\mathcal{S}_p}{[c|\tilde{Q}|d\rangle}\Bigg[\frac{\mathcal{P}_{r2}\,[c\,B]}{\mathcal{P}_Q r_Q[d\,B]-Q_C\,[c\,B]}\Bigg(\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Big([d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle\Big)$$
$$+\frac{1}{24}\Bigg[1+\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Bigg]\Bigg[(8+12r_{\bar{2}}-10d_\delta)[d|\tilde{Q}|d\rangle+(15d_\delta^2-18d_\delta r_{\bar{2}}-16c_\delta)\Big([d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle\Big)\Bigg]$$
$$-\frac{1}{4}\Bigg[\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+2\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\Bigg]\Bigg[2[d|\tilde{Q}|d\rangle+(4r_{\bar{2}}-3d_\delta)\Big([d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle\Big)\Bigg]\Bigg)$$
$$-\frac{\mathcal{P}_Q}{8Q_C}\Big(\big[\mathcal{A}_1\frac{[c\,B]}{[d\,B]}+\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big]\big[(2+4r_Q-3d_\delta)[d|\tilde{Q}|d\rangle-(4r_Q-3d_\delta)[c|\tilde{Q}|c\rangle\big]-4\mathcal{A}_0\frac{[c\,B]^2}{[d\,B]^2}\big[[d|\tilde{Q}|d\rangle-[c|\tilde{Q}|c\rangle\big]\Big)$$
$$+\mathcal{A}_0\frac{\mathcal{P}_Q^2\big[(2+8r_Q-3d_\delta)[d|\tilde{Q}|d\rangle-(8r_Q-3d_\delta)[c|\tilde{Q}|c\rangle\big]-2Q_C\big[(2+3d_\delta)[d|\tilde{Q}|d\rangle-(4+3d_\delta)[c|\tilde{Q}|c\rangle\big]}{8Q_C^2}\frac{[c\,B]}{[d\,B]}\Bigg] \tag{C35}$$

with

$$\mathcal{P}_{\mathcal{S}} = \frac{[d\,B]^2}{[c\,B]^2} + \frac{[d\,B]}{[c\,B]}\Big(\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle} - \frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\Big) - \frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle},$$

$$b_{\mathcal{S}} = -\Big(\frac{[d\,B]}{[c\,B]}\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle} - \frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big)\frac{1}{\mathcal{P}_{\mathcal{S}}} \tag{C36}$$

$$\mathcal{P}_{r2} = \frac{\mathcal{P}_Q}{2} - \frac{[d\,B]}{[c\,B]}\ ,\ r_{\bar{2}} = \Big(\frac{\mathcal{P}_Q r_Q}{2}\Big)\frac{1}{\mathcal{P}_{r2}}\ ,\ b_{\bar{2}} = \Big(Q_C - \mathcal{P}_Q r_Q \frac{[d\,B]}{[c\,B]}\Big)\frac{1}{\mathcal{P}_{\mathcal{S}}} \tag{C37}$$

Finally the double bonus pole pieces are

$$C_{dp:0} = \frac{C_{pre}\mathcal{A}_p\mathcal{S}_p}{2}\frac{[c|\tilde{Q}|d\rangle\,\langle d\,A\rangle\,[c\,B]}{[d|\tilde{Q}|c\rangle\,\langle c\,A\rangle\,[d\,B]}\Bigg[\frac{[c|\tilde{Q}|c\rangle}{[d|\tilde{Q}|c\rangle} + \frac{[d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle} + \frac{\langle d\,A\rangle}{\langle c\,A\rangle} + \frac{[c\,B]}{[d\,B]}\Bigg] \tag{C38}$$

$$C_{dp:ab} = \frac{C_{pre}\mathcal{A}_p\mathcal{S}_p}{\mathcal{D}_Q}\Big(\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big)^4\Bigg[\frac{\big(1-\frac{[A|c|A\rangle^2}{[A|P|A\rangle^2}\big)}{2(\mathcal{E}_Q-1)(\mathcal{E}_S-1)} - \frac{\big(1-\frac{[A|c|A\rangle}{[A|P|A\rangle}\big)(\mathcal{E}_Q+\mathcal{E}_S-2)}{(\mathcal{E}_Q-1)^2(\mathcal{E}_S-1)^2}\Bigg] \tag{C39}$$

$$C_{dp:ax} = \frac{C_{pre}\mathcal{A}_p\mathcal{S}_p}{\mathcal{D}_Q}\Big(\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big)^4\Bigg[\frac{\big(1-\frac{[A|c|A\rangle^3}{[A|P|A\rangle^3}\big)}{3(\mathcal{E}_Q-1)} - \frac{\big(1-\frac{[A|c|A\rangle^2}{[A|P|A\rangle^2}\big)}{2(\mathcal{E}_Q-1)^2} + \frac{\big(1-\frac{[A|c|A\rangle}{[A|P|A\rangle}\big)}{(\mathcal{E}_Q-1)^3}\Bigg] \tag{C40}$$

$$C_{dp:ay} = \frac{C_{pre}\mathcal{A}_p\mathcal{S}_p}{\mathcal{D}_Q}\Big(\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big)^4\Bigg[\frac{\big(1-\frac{[A|c|A\rangle^3}{[A|P|A\rangle^3}\big)}{3(\mathcal{E}_S-1)} - \frac{\big(1-\frac{[A|c|A\rangle^2}{[A|P|A\rangle^2}\big)}{2(\mathcal{E}_S-1)^2} + \frac{\big(1-\frac{[A|c|A\rangle}{[A|P|A\rangle}\big)}{(\mathcal{E}_S-1)^3}\Bigg] \tag{C41}$$

$$C_{dp:axy} = \frac{C_{pre}\mathcal{A}_p\mathcal{S}_p}{4\mathcal{D}_Q}\Big(\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Big)^4\Big(1-\frac{[A|c|A\rangle^4}{[A|P|A\rangle^4}\Big) \tag{C42}$$

with

$$\mathcal{D}_Q = \Big[\frac{\langle c\,A\rangle}{\langle d\,A\rangle} - \frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\Big]\ ,\ \mathcal{E}_Q = -\Bigg[\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle} + \frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\frac{\langle d\,A\rangle}{\langle c\,A\rangle}\Bigg]\frac{1}{\mathcal{D}_Q}\ ,\ \mathcal{E}_S = -\frac{[d\,B]}{[c\,B]}\frac{\langle d\,A\rangle}{\langle c\,A\rangle} \tag{C43}$$

$$C_{dp:sb} = \frac{C_{pre}\mathcal{S}_p\mathcal{A}_p}{2\mathcal{P}_{\mathcal{S}}\mathcal{D}_A}\frac{[c\,B]^2}{[d\,B]^2}\Bigg(1-\frac{[B|d|B\rangle^2}{[B|P|B\rangle^2} - 2(\mathcal{F}_Q+\mathcal{E}_A)\Big[1-\frac{[B|d|B\rangle}{[B|P|B\rangle}\Big]\Bigg) \tag{C44}$$

$$C_{dp:sx} = -\frac{C_{pre}\mathcal{S}_p\mathcal{A}_p}{\mathcal{P}_{\mathcal{S}}}\frac{[c\,B]^3}{[d\,B]^3}\Bigg[\frac{1}{3}\Big(1-\frac{[B|d|B\rangle^3}{[B|P|B\rangle^3}\Big) - \frac{\mathcal{F}_Q}{2}\Big(1-\frac{[B|d|B\rangle^2}{[B|P|B\rangle^2}\Big) + \mathcal{F}_Q^2\Big(1-\frac{[B|d|B\rangle}{[B|P|B\rangle}\Big)\Bigg] \tag{C45}$$

$$C_{dp:sy} = -\frac{C_{pre}\mathcal{S}_p\mathcal{A}_p}{\mathcal{D}_A}\frac{[c\,B]^3}{[d\,B]^3}\frac{1}{\frac{[d\,B]}{[c\,B]}-\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}}\left[\frac{1}{3}\Big(1-\frac{[B|d|B\rangle^3}{[B|P|B\rangle^3}\Big)-\frac{\mathcal{E}_A}{2}\Big(1-\frac{[B|d|B\rangle^2}{[B|P|B\rangle^2}\Big)+\mathcal{E}_A^2\Big(1-\frac{[B|d|B\rangle}{[B|P|B\rangle}\Big)\right] \tag{C46}$$

$$C_{dp:sxy} = C_{pre}\mathcal{S}_p\mathcal{A}_p\frac{[c\,B]^4}{[d\,B]^4}\frac{1}{\frac{[d\,B]}{[c\,B]}-\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}}\frac{1}{4}\Big(1-\frac{[B|d|B\rangle^4}{[B|P|B\rangle^4}\Big) \tag{C47}$$

with

$$\mathcal{F}_Q = \Big(\frac{[d\,B]^2}{[c\,B]^2}-\frac{[d\,B]}{[c\,B]}\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\Big)\frac{1}{\mathcal{G}_Q}\ ,\ \mathcal{D}_A = \frac{[d\,B]}{[c\,B]}+\frac{\langle c\,A\rangle}{\langle d\,A\rangle}\ ,\ \mathcal{E}_A = -\frac{[d\,B]}{[c\,B]}\frac{1}{\mathcal{D}_A} \tag{C48}$$

$$\begin{aligned}C_{dp\gamma:R} = -\frac{1}{4}\mathcal{A}_p\mathcal{S}_pC_{pre}\Bigg(&\frac{\langle d\,A\rangle^2\,(1-2(a_1+b_1))}{\langle c\,A\rangle^2\,\mathcal{P}_{a1}\mathcal{P}_{b1}}-\frac{[c\,B]^2\,(-2(a_1+b_{\bar{2}})-5)}{[d\,B]^2\,\mathcal{P}_{a1}\mathcal{P}_{b2}}\\&+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q+1)}{\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}+\frac{\langle d\,A\rangle^2\,[c\,B]^2\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\Big)}{\langle c\,A\rangle^2\,[d\,B]^2\,Q_C}\Bigg)\end{aligned} \tag{C49}$$

$$\begin{aligned}C_{dp\gamma:Rx} = -\frac{1}{2}\mathcal{A}_p\mathcal{S}_pC_{pre}\Bigg(&\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q+1)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}+\frac{\langle d\,A\rangle^3\,\big({b_1}^2-\frac{1}{2}b_1+\frac{1}{3}\big)}{\langle c\,A\rangle^3\,\mathcal{P}_{b1}}\\&+\frac{[c\,B]^3\,\big((b_{\bar{2}}+1)^2+\frac{1}{2}(b_{\bar{2}}+1)+\frac{1}{3}\big)}{[d\,B]^3\,\mathcal{P}_{b2}}\Bigg)\end{aligned} \tag{C50}$$

$$\begin{aligned}C_{dp\gamma:Ry1} = -\frac{1}{2}\mathcal{A}_p\mathcal{S}_pC_{pre}\Bigg(&-\frac{[c\,B]^2\,(-2(a_1+b_{\bar{2}})-5)}{2\,[d\,B]^2\,\mathcal{P}_{a1}\mathcal{P}_{b2}}+\frac{\langle d\,A\rangle^3\,\big({a_1}^2-\frac{1}{2}a_1+\frac{1}{3}\big)}{\langle c\,A\rangle^3\,\mathcal{P}_{a1}\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Qr_Q\Big)}\\&+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q+1)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}+\frac{\langle d\,A\rangle^2\,[c\,B]^2\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\Big)}{2\,\langle c\,A\rangle^2\,[d\,B]^2\,Q_C}\Bigg)\end{aligned} \tag{C51}$$

$$\begin{aligned}C_{dp\gamma:Ry2} = -\frac{1}{2}\mathcal{A}_p\mathcal{S}_pC_{pre}\Bigg(&\frac{\langle d\,A\rangle^2\,(1-2(a_1+b_1))}{2\,\langle c\,A\rangle^2\,\mathcal{P}_{a1}\mathcal{P}_{b1}}-\frac{[c\,B]^2\,\big({a_1}^2+\frac{5}{2}a_1+\frac{11}{6}\big)}{[d\,B]^2\,\mathcal{P}_{a1}\,\Big(\frac{[d\,B]\mathcal{P}_Qr_Q}{[c\,B]}-Q_C\Big)}\\&+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q+1)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}+\frac{\langle d\,A\rangle^2\,[c\,B]^2\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\Big)}{2\,\langle c\,A\rangle^2\,[d\,B]^2\,Q_C}\Bigg)\end{aligned} \tag{C52}$$

$$\begin{aligned}C_{dp\gamma:Ry12} = -\frac{1}{2}\mathcal{A}_p\mathcal{S}_pC_{pre}\Bigg(&-\frac{[c\,B]^2\,\big({a_1}^2+\frac{5}{2}a_1+\frac{11}{6}\big)}{[d\,B]^2\,\mathcal{P}_{a1}\,\Big(\frac{[d\,B]\mathcal{P}_Qr_Q}{[c\,B]}-Q_C\Big)}+\frac{\langle d\,A\rangle^3\,\big({a_1}^2-\frac{1}{2}a_1+\frac{1}{3}\big)}{\langle c\,A\rangle^3\,\mathcal{P}_{a1}\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Qr_Q\Big)}\\&+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q+1)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}+\frac{\langle d\,A\rangle^2\,[c\,B]^2\,\Big(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\Big)}{2\,\langle c\,A\rangle^2\,[d\,B]^2\,Q_C}\Bigg)\end{aligned} \tag{C53}$$

$$
\begin{aligned}
C_{dp\gamma:Rxy} = -\frac{1}{2}\mathcal{A}_p\mathcal{S}_p C_{pre}\Bigg(& \frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q(2r_Q{+}1)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2} + \frac{\langle d\,A\rangle^2\,[c\,B]^2\left(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\right)}{2\,\langle c\,A\rangle^2\,[d\,B]^2\,Q_C} \\
& + \frac{\langle d\,A\rangle^4}{4\,\langle c\,A\rangle^4\left(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q\right)} - \frac{[c\,B]^3}{4\,[d\,B]^3\left(\frac{[d\,B]\mathcal{P}_Q r_Q}{[c\,B]}-Q_C\right)}\Bigg)
\end{aligned}
\tag{C54}
$$

$$
\begin{aligned}
C_{dp\gamma:I} = -\frac{\mathcal{A}_p\mathcal{S}_p C_{pre}}{4\mathcal{P}_d^s}\Bigg(& -\frac{\langle d\,A\rangle^2\,\mathcal{P}_{r1}\big(\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)(-4a_1-4b_1+4r_1-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{\langle c\,A\rangle^2\,\mathcal{P}_{a1}\mathcal{P}_{b1}} \\
& -\frac{[c\,B]^2\,\mathcal{P}_{r2}\big(\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)(-4a_1-4b_{\bar{2}}+4r_{\bar{2}}-3d_\delta-12)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{[d\,B]^2\,\mathcal{P}_{a1}\mathcal{P}_{b2}} \\
& +\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2\big(\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2} \\
& +\frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q\big(\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C} \\
& -\frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q\big(\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C} \\
& +\frac{\langle d\,A\rangle\,[c\,B]\,\big((-3d_\delta-4)\big(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\big)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\big)}{\langle c\,A\rangle\,[d\,B]\,Q_C}\Bigg)
\end{aligned}
\tag{C55}
$$

$$C_{dp\gamma:Ix} = -\frac{\mathcal{A}_p\mathcal{S}_pC_{pre}}{4\mathcal{P}_d^s}\Bigg(\frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}$$
$$-\frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,((-3d_\delta-4)(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle\,[d\,B]\,Q_C}$$
$$-\frac{\langle d\,A\rangle^3\,\mathcal{P}_{r1}\left(\begin{array}{c}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left(d_\delta(18b_1-18r_1+15d_\delta)-24b_1r_1+24{b_1}^2-16c_\delta\right)\\+\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12b_1+12r_1-10d_\delta+8)\end{array}\right)}{6\,\langle c\,A\rangle^3\,\mathcal{P}_{b1}}$$
$$+\frac{[c\,B]^3\,\mathcal{P}_{r2}\left(\begin{array}{c}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left(d_\delta(18b_{\bar{2}}-18r_{\bar{2}}+15d_\delta)-24b_{\bar{2}}r_{\bar{2}}+24b_{\bar{2}}^2-16c_\delta\right)\\+\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12b_{\bar{2}}+12r_{\bar{2}}-10d_\delta+8)\end{array}\right)}{6\,[d\,B]^3\,\mathcal{P}_{b2}}$$
$$-\frac{3\,[c\,B]^3\,\mathcal{P}_{r2}((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-4b_{\bar{2}}+4r_{\bar{2}}-3d_\delta-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{[d\,B]^3\,\mathcal{P}_{b2}}\Bigg) \tag{C56}$$

$$C_{dp\gamma:Iy1} = -\frac{\mathcal{A}_p\mathcal{S}_pC_{pre}}{4\mathcal{P}_d^s}\Bigg(-\frac{[c\,B]^2\,\mathcal{P}_{r2}((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-4a_1-4b_{\bar{2}}+4r_{\bar{2}}-3d_\delta-12)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{[d\,B]^2\,\mathcal{P}_{a1}\mathcal{P}_{b2}}$$
$$-\frac{\langle d\,A\rangle^3\,\mathcal{P}_{r1}\left(\begin{array}{c}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left((24a_1+18d_\delta)(a_1-r_1)-16c_\delta+15{d_\delta}^2\right)\\+\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12a_1+12r_1-10d_\delta+8)\end{array}\right)}{6\,\langle c\,A\rangle^3\,\mathcal{P}_{a1}\left(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Qr_Q\right)}$$
$$+\frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}$$
$$-\frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,((-3d_\delta-4)(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle\,[d\,B]\,Q_C}\Bigg) \tag{C57}$$

$$
\begin{aligned}
C_{dp\gamma:Iy2} = -\frac{\mathcal{A}_p\mathcal{S}_pC_{pre}}{4\mathcal{P}_d^s}\Bigg(&-\frac{\langle d\,A\rangle^2\,\mathcal{P}_{r1}((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-4a_1-4b_1+4r_1-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle^2\,\mathcal{P}_{a1}\mathcal{P}_{b1}} \\
&-\frac{[c\,B]^2\,\mathcal{P}_{r2}\left(\begin{gathered}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left((24a_1+18d_\delta)(a_1-r_{\bar{2}})-16c_\delta+15{d_\delta}^2\right)\\ -18(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-4a_1+4r_{\bar{2}}-3d_\delta)\\ +\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12a_1+12r_{\bar{2}}-10d_\delta+8)\\ +72(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})-36\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\end{gathered}\right)}{6\,[d\,B]^2\,\mathcal{P}_{a1}\left(\frac{[d\,B]\mathcal{P}_Qr_Q}{[c\,B]}-Q_C\right)} \\
&+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2} \\
&+\frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C} \\
&-\frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C} \\
&+\frac{\langle d\,A\rangle\,[c\,B]\,((-3d_\delta-4)(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle\,[d\,B]\,Q_C}\Bigg)
\end{aligned} \tag{C58}
$$

$$C_{dp\gamma:Iy12} = -\frac{\mathcal{A}_p\mathcal{S}_pC_{pre}}{4\mathcal{P}_d^s}\Bigg(-\frac{\langle d\,A\rangle^3\,\mathcal{P}_{r1}\begin{pmatrix}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left((24a_1+18d_\delta)(a_1-r_1)-16c_\delta+15{d_\delta}^2\right)\\ +\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12a_1+12r_1-10d_\delta+8)\end{pmatrix}}{6\,\langle c\,A\rangle^3\,\mathcal{P}_{a1}\left(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Qr_Q\right)}$$
$$-\frac{[c\,B]^2\,\mathcal{P}_{r2}\begin{pmatrix}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left((24a_1+18d_\delta)(a_1-r_{\bar{2}})-16c_\delta+15{d_\delta}^2\right)\\ -18(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-4a_1+4r_{\bar{2}}-3d_\delta)\\ +\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(-12a_1+12r_{\bar{2}}-10d_\delta+8)\\ +72(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})-36\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\end{pmatrix}}{6\,[d\,B]^2\,\mathcal{P}_{a1}\left(\frac{[d\,B]\mathcal{P}_Qr_Q}{[c\,B]}-Q_C\right)}$$
$$+\frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2}$$
$$-\frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C}$$
$$+\frac{\langle d\,A\rangle\,[c\,B]\,((-3d_\delta-4)(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle\,[d\,B]\,Q_C}\Bigg) \tag{C59}$$

$$
\begin{aligned}
C_{dp\gamma:Ixy} = -\frac{\mathcal{A}_p \mathcal{S}_p C_{pre}}{4\mathcal{P}^s_d}\Bigg(& \frac{\langle d\,A\rangle^2\,[c\,B]\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle^2\,[d\,B]\,Q_C} \\
& + \frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{P}_Q^2((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(8r_Q-3d_\delta)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]\,{Q_C}^2} \\
& - \frac{\langle d\,A\rangle\,[c\,B]^2\,\mathcal{P}_Q((\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(-3d_\delta+4r_Q-4)+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{2\,\langle c\,A\rangle\,[d\,B]^2\,Q_C} \\
& + \frac{\langle d\,A\rangle\,[c\,B]\,((-3d_\delta-4)(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})}{\langle c\,A\rangle\,[d\,B]\,Q_C} \\
& - \frac{\langle d\,A\rangle^4\,\mathcal{P}_{r1}\left(\begin{array}{c}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left(4c_\delta(55d_\delta-32r_1)+120{d_\delta}^2r_1-105{d_\delta}^3\right) \\ +\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\left(-80d_\delta r_1+64r_1-72c_\delta+70{d_\delta}^2-56d_\delta+48\right)\end{array}\right)}{48\,\langle c\,A\rangle^4\left(\frac{\langle c\,A\rangle}{\langle d\,A\rangle}-\mathcal{P}_Q r_Q\right)} \\
& + \frac{[c\,B]^3\,\mathcal{P}_{r2}\left(\begin{array}{c}-\frac{1}{2}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left(-18d_\delta r_{\bar{2}}-16c_\delta+15{d_\delta}^2\right) \\ +\frac{1}{48}(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})\left(4c_\delta(55d_\delta-32r_{\bar{2}})+120{d_\delta}^2r_{\bar{2}}-105{d_\delta}^3\right) \\ +\frac{1}{48}\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\left(-80d_\delta r_{\bar{2}}+64r_{\bar{2}}-72c_\delta+70{d_\delta}^2-56d_\delta+48\right) \\ +3(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})(4r_{\bar{2}}-3d_\delta)-\frac{1}{2}\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}(12r_{\bar{2}}-10d_\delta+8) \\ -4(\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})+6\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}\end{array}\right)}{[d\,B]^3\left(\frac{[d\,B]\mathcal{P}_Q r_Q}{[c\,B]}-Q_C\right)}\Bigg)
\end{aligned}
\tag{C60}
$$

$$
\mathcal{P}^s_d = (\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle})^2+4\frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}\ ,\ d_\delta = -2\frac{\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}(\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}-\frac{[d|\tilde{Q}|d\rangle}{[c|\tilde{Q}|d\rangle})+2\frac{[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}}{\mathcal{P}^s_d}\ ,\ c_\delta = \frac{\frac{[c|\tilde{Q}|c\rangle}{[c|\tilde{Q}|d\rangle}^2}{\mathcal{P}^s_d}
\tag{C61}
$$

b. Complex $\tilde{Q}$

For complex $\tilde{Q}$, c and d are chosen so that $[c|\tilde{Q}|d\rangle = 0$ while $[d|\tilde{Q}|c\rangle \neq 0$. With c and d fixed there are now special cases if either $[BC]$ or $\langle A\,d\rangle$ vanish: The massive momentum is taken to be

$$
\tilde{Q} = P+\bar{\lambda}_c\lambda_X\,. \tag{C62}
$$

$$G_{111} = \begin{cases} G^{cc}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] \neq 0, [Bc] \neq 0, \langle A\, d\rangle \neq 0 \\ G^{cd}_{111} & [B|P|A\rangle = 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] \neq 0, [Bc] \neq 0, \langle A\, d\rangle \neq 0 \\ G^{ce}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] \neq 0, [Bc] \neq 0, \langle A\, d\rangle \neq 0 \\ G^{cx}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] = 0, [Bc] \neq 0, \langle A\, d\rangle \neq 0 \\ G^{cs}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle \neq 0, [B|P\tilde{Q}|B] = 0, [Bc] = 0, \langle A\, d\rangle \neq 0 \\ G^{ct}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] = 0, [Bc] = 0, \langle A\, d\rangle \neq 0 \\ G^{ca}_{111} & [B|P|A\rangle \neq 0, \langle A|P\tilde{Q}|A\rangle = 0, [B|P\tilde{Q}|B] \neq 0, [Bc] \neq 0, \langle A\, d\rangle = 0 \end{cases} \tag{C63}$$

$$G^{cc}_{111} = (C^J_{np:0} + {C^J_{np:\gamma}}^* + C^J_{ap:0} + C^J_{ap:a} + {C^J_{ap:\gamma}}^* + C^J_{sp:0} + C^J_{sp:s} + {C^J_{sp:\gamma}}^* + C^J_{dp:0} + C^J_{dp:s} + C^J_{dp:a} + {C^J_{dp:\gamma}}^*) \tag{C64}$$

$$G^{cd}_{111} = (C^J_{np:0} + {C^J_{np:\gamma}}^* + C^J_{ap:0} + C^J_{ap:a} + {C^J_{ap:\gamma}}^* + C^J_{sp:0} + C^J_{sp:s} + {C^J_{sp:\gamma}}^* + C^J_{dp:0} + C_{dp:ay} + C_{dp:sy} + {C^J_{dp:\gamma}}^*) \tag{C65}$$

$$G^{ce}_{111} = (C^J_{np:0} + {C^J_{np:\gamma}}^* + C^J_{ap:0} + C^J_{ap:ax} + {C^J_{ap:\gamma x}}^* + C^J_{sp:0} + C^J_{sp:s} + {C^J_{sp:\gamma}}^* + C^J_{dp:0} + C^J_{dp:s} + C^J_{dp:ax} + {C^J_{dp:\gamma z}}^*) \tag{C66}$$

$$G^{cx}_{111} = (C^J_{np:0} + {C^J_{np:\gamma}}^* + C^J_{ap:0} + C^J_{ap:a} + {C^J_{ap:\gamma}}^* + C^J_{sp:0} + C^J_{sp:sx} + {C^J_{sp:\gamma x}}^* + C^J_{dp:0} + C^J_{dp:sx} + C^J_{dp:a} + {C^J_{dp:\gamma x}}^*) \tag{C67}$$

$$G^{cs}_{111} = (C^H_{np:0} + {C^H_{np:\gamma}}^* + C^H_{ap:0} + C^H_{ap:a} + {C^H_{ap:\gamma}}^*) \tag{C68}$$

$$G^{ct}_{111} = (C^H_{np:0} + {C^H_{np:\gamma}}^* + C^H_{ap:0} + C^H_{ap:aa} + {C^H_{ap:\gamma a}}^*) \tag{C69}$$

$$G^{ca}_{111} = (C^Y_{np:0} + {C^Y_{np:\gamma}}^* + C^Y_{sp:0} + C^Y_{sp:a} + {C^Y_{sp:\gamma}}^*) \tag{C70}$$

The contributions are: from the no bonus pole piece

$$C^J_{np:0} = \frac{1}{2[d|\tilde{Q}|c\rangle} \left(-\frac{([c|\tilde{Q}|c\rangle + [d|\tilde{Q}|d\rangle)\mathcal{A}_0\mathcal{S}_0}{[d|\tilde{Q}|c\rangle} + \mathcal{A}_0\mathcal{S}_1 + \mathcal{A}_1\mathcal{S}_0 \right) \tag{C71}$$

$$C^H_{np:0} = \frac{C^x_{pre}}{2} \left(\frac{\mathcal{A}_1 T_0 + \mathcal{A}_0 T_1}{[d|\tilde{Q}|c\rangle} - \frac{\mathcal{A}_0 T_0([c|\tilde{Q}|c\rangle + [d|\tilde{Q}|d\rangle)}{[d|\tilde{Q}|c\rangle^2} \right) \tag{C72}$$

with

$$C^x_{pre} = \frac{1}{[dB]}\ ,\ T_x = [cb][cf][cx]\ ,\ T_0 = [dx][db][df]$$
$$T_1 = [dx][db][cf]+[dx][df][cb]+[db][df][cx]\ ,\ T_2 = [dx][cb][cf]+[db][cf][cx]+[df][cb][cx] \tag{C73}$$

$$C^Y_{np:0} = \frac{C^y_{pre}}{2}\left(\frac{\mathcal{S}_1U_0+\mathcal{S}_0U_1}{[d|\tilde{Q}|c\rangle}-\frac{\mathcal{S}_0U_0([c|\tilde{Q}|c\rangle+[d|\tilde{Q}|d\rangle)}{[d|\tilde{Q}|c\rangle^2}\right) \tag{C74}$$

with

$$C^y_{pre} = \frac{1}{\langle c\,A\rangle}\ ,\ U_x = \langle d\,a\rangle\,\langle d\,e\rangle\,\langle d\,y\rangle\ \ ,\ U_0 = \langle c\,a\rangle\,\langle c\,e\rangle\,\langle c\,y\rangle$$
$$U_1 = \langle d\,a\rangle\,\langle c\,e\rangle\,\langle c\,y\rangle+\langle c\,a\rangle\,\langle c\,y\rangle\,\langle d\,e\rangle+\langle c\,a\rangle\,\langle c\,e\rangle\,\langle d\,y\rangle$$
$$U_2 = \langle d\,a\rangle\,\langle c\,y\rangle\,\langle d\,e\rangle+\langle c\,a\rangle\,\langle d\,e\rangle\,\langle d\,y\rangle+\langle d\,a\rangle\,\langle c\,e\rangle\,\langle d\,y\rangle \tag{C75}$$

$$\begin{aligned}C^J_{np:\gamma} = \frac{1}{2\,[d\,c]}\Bigg(&\frac{(2\,\langle c\,d\rangle+\langle X\,d\rangle)\mathcal{A}_0\mathcal{S}_0}{\langle X\,c\rangle^2}-\frac{\mathcal{A}_0\mathcal{S}_1}{\langle X\,c\rangle}-\frac{(2\,\langle c\,d\rangle+3\,\langle X\,d\rangle)\mathcal{A}_0\mathcal{S}_x}{\langle X\,d\rangle^2}\\ &-\frac{\mathcal{A}_1\mathcal{S}_0}{\langle X\,c\rangle}+\frac{(2\,\langle c\,d\rangle+\langle X\,d\rangle)\mathcal{A}_1\mathcal{S}_1}{\langle X\,d\rangle^2}+\frac{\langle X\,c\rangle\,(6\,\langle c\,d\rangle+5\,\langle X\,d\rangle)\mathcal{A}_1\mathcal{S}_x}{\langle X\,d\rangle^3}\\ &+\frac{(\langle X\,d\rangle-2\,\langle c\,d\rangle)\mathcal{A}_x\mathcal{S}_0}{\langle X\,d\rangle^2}-\frac{\langle X\,c\rangle\,(6\,\langle c\,d\rangle+\langle X\,d\rangle)\mathcal{A}_x\mathcal{S}_1}{\langle X\,d\rangle^3}-\frac{6\,\langle X\,c\rangle^2\,(2\,\langle c\,d\rangle+\langle X\,d\rangle)\mathcal{A}_x\mathcal{S}_x}{\langle X\,d\rangle^4}\Bigg)\end{aligned} \tag{C76}$$

where

$$\mathcal{A}_x = \frac{\langle d\,a\rangle\,\langle d\,e\rangle\,\langle d\,y\rangle}{\langle d\,A\rangle}\ ,\ \mathcal{S}_x = \frac{[cb][cf][cx]}{[cB]} \tag{C77}$$

$$
\begin{aligned}
C^{H}_{np:\gamma} =& \frac{C^{x}_{pre}}{6[c|\tilde{Q}|c\rangle([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^5[d|\tilde{Q}|c\rangle^2} \\
\times & \left(\begin{array}{l}
3\mathcal{A}_0([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^2 \\
\quad \times \left(\begin{array}{c}
T_0[c|\tilde{Q}|c\rangle^5-(2[d|\tilde{Q}|d\rangle T_0+[d|\tilde{Q}|c\rangle T_1)[c|\tilde{Q}|c\rangle^4+[d|\tilde{Q}|c\rangle(3[d|\tilde{Q}|d\rangle T_1+[d|\tilde{Q}|c\rangle T_2)[c|\tilde{Q}|c\rangle^3 \\
+\left(2T_0[d|\tilde{Q}|d\rangle^3-3[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2-4[d|\tilde{Q}|c\rangle^2 T_2[d|\tilde{Q}|d\rangle-[d|\tilde{Q}|c\rangle^3 T_x\right)[c|\tilde{Q}|c\rangle^2 \\
+[d|\tilde{Q}|d\rangle\left(-T_0[d|\tilde{Q}|d\rangle^3+[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2+3[d|\tilde{Q}|c\rangle^2 T_2[d|\tilde{Q}|d\rangle+5[d|\tilde{Q}|c\rangle^3 T_x\right)[c|\tilde{Q}|c\rangle \\
+2[d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right) \\
+[d|\tilde{Q}|c\rangle \\
\quad \times \left(\begin{array}{l}
\mathcal{A}_x[d|\tilde{Q}|c\rangle \left(\begin{array}{c}
-9T_0[c|\tilde{Q}|c\rangle^5+15(2[d|\tilde{Q}|d\rangle T_0+[d|\tilde{Q}|c\rangle T_1)[c|\tilde{Q}|c\rangle^4 \\
-9\left(4T_0[d|\tilde{Q}|d\rangle^2+3[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle+2[d|\tilde{Q}|c\rangle^2 T_2\right)[c|\tilde{Q}|c\rangle^3 \\
+\left(18T_0[d|\tilde{Q}|d\rangle^3+9[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2+20[d|\tilde{Q}|c\rangle^3 T_x\right)[c|\tilde{Q}|c\rangle^2 \\
+[d|\tilde{Q}|d\rangle\left(\begin{array}{c}
-3T_0[d|\tilde{Q}|d\rangle^3+3[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2 \\
+18[d|\tilde{Q}|c\rangle^2 T_2[d|\tilde{Q}|d\rangle \\
+38[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right)[c|\tilde{Q}|c\rangle \\
+2[d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right) \\
-3\mathcal{A}_1([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle) \\
\quad \times \left(\begin{array}{c}
T_0[c|\tilde{Q}|c\rangle^5-(4[d|\tilde{Q}|d\rangle T_0+[d|\tilde{Q}|c\rangle T_1)[c|\tilde{Q}|c\rangle^4 \\
+\left(6T_0[d|\tilde{Q}|d\rangle^2+[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle+[d|\tilde{Q}|c\rangle^2 T_2\right)[c|\tilde{Q}|c\rangle^3 \\
+\left(\begin{array}{c}
-4T_0[d|\tilde{Q}|d\rangle^3+[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2 \\
+4[d|\tilde{Q}|c\rangle^2 T_2[d|\tilde{Q}|d\rangle-[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right)[c|\tilde{Q}|c\rangle^2 \\
+[d|\tilde{Q}|d\rangle\left(\begin{array}{c}
T_0[d|\tilde{Q}|d\rangle^3-[d|\tilde{Q}|c\rangle T_1[d|\tilde{Q}|d\rangle^2 \\
-5[d|\tilde{Q}|c\rangle^2 T_2[d|\tilde{Q}|d\rangle-10[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right)[c|\tilde{Q}|c\rangle \\
-[d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle^3 T_x
\end{array}\right)
\end{array}\right)
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
C^{Y}_{np:\gamma} =& \frac{C^{y}_{pre}}{6([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^5[d|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle^2} \\
&\times\Bigg(3\mathcal{S}_0([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^2 \\
&\times\left(\begin{array}{c}
[d|\tilde{Q}|d\rangle U_0[c|\tilde{Q}|c\rangle^4 \\
-[d|\tilde{Q}|d\rangle(2[d|\tilde{Q}|d\rangle U_0+[d|\tilde{Q}|c\rangle U_1)[c|\tilde{Q}|c\rangle^3 \\
+[d|\tilde{Q}|c\rangle\left(3U_1[d|\tilde{Q}|d\rangle^2-3[d|\tilde{Q}|c\rangle U_2[d|\tilde{Q}|d\rangle-2[d|\tilde{Q}|c\rangle^2U_x\right)[c|\tilde{Q}|c\rangle^2 \\
+[d|\tilde{Q}|d\rangle\left(2U_0[d|\tilde{Q}|d\rangle^3-3[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2+4[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle-5[d|\tilde{Q}|c\rangle^3U_x\right)[c|\tilde{Q}|c\rangle \\
+[d|\tilde{Q}|d\rangle^2\left(-U_0[d|\tilde{Q}|d\rangle^3+[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2-[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle+[d|\tilde{Q}|c\rangle^3U_x\right)
\end{array}\right) \\
&-[d|\tilde{Q}|c\rangle \\
&\times\left(\begin{array}{l}
3\mathcal{S}_1([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle) \\
\times\left(\begin{array}{c}
[d|\tilde{Q}|d\rangle U_0[c|\tilde{Q}|c\rangle^4-[d|\tilde{Q}|d\rangle(4[d|\tilde{Q}|d\rangle U_0+[d|\tilde{Q}|c\rangle U_1)[c|\tilde{Q}|c\rangle^3 \\
+\left(6U_0[d|\tilde{Q}|d\rangle^3+[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2-5[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle-[d|\tilde{Q}|c\rangle^3U_x\right)[c|\tilde{Q}|c\rangle^2 \\
+[d|\tilde{Q}|d\rangle\left(-4U_0[d|\tilde{Q}|d\rangle^3+[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2+4[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle-10[d|\tilde{Q}|c\rangle^3U_x\right)[c|\tilde{Q}|c\rangle \\
+[d|\tilde{Q}|d\rangle^2\left(U_0[d|\tilde{Q}|d\rangle^3-[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2+[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle-[d|\tilde{Q}|c\rangle^3U_x\right)
\end{array}\right) \\
+\mathcal{S}_x[d|\tilde{Q}|c\rangle \\
\times\left(\begin{array}{c}
-3[d|\tilde{Q}|d\rangle U_0[c|\tilde{Q}|c\rangle^4+3[d|\tilde{Q}|d\rangle(6[d|\tilde{Q}|d\rangle U_0+[d|\tilde{Q}|c\rangle U_1)[c|\tilde{Q}|c\rangle^3 \\
+\left(-36U_0[d|\tilde{Q}|d\rangle^3+9[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2+18[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle+2[d|\tilde{Q}|c\rangle^3U_x\right)[c|\tilde{Q}|c\rangle^2 \\
+\left(30U_0[d|\tilde{Q}|d\rangle^4-27[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^3+38[d|\tilde{Q}|c\rangle^3U_x[d|\tilde{Q}|d\rangle\right)[c|\tilde{Q}|c\rangle \\
+[d|\tilde{Q}|d\rangle^2\left(-9U_0[d|\tilde{Q}|d\rangle^3+15[d|\tilde{Q}|c\rangle U_1[d|\tilde{Q}|d\rangle^2-18[d|\tilde{Q}|c\rangle^2U_2[d|\tilde{Q}|d\rangle+20[d|\tilde{Q}|c\rangle^3U_x\right)
\end{array}\right)
\end{array}\right)\Bigg)
\end{aligned}
$$

From the bonus angle pole piece:

$$
C^{J}_{ap:0} = \frac{\langle d\,A\rangle\,\mathcal{A}_p}{2\,\langle c\,A\rangle\,[d|\tilde{Q}|c\rangle}\left(\mathcal{S}_1-\left(\frac{\langle d\,A\rangle}{\langle c\,A\rangle}+\frac{[c|\tilde{Q}|c\rangle+[d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\right)\mathcal{S}_0\right) \tag{C78}
$$

$$
C^{J}_{ap:a} = \frac{\langle d\,A\rangle^2\,\mathcal{A}_p}{\langle c\,A\rangle^2\,\mathcal{P}^{J}_{\mathcal{A}}}\left(\begin{array}{c}\left(\frac{\langle d\,A\rangle^2[A\,d]^2}{2[A|P|A\rangle^2}-\frac{\langle d\,A\rangle[A\,d]b^{J}_{\mathcal{A}}}{[A|P|A\rangle}\right)\left(\frac{\langle c\,A\rangle\mathcal{S}_1}{\langle d\,A\rangle}+\frac{\langle c\,A\rangle^2\mathcal{S}_x}{\langle d\,A\rangle^2}+\mathcal{S}_0\right) \\ -\frac{\langle d\,A\rangle[A\,d]\left(\frac{\langle c\,A\rangle\mathcal{S}_1}{\langle d\,A\rangle}+2\mathcal{S}_0\right)}{[A|P|A\rangle}\end{array}\right) \tag{C79}
$$

with

$$\mathcal{P}^J_{\mathcal{A}} = -\frac{\langle c\, d\rangle \, \langle A|P\tilde{Q}|A\rangle}{\langle d\, A\rangle^2}\ ,\ \ b^J_{\mathcal{A}} = -\frac{\langle c\, d\rangle \, [d|\tilde{Q}|A\rangle}{\langle d\, A\rangle}\frac{1}{\mathcal{P}_{\mathcal{A}}} \tag{C80}$$

$$C^J_{ap:\gamma} = \frac{\mathcal{A}_p}{2[d|\tilde{Q}|c\rangle^2 \mathcal{P}^J_{\mathcal{A}}}\begin{pmatrix} ([c|\tilde{Q}|c\rangle - [d|\tilde{Q}|d\rangle)([c|\tilde{Q}|c\rangle(2b^J_{\mathcal{A}}-1) - [d|\tilde{Q}|d\rangle(2b^J_{\mathcal{A}}+3))\mathcal{S}_0 \\ +[d|\tilde{Q}|c\rangle(([c|\tilde{Q}|c\rangle(1-2b^J_{\mathcal{A}}) + [d|\tilde{Q}|d\rangle(2b^J_{\mathcal{A}}+1))\mathcal{S}_1 + [d|\tilde{Q}|c\rangle(2b^J_{\mathcal{A}}-1)\mathcal{S}_x) \end{pmatrix} \tag{C81}$$

$$C^J_{ap:ax} = -\frac{\langle d\, A\rangle^3 \, \mathcal{A}_p}{\langle c\, A\rangle^2 \, \langle c\, d\rangle \, [d|\tilde{Q}|A\rangle}\begin{pmatrix} \frac{\langle d\, A\rangle^3 [A\, d]^3 \left(\frac{\langle c\, A\rangle \mathcal{S}_1}{\langle d\, A\rangle} + \frac{\langle c\, A\rangle^2 \mathcal{S}_x}{\langle d\, A\rangle^2} + \mathcal{S}_0\right)}{3[A|P|A\rangle^3} \\ -\frac{\langle d\, A\rangle^2 [A\, d]^2 \left(\frac{\langle c\, A\rangle \mathcal{S}_1}{\langle d\, A\rangle} + 2\mathcal{S}_0\right)}{2[A|P|A\rangle^2} + \frac{\langle d\, A\rangle [A\, d]\mathcal{S}_0}{[A|P|A\rangle} \end{pmatrix} \tag{C82}$$

$$C^J_{ap:\gamma x} = \frac{\langle d\, A\rangle \, \mathcal{A}_p}{6\, \langle c\, d\rangle \, [d|\tilde{Q}|A\rangle [d|\tilde{Q}|c\rangle^2}\begin{pmatrix} 2\left([c|\tilde{Q}|c\rangle [d|\tilde{Q}|d\rangle + [c|\tilde{Q}|c\rangle^2 + [d|\tilde{Q}|d\rangle^2\right)\mathcal{S}_0 \\ +[d|\tilde{Q}|c\rangle(2[d|\tilde{Q}|c\rangle \mathcal{S}_x - (2[c|\tilde{Q}|c\rangle + [d|\tilde{Q}|d\rangle)\mathcal{S}_1) \end{pmatrix} \tag{C83}$$

$$C^H_{ap:0} = \frac{\langle d\, A\rangle \, \mathcal{A}_p C^x_{pre}}{2\, \langle c\, A\rangle \, [d|\tilde{Q}|c\rangle}\left(T_1 - T_0\left(\frac{\langle d\, A\rangle}{\langle c\, A\rangle} + \frac{[c|\tilde{Q}|c\rangle + [d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\right)\right) \tag{C84}$$

$$\begin{aligned} C^H_{ap:a} = & \frac{[A\, d]\, \mathcal{A}_p C^x_{pre}}{2\, \langle c\, A\rangle^2 \, \mathcal{P}^J_{\mathcal{A}}(\langle c\, A\rangle \, [A\, c] + \langle d\, A\rangle \, [A\, d])^2} \\ & \times \begin{pmatrix} \langle d\, A\rangle \begin{pmatrix} \langle d\, A\rangle \, (2c^J_{\mathcal{A}}(\langle c\, A\rangle \, [A\, c] + \langle d\, A\rangle \, [A\, d])(\langle d\, A\rangle \, T_0 + \langle c\, A\rangle \, T_1) \\ + \langle d\, A\rangle \, (\langle c\, A\rangle \, [A\, d]\, T_1 - (2\, \langle c\, A\rangle \, [A\, c] + \langle d\, A\rangle \, [A\, d])T_0)) \\ + \langle c\, A\rangle^2 \, T_2(2\, \langle c\, A\rangle \, [A\, c]\, (c^J_{\mathcal{A}}+1) + \langle d\, A\rangle \, [A\, d]\, (2c^J_{\mathcal{A}}+3)) \end{pmatrix} \\ + \langle c\, A\rangle^3 \, T_x(2\, \langle c\, A\rangle \, [A\, c]\, (c^J_{\mathcal{A}}+2) + \langle d\, A\rangle \, [A\, d]\, (2c^J_{\mathcal{A}}+5)) \end{pmatrix} \end{aligned} \tag{C85}$$

with

$$c^J_{\mathcal{A}} = \frac{\langle c\, A\rangle \, [c|\tilde{Q}|c\rangle}{\langle d\, A\rangle \, \mathcal{P}^J_{\mathcal{A}}} \tag{C86}$$

$$\begin{aligned} C^H_{ap:aa} = & -\frac{\langle d\, A\rangle \, [A\, d]\, \mathcal{A}_p C^x_{pre}}{6\, \langle c\, A\rangle^3 \, [c|\tilde{Q}|c\rangle(\langle c\, A\rangle \, [A\, c] + \langle d\, A\rangle \, [A\, d])^3} \\ & \times \begin{pmatrix} \langle d\, A\rangle^3 \begin{pmatrix} 2\left(3\, \langle c\, A\rangle \, [A\, c]\, \langle d\, A\rangle \, [A\, d] + 3\, \langle c\, A\rangle^2 \, [A\, c]^2 + \langle d\, A\rangle^2 \, [A\, d]^2\right) T_0 \\ + \langle c\, A\rangle \, [A\, d]\, (2\, \langle c\, A\rangle \, [A\, d]\, T_2 - (3\, \langle c\, A\rangle \, [A\, c] + \langle d\, A\rangle \, [A\, d])T_1) \end{pmatrix} \\ + \langle c\, A\rangle^3 \left(15\, \langle c\, A\rangle \, [A\, c]\, \langle d\, A\rangle \, [A\, d] + 6\, \langle c\, A\rangle^2 \, [A\, c]^2 + 11\, \langle d\, A\rangle^2 \, [A\, d]^2\right) T_x \end{pmatrix} \end{aligned} \tag{C87}$$

$$C^{H}_{ap:\gamma} = \frac{\mathcal{A}_p C^x_{pre}}{2\mathcal{P}^J_{\mathcal{A}}} \begin{pmatrix} \frac{2[c|\tilde{Q}|c\rangle[d|\tilde{Q}|d\rangle(2c^J_A+1)T_0}{[d|\tilde{Q}|c\rangle^2}+\frac{[d|\tilde{Q}|c\rangle T_x([c|\tilde{Q}|c\rangle(2c^J_A+3)-[d|\tilde{Q}|d\rangle(2c^J_A+5))}{([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^2} \\ -\frac{[c|\tilde{Q}|c\rangle^2(2c^J_A+3)T_0}{[d|\tilde{Q}|c\rangle^2}+\frac{[c|\tilde{Q}|c\rangle(2c^J_A+3)T_1}{[d|\tilde{Q}|c\rangle}-\frac{[d|\tilde{Q}|d\rangle^2(2c^J_A-1)T_0}{[d|\tilde{Q}|c\rangle^2}-\frac{[d|\tilde{Q}|d\rangle(2c^J_A+1)T_1}{[d|\tilde{Q}|c\rangle}-(2c^J_A+3)T_2 \end{pmatrix} \tag{C88}$$

$$\begin{aligned} C^{H}_{ap:\gamma a} = {} & \frac{\langle d\, A\rangle\, \mathcal{A}_p C^x_{pre}}{6\,\langle c\, A\rangle\, [c|\tilde{Q}|c\rangle[d|\tilde{Q}|c\rangle^2([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^3} \\ & \times \begin{pmatrix} -[c|\tilde{Q}|c\rangle^2\left(3[d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle T_1+6[d|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle^2T_2+2[d|\tilde{Q}|d\rangle^3T_0+2[d|\tilde{Q}|c\rangle^3T_x\right) \\ +[c|\tilde{Q}|c\rangle[d|\tilde{Q}|d\rangle\left(-[d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle T_1+6[d|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle^2T_2+4[d|\tilde{Q}|d\rangle^3T_0+7[d|\tilde{Q}|c\rangle^3T_x\right) \\ +[c|\tilde{Q}|c\rangle^3\left(5[d|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle T_1+2[d|\tilde{Q}|d\rangle^2T_0+2[d|\tilde{Q}|c\rangle^2T_2\right) \\ -2[c|\tilde{Q}|c\rangle^4(2[d|\tilde{Q}|d\rangle T_0+[d|\tilde{Q}|c\rangle T_1) \\ +2[c|\tilde{Q}|c\rangle^5T_0 \\ +[d|\tilde{Q}|d\rangle^2\left([d|\tilde{Q}|d\rangle^2[d|\tilde{Q}|c\rangle T_1-2[d|\tilde{Q}|d\rangle[d|\tilde{Q}|c\rangle^2T_2-2[d|\tilde{Q}|d\rangle^3T_0-11[d|\tilde{Q}|c\rangle^3T_x\right) \end{pmatrix} \end{aligned} \tag{C89}$$

From the bonus square pole piece:

$$C^{J}_{sp:0} = \frac{[c\, B]\,\mathcal{S}_p}{2\,[d\, B]\,[d|\tilde{Q}|c\rangle}\left(\mathcal{A}_1-\mathcal{A}_0\left(\frac{[c\, B]}{[d\, B]}+\frac{[c|\tilde{Q}|c\rangle+[d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\right)\right) \tag{C90}$$

$$C^{J}_{sp:s} = \frac{[c\, B]^2\,\mathcal{S}_p}{[d\, B]^2\,\mathcal{P}^J_S}\begin{pmatrix} \left(\frac{\langle c\, B\rangle^2[B\, c]^2}{2[B|P|B\rangle^2}-\frac{\langle c\, B\rangle[B\, c]b^J_s}{[B|P|B\rangle}\right)\left(\frac{[d\, B]\mathcal{A}_1}{[c\, B]}+\frac{[d\, B]^2\mathcal{A}_x}{[c\, B]^2}+\mathcal{A}_0\right) \\ -\frac{\langle c\, B\rangle[B\, c]\left(\frac{[d\, B]\mathcal{A}_1}{[c\, B]}+2\mathcal{A}_0\right)}{[B|P|B\rangle} \end{pmatrix} \tag{C91}$$

with

$$\mathcal{P}^J_S = \frac{[c\, d]\,([B\, c]\,[B|\tilde{Q}|c\rangle+[B\, d]\,[B|\tilde{Q}|d\rangle)}{[c\, B]^2}\ ,\ \ b^J_S = -\frac{\frac{[d\, B][c|\tilde{Q}|c\rangle}{[c\, B]}-[d|\tilde{Q}|c\rangle}{\mathcal{P}^J_S} \tag{C92}$$

$$C^{J}_{sp:\gamma} = \frac{\mathcal{S}_p}{2[d|\tilde{Q}|c\rangle^2\mathcal{P}^J_S}\begin{pmatrix} ([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)([c|\tilde{Q}|c\rangle(2b^J_s+3)+[d|\tilde{Q}|d\rangle(1-2b^J_s))\mathcal{A}_0 \\ +[d|\tilde{Q}|c\rangle(([c|\tilde{Q}|c\rangle(2b^J_s+1)+[d|\tilde{Q}|d\rangle(1-2b^J_s))\mathcal{A}_1+[d|\tilde{Q}|c\rangle(2b^J_s-1)\mathcal{A}_x) \end{pmatrix} \tag{C93}$$

$$\begin{aligned} C^{J}_{sp:sx} = {} & \frac{\langle c\, B\rangle\,[B\, c]\,[c\, B]\,\mathcal{S}_p}{6\,[d\, B]^2\,[B|P|B\rangle^3\,[c\, d]\,[B|\tilde{Q}|c\rangle} \\ & \times\begin{pmatrix} 2\,[c\, B]^2\left(-3\,\langle c\, B\rangle\,[B\, c]\,[B|P|B\rangle+\langle c\, B\rangle^2\,[B\, c]^2+3[B|P|B\rangle^2\right)\mathcal{A}_0 \\ +\,\langle c\, B\rangle\,[B\, c]\,[d\, B]\,([c\, B]\,(2\,\langle c\, B\rangle\,[B\, c]-3[B|P|B\rangle)\mathcal{A}_1+2\,\langle c\, B\rangle\,[B\, c]\,[d\, B]\,\mathcal{A}_x) \end{pmatrix} \end{aligned} \tag{C94}$$

$$C^J_{sp:\gamma x} = -\frac{[c\,B]\,\mathcal{S}_p}{6\,[c\,d]\,[B|\tilde{Q}|c\rangle [d|\tilde{Q}|c\rangle^2} \begin{pmatrix} 2\left([c|\tilde{Q}|c\rangle [d|\tilde{Q}|d\rangle + [c|\tilde{Q}|c\rangle^2 + [d|\tilde{Q}|d\rangle^2\right)\mathcal{A}_0 \\ +[d|\tilde{Q}|c\rangle(2[d|\tilde{Q}|c\rangle\mathcal{A}_x - ([c|\tilde{Q}|c\rangle + 2[d|\tilde{Q}|d\rangle)\mathcal{A}_1) \end{pmatrix} \tag{C95}$$

$$C^Y_{sp:0} = \frac{C^y_{pre}\,[c\,B]\,\mathcal{S}_p}{2\,[d\,B]\,[d|\tilde{Q}|c\rangle}\left(U_1 - U_0\left(\frac{[c\,B]}{[d\,B]} + \frac{[c|\tilde{Q}|c\rangle + [d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\right)\right) \tag{C96}$$

$$C^Y_{sp:s} = \frac{C^y_{pre}\,\langle c\,B\rangle\,[B\,c]\,\mathcal{S}_p}{2\,[c\,B]\,[d\,B]^2\,\mathcal{P}^J_S(\langle c\,B\rangle\,[B\,c] + \langle d\,B\rangle\,[B\,d])^2} \\ \times \begin{pmatrix} [c\,B]^3\,U_0(\langle c\,B\rangle\,[B\,c]\,(2c^J_S - 1) + 2\,\langle d\,B\rangle\,[B\,d]\,(c^J_S - 1)) \\ +\,[d\,B] \begin{pmatrix} [c\,B]^2\,U_1(\langle c\,B\rangle\,[B\,c]\,(2c^J_S + 1) + 2\,\langle d\,B\rangle\,[B\,d]\,c^J_S) \\ +\,[d\,B]\,([c\,B]\,U_2(\langle c\,B\rangle\,[B\,c]\,(2c^J_S + 3) + 2\,\langle d\,B\rangle\,[B\,d]\,(c^J_S + 1)) \\ +\,[d\,B]\,U_x(\langle c\,B\rangle\,[B\,c]\,(2c^J_S + 5) + 2\,\langle d\,B\rangle\,[B\,d]\,(c^J_S + 2))) \end{pmatrix} \end{pmatrix} \tag{C97}$$

with

$$c^J_S = \frac{[d\,B]\,[d|\tilde{Q}|d\rangle}{[c\,B]\,\mathcal{P}^J_S} \tag{C98}$$

$$C^Y_{sp:\gamma} = \frac{C^y_{pre}\mathcal{S}_p}{2\mathcal{P}^J_S} \begin{pmatrix} \frac{2[c|\tilde{Q}|c\rangle[d|\tilde{Q}|d\rangle(2c^J_S+1)U_0}{[d|\tilde{Q}|c\rangle^2} - \frac{[d|\tilde{Q}|c\rangle U_x([c|\tilde{Q}|c\rangle(2c^J_S+5) - [d|\tilde{Q}|d\rangle(2c^J_S+3))}{([c|\tilde{Q}|c\rangle - [d|\tilde{Q}|d\rangle)^2} \\ -\frac{[c|\tilde{Q}|c\rangle^2(2c^J_S-1)U_0}{[d|\tilde{Q}|c\rangle^2} - \frac{[c|\tilde{Q}|c\rangle(2c^J_S+1)U_1}{[d|\tilde{Q}|c\rangle} - \frac{[d|\tilde{Q}|d\rangle^2(2c^J_S+3)U_0}{[d|\tilde{Q}|c\rangle^2} + \frac{[d|\tilde{Q}|d\rangle(2c^J_S+3)U_1}{[d|\tilde{Q}|c\rangle} - (2c^J_S+3)U_2 \end{pmatrix} \tag{C99}$$

Finally, from the double bonus pole piece:

$$C^J_{dp:0} = -\frac{\langle d\,A\rangle\,[c\,B]\,\mathcal{A}_p\mathcal{S}_p}{2\,\langle c\,A\rangle\,[d\,B]\,[d|\tilde{Q}|c\rangle}\left(\frac{\langle d\,A\rangle}{\langle c\,A\rangle} + \frac{[c\,B]}{[d\,B]} + \frac{[c|\tilde{Q}|c\rangle}{[d|\tilde{Q}|c\rangle} + \frac{[d|\tilde{Q}|d\rangle}{[d|\tilde{Q}|c\rangle}\right) \tag{C100}$$

$$C^J_{dp:a} = -\frac{C^J_{pre:dpa}}{2(\mathcal{E}^J_S - 1)(\mathcal{E}^J_Q - 1)}\left(\frac{\langle c\,A\rangle\,[A\,c]}{[A|P|A\rangle} - 1\right)\left(\left(\frac{\langle c\,A\rangle\,[A\,c]}{[A|P|A\rangle} + 1\right) - \frac{2(\mathcal{E}^J_S + \mathcal{E}^J_Q - 2)}{(\mathcal{E}^J_S - 1)(\mathcal{E}^J_Q - 1)}\right) \tag{C101}$$

with

$$\begin{aligned} C^J_{pre:dpa} &= \frac{\langle a\,A\rangle\,\langle e\,A\rangle\,\langle y\,A\rangle\,[b\,B]\,[f\,B]\,[x\,B]\,\langle c\,d\rangle^3\,[d\,c]^3}{\langle c\,A\rangle^4\,[c\,B]^4\,[c|\tilde{Q}|c\rangle} \\ \mathcal{E}^J_S &= -\frac{\langle d\,A\rangle\,[d\,B]}{\langle c\,A\rangle\,[c\,B]}\ ,\ \mathcal{E}^J_Q = \frac{[d|\tilde{Q}|d\rangle\,\langle c\,A\rangle - \langle d\,A\rangle\,[d|\tilde{Q}|c\rangle}{[c|\tilde{Q}|c\rangle\,\langle c\,A\rangle} \end{aligned} \tag{C102}$$

$$C^J_{dp:ax} = \frac{\langle d\,A\rangle\,[A\,d]\,C^J_{pre:dpa}}{6(\mathcal{E}^J_S-1)^2(\langle c\,A\rangle\,[A\,c]+\langle d\,A\rangle\,[A\,d])^3(\mathcal{E}^J_S-\mathcal{E}^J_Q)}$$
$$\times\left(6\,\langle c\,A\rangle^2\,[A\,c]^2\left({\mathcal{E}^J_S}^2-3\mathcal{E}^J_S+3\right)+3\,\langle c\,A\rangle\,[A\,c]\,\langle d\,A\rangle\,[A\,d]\left(2{\mathcal{E}^J_S}^2-7\mathcal{E}^J_S+9\right)+\langle d\,A\rangle^2\,[A\,d]^2\left(2{\mathcal{E}^J_S}^2-7\mathcal{E}^J_S+11\right)\right) \tag{C103}$$

$$C_{dp:ay} = \frac{\langle d\,A\rangle\,[A\,d]\,C^J_{pre:dpa}}{6(\mathcal{E}^J_Q-1)^2(\langle c\,A\rangle\,[A\,c]+\langle d\,A\rangle\,[A\,d])^3(\mathcal{E}^J_Q-\mathcal{E}^J_S)}$$
$$\times\left(6\,\langle c\,A\rangle^2\,[A\,c]^2\left({\mathcal{E}^J_Q}^2-3\mathcal{E}^J_Q+3\right)+3\,\langle c\,A\rangle\,[A\,c]\,\langle d\,A\rangle\,[A\,d]\left(2{\mathcal{E}^J_Q}^2-7\mathcal{E}^J_Q+9\right)+\langle d\,A\rangle^2\,[A\,d]^2\left(2{\mathcal{E}^J_Q}^2-7\mathcal{E}^J_Q+11\right)\right) \tag{C104}$$

$$C^J_{dp:s} = \frac{[c\,B]^2\,\mathcal{A}_p\mathcal{S}_p}{2\,[d\,B]^2\,\mathcal{D}^J_A\mathcal{G}^J_Q}\left(-\frac{2\,\langle c\,B\rangle\,[B\,c]\,(\mathcal{E}^J_A+\mathcal{F}^J_Q)}{[B|P|B\rangle}-\frac{\langle d\,B\rangle^2\,[B\,d]^2}{[B|P|B\rangle^2}+1\right) \tag{C105}$$

with

$$\mathcal{D}^J_A = \frac{\langle c\,A\rangle}{\langle d\,A\rangle}+\frac{[d\,B]}{[c\,B]}\ ,\ \mathcal{E}^J_A = -\frac{\langle d\,A\rangle\,[d\,B]}{\langle c\,A\rangle\,[c\,B]+\langle d\,A\rangle\,[d\,B]}$$
$$\mathcal{F}^J_Q = \frac{[c\,B]\,[d\,B]\,[d|\tilde{Q}|d\rangle}{[c\,d]\,([B\,c]\,[B|\tilde{Q}|c\rangle+[B\,d]\,[B|\tilde{Q}|d\rangle)}\ ,\ \mathcal{G}^J_Q = -\frac{[c\,d]\,([B\,c]\,[B|\tilde{Q}|c\rangle+[B\,d]\,[B|\tilde{Q}|d\rangle)}{[c\,B]^2} \tag{C106}$$

$$C^J_{dp:sx} = \frac{\langle c\,B\rangle\,[c\,B]^4\,\mathcal{A}_p\mathcal{S}_p}{6\,[d\,B]^3\,[d|\tilde{Q}|d\rangle\mathcal{D}^J_A(\langle c\,B\rangle\,[B\,c]+\langle d\,B\rangle\,[B\,d])^3}$$
$$\times\left(\begin{array}{c}\langle c\,B\rangle^2\,[B\,c]^2\left(6{\mathcal{E}^J_A}^2-3\mathcal{E}^J_A+2\right)+3\,\langle c\,B\rangle\,[B\,c]\,\langle d\,B\rangle\,[B\,d]\left(4{\mathcal{E}^J_A}^2-3\mathcal{E}^J_A+2\right)\\ +6\,\langle d\,B\rangle^2\,[B\,d]^2\left({\mathcal{E}^J_A}^2-\mathcal{E}^J_A+1\right)\end{array}\right) \tag{C107}$$

$$C_{dp:sy} = \frac{\langle c\,B\rangle\,[c\,B]^4\,\mathcal{A}_p\mathcal{S}_p}{6\,[d\,B]^3\,\mathcal{G}^J_Q(\langle c\,B\rangle\,[B\,c]+\langle d\,B\rangle\,[B\,d])^3}$$
$$\times\left(\begin{array}{c}\langle c\,B\rangle^2\,[B\,c]^2\left(6{\mathcal{F}^J_Q}^2-3\mathcal{F}^J_Q+2\right)+3\,\langle c\,B\rangle\,[B\,c]\,\langle d\,B\rangle\,[B\,d]\left(4{\mathcal{F}^J_Q}^2-3\mathcal{F}^J_Q+2\right)\\ +6\,\langle d\,B\rangle^2\,[B\,d]^2\left({\mathcal{F}^J_Q}^2-\mathcal{F}^J_Q+1\right)\end{array}\right) \tag{C108}$$

$$C^J_{dp:\gamma} = \frac{([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)^2\mathcal{A}_p\mathcal{S}_p}{[d|\tilde{Q}|c\rangle^2\mathcal{P}^J_{\mathcal{A}}\mathcal{P}^J_S}\left(([c|\tilde{Q}|c\rangle-[d|\tilde{Q}|d\rangle)\left(-c^J_A-b^J_s+\frac{1}{2}\right)-3[c|\tilde{Q}|c\rangle\right) \tag{C109}$$

$$C^J_{dp:\gamma x}=\frac{[c\,B]\,([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)\mathcal{A}_p\mathcal{S}_p}{[c\,d]\,[B|\tilde Q|c\rangle[d|\tilde Q|c\rangle^2\mathcal{P}^J_{\mathcal{A}}}\begin{pmatrix}\frac{3}{2}[c|\tilde Q|c\rangle(2c^J_A-1)([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)\\ +\left(c^{J\,2}_A-\frac{1}{2}c^J_A+\frac{1}{3}\right)([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)^2+3[c|\tilde Q|c\rangle^2\end{pmatrix}\tag{C110}$$

$$C^J_{dp:\gamma z}=\frac{\langle d\,A\rangle\,([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)\mathcal{A}_p\mathcal{S}_p}{\langle c\,A\rangle\,[c|\tilde Q|c\rangle[d|\tilde Q|c\rangle^2\mathcal{P}^J_{S}}\begin{pmatrix}\frac{3}{2}[c|\tilde Q|c\rangle(2b^J_s-1)([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)\\ +\left(b^{J\,2}_s-\frac{1}{2}b^J_s+\frac{1}{3}\right)([c|\tilde Q|c\rangle-[d|\tilde Q|d\rangle)^2+3[c|\tilde Q|c\rangle^2\end{pmatrix}\tag{C111}$$

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