Hyper Natural Deduction

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***Abstract*—We introduce a Hyper Natural Deduction system as an extension of Gentzen's Natural Deduction system. A Hyper Natural Deduction consists of a finite set of derivations which may use, beside typical Natural Deduction rules, additional rules providing means for communication between derivations. We show that our Hyper Natural Deduction system is sound and complete for infinite-valued propositional Gödel Logic, by giving translations to and from Avron's Hypersequent Calculus. We also provide conversions for normalisation and prove the existence of normal forms for our Hyper Natural Deduction system.**

I. Introduction

Curry-Howard correspondences represent a fruitful paradigm in the development of proof theory and computer science, which are well-established for sequential programming. The original Curry-Howard correspondence connects intuitionistic natural deduction and λ-calculus. Following the seminal work by Griffen [1], much work has been done extending Curry-Howard correspondences to other types of logics [2]–[4]. Current research is aiming for extensions to concurrent programming [5], [6]. Herein, process calculi [7] and their connections to linear logic have taken a prominent role [8], [9], but linear logic's lack of good algebraic/relational semantics creates a hurdle. A different approach has been taken in [10], pointing out the relation between process calculi and linear intermediate logics, aka Gödel logics.

On the proof theoretic side, the Hypersequent Calculus introduced by Avron [11] is a well-studied formalism for investigating intermediate logics. Starting with Gentzen's sequent calculus [12] for intuitionistic and classical logic, sequent systems have developed into the framework of choice for proof-theoretic investigations of a wide variety of logics. The extension to hypersequents by Avron has started a new rush of activity, and nowadays Hypersequent calculi for a wide variety of non-classical logics as well as modal logics have been found [13]–[15].

Avron's Hypersequent Calculus is conjectured to also model a form of communication between processes. Previous approaches to explicate this conjecture can be distinguished into either semantical (computational side) or syntactic (proof theoretic side) ones. Examples of the first category are game theoretic interpretation [16] and extensions of the λ-calculus to work with Gödel logics [17]. While these approaches from the semantical side look tempting, they either failed to connect to the proof-theoretic side, or didn't provide a computational interpretation.

On the syntactical side, Baaz et al. [18] introduced a Hyper Natural Deduction system which operates on the level of hyper derivations, that is, hyper sequences of derivations. Their system lacks a normalisation procedure via conversions — normalisation is proved by translation into Hypersequent Calculus, followed by cut-elimination and re-translation. Consequently, they do not obtain a computational interpretation via a Curry-Howard correspondence. In contrast to this, our Hyper Natural Deduction system operates on the level of derivations. This allows us to define a normalisation procedure based on conversions, which has the potential for having a meaningful computational interpretation. The latter is the object on ongoing research.

A. Our program

The original Curry-Howard correspondence (upper part of Figure 1) starts from Intuitionistic Logic (IL), IL's sequent calculus (LJ) and related natural deduction system (NJ), and connects these with the λ-calculus.

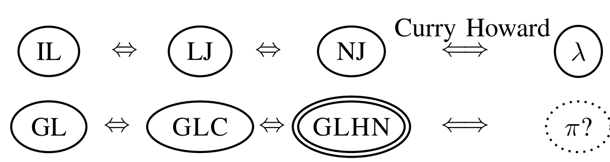

Fig. 1. Curry-Howard correspondence and the proposed approach

We propose a similar approach for (infinite-valued propositional) Gödel Logic (GL) and its (hyper) sequent calculus (GLC). As a first step to establish similar correspondences here, the work in this paper introduces a Hyper Natural Deduction system (GLHN), that is equivalent to the Hypersequent Calculus, and thus sound and complete for Gödel logics.

Future work will be dedicated to complete the Curry-Howard correspondence for Gödel logics, by seeking a term system representing the Hyper Natural Deduction system presented here, and trying to relate it to process calculi.

B. Layout of the article

In Section II, we start by briefly reviewing the main results about Gödel logics, to provide the context for Hypersequent

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Calculus — more details can be found in the handbook article on Gödel logics [19]. In Section III we recall Avron's Hypersequent Calculus (GLC). Section IV defines Hyper Natural Deductions (GLHN) and shows that these are closed under rules corresponding to the inferences of the Hypersequent Calculus. Section V provides translations between GLC and GLHN, proving soundness and completeness of the new system with respect to infinite-valued propositional Gödel logic. Section VI provides a proof of weak normalisation for GLHN. Conclusions and future work are discussed in Section VII.

II. Gödel logics

Gödel [20] introduced propositional finite-valued Gödel logics to show that intuitionistic logic does not have a finite characteristic matrix. These logics provide the first examples of intermediate logics, which lie in strength between classical and intuitionistic logics. Dummett [21] was the first to study infinite-valued propositional Gödel logics. He axiomatised this logic as intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Infinite-valued propositional Gödel logic is also called Gödel-Dummett logic or Dummett's LC. In terms of Kripke semantics, the linearity axiom selects those accessibility relations which are linear orders.

Extending Gödel logics to first-order logic has some surprising consequences: Whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different logics at the first-order level [22]–[24], and, surprisingly, only countably many [25]. Scarpellini [26] has provided a general method to show non-axiomatisability of many infinite-valued intermediate logics. Thus, it is interesting to note that some infinite-valued Gödel logics belong to the limited class of recursively enumerable logics based on linearly ordered truth values [27], [28]. In particular the Gödel logic based on $[0, 1]$ is characterised by the first order Hypersequent Calculus [11].

We consider propositional Gödel logics. Truth values V form a subset of the real interval $[0, 1]$ that include 0 and 1. The valuation of conjunction and disjunction is given by $\min$ and $\max$, respectively. The valuation of the implication gives 1 if the antecedent evaluates to a value less than or equal to the valuation of the succedent, otherwise to the valuation of the succedent. A formula is called valid with respect to V, if it is mapped to 1 for all valuations based on V.

The set of all formulas which are valid with respect to V is called the propositional Gödel logic based on V, and will be denoted by G^0_V. The validity of a formula A with respect to V will be denoted by $\models^0_V A$ or $\models_{\mathsf{G}^0_V} A$. *Infinite-valued propositional Gödel logic* GL is given by the propositional Gödel logic $\mathsf{G}^0_{[0,1]}$ based on the full interval $[0, 1]$.

III. Hypersequent Calculus

In the following a formula will always be a propositional formula. We shall use $A, B, C \dots$ to range over formulas, and $\Gamma, \Delta, \Xi \dots$ to range over finite sets of formulas.

We describe a version of Avron's Hypersequent Calculus [11] following [18]. As the version based on multi-conclusion sequents does not play a role for our exposition, we only define the single-conclusion version here. Thus, a *sequent* is an expression of the form $\Gamma \Rightarrow A$, where Γ is a finite set of formulas, and A is a formula. In particular, Γ being a set implies that structural rules of exchange, contraction and expansion (the converse of contraction) are build into our calculus. A *hypersequent*, in turn, is a finite multiset of sequents, which implies that the external version of exchange, but not contraction and expansion, is also build into our calculus. We shall use the usual hypersequent notation $s_1 \mid \dots \mid s_n$ (for the multiset consisting of $s_1, \dots, s_n$), and $\mathcal{H}, \mathcal{H}' \dots$ to range over hypersequents. We also employ standard notations, e.g. $\Gamma, A \Rightarrow B$ for $\Gamma \cup \{A\} \Rightarrow \{B\}$, and $\mathcal{H} \mid s$ instead of $\mathcal{H} \cup \{s\}$, etc.

The *Hypersequent Calculus for propositional Gödel logic*, GLC, is given by the following axioms and rules:

Axioms: (id) $A \Rightarrow A$ and $(\bot)\ \bot \Rightarrow A$

Cut Rule: $$\textit{cut}\ \frac{\Gamma \Rightarrow A \mid \mathcal{H}_1 \qquad A, \Gamma \Rightarrow C \mid \mathcal{H}_2}{\Gamma \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2}$$

Internal Structural Rule: $$w\ \frac{\Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma, B \Rightarrow C \mid \mathcal{H}}$$

External Structural Rules:

$$\textit{EW}\ \frac{\Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma \Rightarrow C \mid \Gamma' \Rightarrow C' \mid \mathcal{H}}$$

$$\textit{EC}\ \frac{\Gamma \Rightarrow C \mid \Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma \Rightarrow C \mid \mathcal{H}}$$

Logical rules

$$\to{,}l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \to B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'}$$

$$\to{,}r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \to B \mid \mathcal{H}}$$

$$\vee{,}l\ \frac{\Gamma, A \Rightarrow C \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \vee B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'}$$

$$\vee_i{,}r\ \frac{\Gamma \Rightarrow A_i \mid \mathcal{H}}{\Gamma \Rightarrow A_1 \vee A_2 \mid \mathcal{H}} \quad i \in \{1, 2\}$$

$$\wedge_i{,}l\ \frac{\Gamma, A_i \Rightarrow C \mid \mathcal{H}}{\Gamma, A_1 \wedge A_2 \Rightarrow C \mid \mathcal{H}} \quad i \in \{1, 2\}$$

$$\wedge{,}r\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma \Rightarrow B \mid \mathcal{H}'}{\Gamma \Rightarrow A \wedge B \mid \mathcal{H} \mid \mathcal{H}'}$$

Communication:

$$\textit{com}\ \frac{\Gamma_1 \Rightarrow A_1 \mid \mathcal{H} \qquad \Gamma_2 \Rightarrow A_2 \mid \mathcal{H}'}{\Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1 \mid \mathcal{H} \mid \mathcal{H}'}$$

Split: $$\textit{split}\ \frac{\Pi, \Gamma \Rightarrow A \mid \mathcal{H}}{\Pi \Rightarrow A \mid \Gamma \Rightarrow A \mid \mathcal{H}}$$

Theorem 1 ([11]). GLC *is sound and complete for infinitary propositional Gödel logic GL.*

As pointed out by Avron [11], a hypersequent can be thought of as a specification of a multi-process. With respect to this interpretation the communication rule models the exchange of information within multi-processes. In the following we will define an extension of natural deduction (denoted Hyper Natural Deduction GLHN) which will be equivalent to GLC. One motivation for defining and studying GLHN is to make the kind of communication inherent in GLC under the above interpretation more explicit.

IV. Hyper Natural Deduction

The proposed Hyper Natural Deduction system extends Gentzen's Natural Deduction system (NJ) by two main adaptations: First, we extend Gentzen's NJ by three more rules (Definition 1) intended to model split, communication and external contraction in Avron's Hypersequent Calculus. The derivation trees obtained in this system will be called 'GLHN pre-derivation' (Definition 2) — 'pre-derivation' as the new rules are meaningless without further ingredients. In the simplest case where the additional rules are not used, a GLHN pre-derivation is also a NJ derivation.

The second adaptation is that we consider sets of GLHN pre-derivations. Not every set of GLHN pre-derivations will provide a structure that can be interpreted as a meaningful proof in Gödel logic. We will define 'GLHN deductions' as finite sets of GLHN pre-derivations satisfying several conditions (Definition 7). We then show that GLHN deductions correspond to GLC derivations by providing translations between the two (Theorem 3 and Corollary 2).

A. Gentzen's Natural Deduction

Our system of Hyper Natural Deductions will be based on Gentzen's Natural Deduction system NJ [12]. We will present NJ in the version given in [29]. As usual, a Natural Deduction style *derivation* consists of an upward rooted tree, where the nodes are formulas. Formulas at leaf nodes are called *assumptions*. All non-leaf nodes are carrying labels providing information about the rule which has been applied plus some other information (like which assumptions have been closed). We say that A is derivable from assumptions Γ and write

$$\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}$$

if there is a derivation σ with root A such that the union of all open assumptions of σ is a subset of Γ. Derivations in NJ are generated inductively using the following initial, introduction and elimination rules:

Any formula, viewed as a tree consisting of one node, is a derivation.

$$\wedge\text{-}i\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ B\end{matrix}}{A\wedge B}\qquad\qquad \wedge\text{-}e\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\wedge B\end{matrix}}{A}\qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ A\wedge B\end{matrix}}{B}$$

$$\vee\text{-}i\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{A\vee B}\qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ B\end{matrix}}{A\vee B}$$

$$^{k}\vee\text{-}e\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\vee B\end{matrix}\qquad\begin{matrix}\Delta,{}^{k}[A]\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\Pi,{}^{k}[B]\\ \vdots\\ C\end{matrix}}{C}$$

$$^{k}\rightarrow\text{-}i\ \frac{\begin{matrix}\Gamma,{}^{k}[A]\\ \vdots\\ B\end{matrix}}{A\rightarrow B}\qquad\qquad \rightarrow\text{-}e\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ A\rightarrow B\end{matrix}}{B}$$

$$\bot\text{-}i\ \frac{\begin{matrix}\Gamma\\ \vdots\\ \bot\end{matrix}}{A}$$

For any rule occurrence in a derivation, we employ notions like *immediate sub-derivation*, *upper derivations*, *upper left (middle, right) derivation* etc. in the natural way related to the pictorial definition of the rules.

B. Rules for Hyper Natural Deduction

To define a Hyper Natural Deduction system for Gödel Logic (GLHN), we expand NJ by three new rules. Besides communication and split rules which correspond to those in GLC, we will also need a contraction rule to be able to define all conversion rules needed for normalisation. The new communication and split rules employ ideas from process algebra: They come in pairs of duals (technically realised by using so called *names* in the following), and the idea is that such dual labels form a connection which will be used during normalisation to "communicate" sub-derivations.

Definition 1 (Rules of GLHN)**.** Let X be a countable infinite set of *names*. Let $x \mapsto \bar{x}$ be a function from X to X such that for each name x its *dual name* $\bar{x}$ is distinct from x, and x and $\bar{x}$ are duals of each other, that is, $\bar{\bar{x}} = x$.

The set of rules of GLHN consists of the above rules for Natural Deduction, plus the following three rules:

$$^{k}S^{x}_{\Gamma,\Delta}\ \frac{\begin{matrix}[\Gamma]^{k},\Delta\\ \vdots\\ A\end{matrix}}{A}\qquad\qquad \mathrm{com}^{x}_{A,B}\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{B}\qquad\qquad \textit{contr}\ \frac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ A\end{matrix}}{A}$$

where x is a name in X.

The superscript k in the left rule is used to connect discharged assumptions to rule applications in the usual way; most of the time we will drop it in the following.

Definition 2 (GLHN pre-derivation)**.** A GLHN *pre-derivation* is a tree defined using the rules of GLHN. A GLHN pre-derivation is often just called *pre-derivation.*

We shall use $\rho, \sigma \dots$ to range over pre-derivations. Labels of the form $S^{x}_{\Gamma,\Delta}$ are called *splitting labels* and the rule introducing them *splitting rules* or simply *splitting*. Those of the form $\mathrm{com}^{x}_{A,B}$ are called *communication labels* and rules introducing them *communication rules* or simply *communication*. Communication and splitting labels are often jointly referred to as *labels*.

Furthermore, we stipulate a duality on labels as follows: The dual of $S^x_{\Gamma,\Delta}$ (resp. $\mathrm{com}^x_{A,B}$), denoted $\overline{S^x_{\Gamma,\Delta}}$ (resp. $\overline{\mathrm{com}^x_{A,B}}$), is $S^{\bar{x}}_{\Delta,\Gamma}$ (resp. $\mathrm{com}^{\bar{x}}_{B,A}$).

The following notations and concepts will be useful:

Definition 3. For pre-derivations $\rho, \rho_1, \dots, \rho_n$ we define the following formulas and sets: $\mathrm{Labels}(\rho_1, \dots, \rho_n)$ is the set of labels occurring in any of the pre-derivations $\rho_1, \dots, \rho_n$; $\mathrm{Assum}(\rho)$ is the set of not discharged assumptions of ρ; $\mathrm{Conc}(\rho)$ is the final conclusion, i.e., last formula, of ρ. For an occurrence of a formula A in a pre-derivation, we define the *sub-pre-derivation rooted in* A as the sub-tree up to and including A. An *immediate sub-pre-derivation of* ρ is a sub-pre-derivation rooted in one of the premises of the final rule of ρ.

We want to define Hyper Natural Deductions as sets of pre-derivations, with the aim to establish a correspondence to GLC. The idea is that communication and splitting labels come in pairs of duals which together correspond to instances of communication and splitting, resp., in GLC. So far, the use of communication and splitting labels in pre-derivations is not restricted, and we will have to impose various restrictions on their occurrence for the above idea to work. These restriction will impose conditions on GLHN deductions which mainly intend to model independence of parts of GLHN deductions: Non-unary rules on the sequent-level in GLC require that the derivations of their premises are independent. We will have to model this through our conditions on GLHN: For example, for each pair of dual communication labels in an GLHN deduction, the conditions will have to allow us to identify 'independent' parts of the deduction which independently justify the premises of the communication application. The following equivalence relation on pre-derivations will be needed to achieve this.

Definition 4 (Induced equivalence relation and partition)**.** For a finite set $R = \{\rho_1, \dots, \rho_n\}$ of pre-derivations, define the following relation $\sim_R$ on R: $\rho_i \sim_R \rho_j$ if there is an $l \in \mathrm{Labels}(\rho_i)$ and $l' \in \mathrm{Labels}(\rho_j)$ such that l and l' are dual to each other. Note that a label is never dual to itself, thus we may have $\rho_i \nsim_R \rho_i$ even if $\mathrm{Labels}(\rho_i) \neq \emptyset$.

The transitive and reflexive closure of $\sim_R$, denoted by $\approx_R$, is an equivalence relation on R.

For an arbitrary pre-derivation σ (not necessary in R) let $\mathrm{lnk}_R(\sigma)$, the *set of pre-derivations in* R *linked to* σ, be $[\sigma]_{\approx_{R\cup\{\sigma\}}} \setminus \{\sigma\}$.

The contraction rule, which is necessary to formulate the conversion rules for the normalisation procedure, provides a challenge for expressing the above mentioned independence condition as it is the only rule which allows to combine dependent pre-derivations and thus effects equivalent classes obtained from the previously defined equivalence relation. In order to obtain a working definition of independence, we introduce the concept of expansions of pre-derivations which removes contractions and in this way splits dependent sub-pre-derivations into separate pre-derivations. The following definition is used *only* to later define appropriate equivalence relations for characterising independence. The apparent asymmetry in the definition is harmless: both options can be chosen without an effect on the induced independence of pre-derivations.

Definition 5 (Expansion of pre-derivations)**.** With each pre-derivation ρ we associate the *expansion of* ρ, a set of contraction-free pre-derivations ρ^*, inductively as follows: If contraction does not occur in ρ, then $\rho^* = \{\rho\}$. Otherwise choose an uppermost occurrence of contraction in ρ, and let σ_1 and σ_2 be the upper left and right, respectively, sub-pre-derivation at this occurrence. Then ρ^* is defined as the union of $\{\sigma_1\}$ and the expansion of the pre-derivation obtained from ρ by replacing the sub-pre-derivation rooted in the conclusion of the contraction occurrence, with σ_2.

For a set of pre-derivations $R = \{\rho_1, \dots, \rho_n\}$, we define the expansion of R as $R^* = \bigcup_{i=1}^n \rho_i^*$.

Observe that expansions of pre-derivation do not contain contractions.

The definition of Hyper Natural Deduction will require a total order on the set of labels, which ignores duality. To express this, the following definition will be helpful:

Definition 6 (Induced orders on labels)**.** Let L be a set of labels closed under taking duals. Let $L/\_$ be the set of classes of dual labels in L, and let

$$L \ni x \quad \mapsto \quad [x]_\_ := \{x, \bar{x}\} \in L/\_$$

be an embedding of L into $L/\_$. A total order $\precsim$ on $L/\_$ induces a total pre-order $\preceq$ on L in the obvious way by

$$x \preceq y \quad \Leftrightarrow \quad [x]_\_ \precsim [y]_\_$$

Obviously, dual labels are at the same position of the induced total pre-order.

C. Hyper Natural Deduction

We are now in the position to give the definition of our Hyper Natural Deduction system GLHN.

Definition 7 (Hyper Natural Deduction for Gödel Logic)**.** Let $R = \{\rho_1, \dots, \rho_n\}$ be a finite set of pre-derivations, and let $\precsim$ be a total order on $\mathrm{Labels}(R)/\_$. $(R, \precsim)$ is called a ***GL Hyper Natural Deduction***, or simply *Hyper Natural Deduction*, and is denoted GLHN, if the following conditions are satisfied:

1) **(Dual labels)** $\mathrm{Labels}(R)$ is closed under taking duals. That is, if $l \in \mathrm{Labels}(R)$, then also $\bar{l} \in \mathrm{Labels}(R)$. Recall the definition of dual labels:

$$\overline{S^x_{\Delta,\Gamma}} = S^{\bar{x}}_{\Gamma,\Delta} \qquad \text{and} \qquad \overline{\mathrm{com}^x_{B,A}} = \mathrm{com}^{\bar{x}}_{A,B}$$

2) **(Separation of pre-derivations)** If $l \in \mathrm{Labels}(R)$, and if ρ and ρ' are sub-pre-derivations of pre-derivations in R rooted in the premise of rules labelled l and $\bar{l}$,

respectively, then neither ρ is a sub-tree of ρ', nor ρ' a sub-tree of ρ.

3) **(Consistent communication labelling)** If $l \in \mathrm{Labels}(R)$ is a communication label, then all sub-pre-derivations of pre-derivations in R rooted in premises of rules labelled by l are different occurrences of the same pre-derivation.
4) **(Consistent splitting labelling)** If $l \in \mathrm{Labels}(R)$ is a splitting label, then all sub-pre-derivations of pre-derivations in R rooted in premises of rules labelled by l or $\bar{l}$ are different occurrences of the same pre-derivation.
5) **(Label ordering)** The order of labels occurring on any path through any pre-derivation in R respects the total pre-order induced by $\precsim$. The ordering on paths through pre-derivations is from assumptions to conclusions.
6) **(Independence of premises)** For any non-unary logical rule occurring in R, that is $\wedge$-i, $\vee$-e, $\rightarrow$-e, and any pair of dually labelled communication rules occurring in R, we require an independence of their immediate sub-pre-derivations in the following way. For occurrences of $\wedge$-i, $\rightarrow$-e, denote the two upper pre-derivations with σ_i for $i = 1, 2$. For occurrences of $\vee$-e, denote the three upper pre-derivations with σ_i $i = 1, 2, 3$. In the case of a pair of dually labelled communication rules occurring in R, denote one sub-pre-derivation by σ_1 and the other by σ_2. Furthermore, assume w.l.o.g. that in the case of a logical rule, this rule occurs in ρ_1, that is, $\sigma_1, \sigma_2(, \sigma_3)$ are part of ρ_1, and in case of communication σ_1 and σ_2 appear in ρ_1 and ρ_2, respectively.

   That is, we are in one of the following situations:

$$\begin{matrix} \wedge\text{-}i \text{ or } \rightarrow\text{-}e \; \dfrac{\begin{matrix}\Gamma_1\\ \sigma_1 \vdots\\ A_1\end{matrix} \qquad \begin{matrix}\Gamma_2\\ \sigma_2 \vdots\\ A_2\end{matrix}}{B} \\ \vdots \\ \rho_1 \end{matrix} \qquad \text{or}$$

$$\begin{matrix}\Gamma_1\\ \sigma_1 \vdots\\ \mathrm{com}^x_{A,B}\,\dfrac{A}{B} \\ \vdots \\ \rho_1\end{matrix} \qquad \begin{matrix}\Gamma_2\\ \sigma_2 \vdots\\ \overline{\mathrm{com}^x_{A,B}}\,\dfrac{B}{A} \\ \vdots \\ \rho_2\end{matrix} \qquad \text{or}$$

$$\vee\text{-}e \; \dfrac{\begin{matrix}\Gamma\\ \sigma_1 \vdots\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \sigma_2 \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \sigma_3 \vdots\\ C\end{matrix}}{C}$$

   Denote with R' the remaining pre-derivations, i.e., for the logical rules $R' = \{\rho_2, \ldots, \rho_n\}$, and for communication $R' = \{\rho_3, \ldots, \rho_n\}$. Denote with $R_i = R' \cup \{\sigma_i\}$ for $i = 1, 2(, 3)$.

   Then we require for any such rule occurrence, that the equivalence classes of σ_i with respect to $\approx_{R_i^*}$ as sets of pre-derivations are pairwise disjoint, that is,

$$\forall i \neq j \in \{1, 2(, 3)\} \,:\, \mathrm{lnk}_{R_i^*}(\sigma_i) \cap \mathrm{lnk}_{R_j^*}(\sigma_j) = \varnothing$$

   It is important to note that the equivalence classes are build with respect to the *expansions* R_i^* of the pre-derivations R_i.
7) **(Local dependence of contraction premises)** For any occurrence of a contraction rule, we stipulate a dependency of the immediate sub-pre-derivations. With the notions from the previous condition, that is, σ_1 and σ_2 being the upper left and right, respectively, sub-pre-derivations of this contraction occurrence, and R', R_1, R_2 being defined as before, we require that the two equivalence classes coincide:

$$\mathrm{lnk}_{R_1^*}(\sigma_1) = \mathrm{lnk}_{R_2^*}(\sigma_2)$$

8) **(Global dependence of pre-derivations)** The whole set R forms one equivalence class: $R = [\rho_1]_{\approx_R}$. Note that we do not build the equivalence class with respect to expansion R^*, but with respect to R.

We require that a total order on the labels is given as part of the definition of a Hyper Natural Deduction. This is not necessary as there is an efficient algorithm which, given a set of pre-derivations, checks whether such an order exists (and in this case computes it). Thus, in the following we will usually not mention this order when giving a Hyper Natural Deduction.

Observe that the previous definition induces an efficient (logspace) decision procedure to determine whether a given set of pre-derivations forms a Hyper Natural Deduction. As discussed above this is true even if the total order is omitted, as it can be computed efficiently from the set of pre-derivations.

D. Hyper rules for Hyper Natural Deduction

We will now define rules which operate on Hyper Natural Deductions. They will be useful for building Hyper Natural Deductions, needed for example in the translation from hypersequent derivations to Hyper Natural Deductions.

We start by defining a general notion of a rule operating on sets of pre-derivations which we denote a *hyper rule*. In the following, when saying "set of pre-derivations" we mean "*finite* set of pre-derivations".

Definition 8 (Hyper rule)**.** A *hyper rule* $\mathbf{r}$ *of arity* k is a $k{+}1$-ary relation where the arguments are sets of pre-derivations. If a hyper rule $\mathbf{r}$ of arity $k > 1$ is defined on some $R_1 \ldots, R_k, R$ where $R_1 \ldots, R_k, R$ are sets of pre-derivations, then we say that $\mathbf{r}$ *applied to* $R_1 \ldots, R_k$ *yields* R.

A hyper rule of arity $k > 1$ will always satisfy that if it yields R from $R_1 \ldots, R_k$, then the sets of labels for R_i, $\mathrm{Labels}(R_i)$ for $i \leq k$, are pairwise disjoint.

We now define hyper rules based on rules for NJ, and on communication, splitting and contraction rules. Let k be the arity of the rule under consideration. For the following, assume that R_i is a set of pre-derivations, and that $\rho_i \in R_i$, for $i \leq k$. Furthermore, assume that $\mathrm{Labels}(R_i)$ for $i \leq k$ are pairwise disjoint.

Each NJ-rule r induces a corresponding hyper rule $\mathbf{r}$. If r has k premises, then $\mathbf{r}$ has arity k. In addition to these, we have a hyper communication rule $\mathbf{com}$ of arity 2, a hyper splitting rule $\mathbf{split}$ of arity 1, and a hyper contraction rule $\mathbf{contr}$ of arity 1.

Hyper rule r for NJ rule r of arity k: Assume that $\rho_1, \ldots, \rho_k$ together with the NJ rule r form a pre-derivation ρ with conclusion A, that is

$$\rho \quad = \quad r\,\frac{\rho_1\ \ldots \rho_k}{A}$$

Then $\mathbf{r}$ applied to $R_1, \ldots, R_k$ yields

$$\{\rho\}\ \cup\ R_1\backslash\{\rho_1\}\ \cup \cdots \cup\ R_k\backslash\{\rho_k\}\ .$$

Hyper communication rule: Assume $\mathrm{Conc}(\rho_1) = A$, $\mathrm{Conc}(\rho_2) = B$, and let x be a fresh name. Let

$$\bar{\rho}_1 := \quad \mathrm{com}^{x}_{A,B}\,\frac{\begin{array}{c}\rho_1 \vdots \\ A\end{array}}{B} \qquad \text{and} \qquad \bar{\rho}_2 := \quad \mathrm{com}^{\bar{x}}_{B,A}\,\frac{\begin{array}{c}\rho_2 \vdots \\ B\end{array}}{A}$$

Then $\mathbf{com}$ applied to R_1, R_2 yields

$$\{\bar{\rho}_1, \bar{\rho}_2\}\ \cup\ R_1\backslash\{\rho_1\}\ \cup\ R_2\backslash\{\rho_2\}\ .$$

Hyper splitting rule: Assume

$$\rho_1 \quad = \quad \begin{array}{c}\Gamma, \Delta \\ \vdots \\ A\end{array}$$

and let x be a fresh name. Let

$$\bar{\rho}_{1,1} := \quad {}^{k}S^{x}_{\Gamma,\Delta}\,\frac{\begin{array}{c}\Gamma, {}^{k}[\Delta] \\ \rho_1 \vdots \\ A\end{array}}{A} \qquad \text{and} \qquad \bar{\rho}_{1,2} := \quad {}^{\ell}S^{\bar{x}}_{\Delta,\Gamma}\,\frac{\begin{array}{c}{}^{\ell}[\Gamma], \Delta \\ \rho_1 \vdots \\ A\end{array}}{A}$$

Then $\mathbf{split}$ applied to R_1 yields

$$\{\bar{\rho}_{1,1}, \bar{\rho}_{1,2}\}\ \cup\ R_1\backslash\{\rho_1\}\ .$$

Hyper contraction rule: Assume ρ_2 is a second element in R_1, i.e. $\rho_2 \in R_1 \setminus \{\rho_1\}$. Furthermore, assume $\mathrm{Conc}(\rho_1) = A = \mathrm{Conc}(\rho_2)$. Let

$$\bar{\rho}_1 \quad := \quad contr\,\frac{\begin{array}{c}\rho_1 \vdots \\ A\end{array} \quad \begin{array}{c}\rho_2 \vdots \\ A\end{array}}{A}$$

Then $\mathbf{contr}$ applied to R_1 yields

$$\{\bar{\rho}_1\}\ \cup\ R_1\backslash\{\rho_1, \rho_2\}\ .$$

Definition 9 (GLHN hyper rules)**.** The GLHN *hyper rules* are given by the hyper rules corresponding to NJ rules, plus the hyper communication rule $\mathbf{com}$, the hyper splitting rule $\mathbf{split}$, and the hyper contraction rule $\mathbf{contr}$.

We have defined GLHN in an *a priori* or *explicit* way by requiring that certain properties hold for a set of pre-derivations. This is in contrast to a procedural definition that is often used in proof-theoretic settings. The following Lemma shows that we can take also this procedural point of view, that is, if we start from Hyper Natural Deductions and apply a GLHN hyper rule, then we again obtain a Hyper Natural Deduction. This shows that in a natural way we can create Hyper Natural Deductions by locally adding rules to pre-derivations without destroying the global properties needed for being a Hyper Natural Deduction. For example, the independence of premises requirements are preserved when applying GLHN hyper rules.

Lemma 1 (Stability)**.** GLHN *is closed under applying* GLHN *hyper rules.*

That is, assume $\mathbf{r}$ *is a* GLHN *hyper rule of arity k, and $R_1 \ldots, R_k$ are* GLHN *deductions, such that the label sets* $\mathrm{Labels}(R_i)$ *are pairwise disjoint. Assume as well that* $\mathbf{r}$ *applied to $R_1, \ldots, R_k$ yields R. Then, R is again a* GLHN *deduction.*

Proof sketch: The proof of the stability lemma consists of meticulously checking for each GLHN rule that the resulting set of pre-derivations again forms a GLHN if the original set of pre-derivations formed GLHNs. We will only mention a few cases in detail. Let $R_i = \{\rho^i_1, \ldots, \rho^i_{n_i}\}$ be GLHNs, such that $\mathrm{Labels}(R_i)$, $i = 1, \ldots, k$, are pairwise disjoint. In the following conditions 1–8 are referring to those of Definition 7.

Unary hyper rules

For unary rules it is obvious that conditions 1–8 are satisfied and the new figure forms a GLHN.

$\wedge$-i, $\vee$-e, $\rightarrow$-e, communication

For $\wedge$-i, $\vee$-e, $\rightarrow$-e, and communication, the proof works similar, we only treat $\wedge$-i: Let us assume that

$$\rho^1_1 : \quad \begin{array}{c}\Pi \\ \vdots \\ A\end{array} \qquad\qquad \rho^2_1 : \quad \begin{array}{c}\Gamma \\ \vdots \\ B\end{array}$$

We show that $R_\wedge = \{\rho_\wedge, \rho^i_2, \ldots, \rho^i_{n_i} : i = 1, 2\}$ is a GLHN, where $\rho_\wedge$ is

$$\rho_\wedge : \quad \wedge\text{-}i\,\frac{\begin{array}{c}\Pi \\ \rho^1_1 \vdots \\ A\end{array} \quad \begin{array}{c}\Gamma \\ \rho^2_1 \vdots \\ B\end{array}}{A \wedge B}$$

Skipping the treatment of most conditions, we only discuss the critical condition 6 (indep. premises) for the case of an $\wedge$-i rule occurrence in R. If this occurrence is the same as the final rule in $\rho_\wedge$, then the assertion follows from the assumption that $\mathrm{Labels}(R_1)$ and $\mathrm{Labels}(R_2)$ are disjoint. Otherwise, it has to appear in either the left-upper part of $\rho_\wedge$, i.e., ρ^1_1, or the right-upper part, i.e., ρ^2_1, or one of the ρ^i_j where $i = 1, 2$ and $j = 2, \ldots, n_i$. If it is one of the first two cases, i.e., appearing in $\rho_\wedge$, then by the assumptions of R_i being GLHNs and the disjointness of the respective label sets, also in R condition 6 (indep. premises) is satisfied for this rule occurrence.

On the other hand, if the rule appears in one of the ρ^i_j, then we have the following figure:

$$\begin{array}{ccc} \dfrac{\overset{\Pi}{\underset{\rho_1^1\,\vdots}{A}} \quad \overset{\Pi}{\underset{\rho_1^2\,\vdots}{B}}}{A \wedge B}\;\wedge\text{-}i & & \begin{array}{c} \dfrac{\overset{\Delta}{\underset{\sigma_1\,\vdots}{F}} \quad \overset{\Xi}{\underset{\sigma_2\,\vdots}{G}}}{F \wedge G}\;\wedge\text{-}i \\ \vdots \end{array} \\ \rho_\wedge & \dots & \rho_j^i \end{array}$$

Assume without loss of generality that $\rho_j^i = \rho_{n_1}^1$. Let

$$\begin{aligned} S &= \{\rho_1^1, \dots, \rho_{n_1-1}^1, \rho_{n_1}^1\} \\ S_1 &= \{\rho_1^1, \dots, \rho_{n_1-1}^1, \sigma_1\} \\ S_2 &= \{\rho_1^1, \dots, \rho_{n_1-1}^1, \sigma_2\} \end{aligned}$$

As R_1 is a GLHN deduction, the critical condition holds for the displayed $\wedge$-i rule occurrence in ρ_j^i in R_1. Hence we obtain that

$$\operatorname{lnk}_{S_1^*}(\sigma_1) \cap \operatorname{lnk}_{S_2^*}(\sigma_2) = \varnothing$$

Now consider the same $\wedge$-i rule, but in the new set of pre-derivations, and let

$$\begin{aligned} S' &= \{\rho_\wedge, \rho_2^1, \dots, \rho_{n_1}^1, \rho_2^2, \dots, \rho_{n_2}^2\} \\ S_1' &= \{\rho_\wedge, \rho_2^1, \dots, \qquad \rho_2^2, \dots, \rho_{n_2}^2, \sigma_1\} \\ S_2' &= \{\rho_\wedge, \rho_2^1, \dots, \qquad \rho_2^2, \dots, \rho_{n_2}^2, \sigma_2\} \end{aligned}$$

We need to show that

$$\operatorname{lnk}_{S_1'^*}(\sigma_1) \cap \operatorname{lnk}_{S_2'^*}(\sigma_2) = \varnothing$$

Assume for the sake of contradiction, that the intersection contains a pre-derivation. There are three cases to be considered: (i) this pre-derivation is contained in $\rho_\wedge^*$, (ii) it is contained in one of the ρ_i^{2*} ($i = 2, \dots, n_2$), or (iii) it is contained in one of the ρ_i^{1*} ($i = 2, \dots, n_2 - 1$).

(i) If this pre-derivation is contained in $\rho_\wedge^*$, since $\mathrm{Labels}(R_1)$ and $\mathrm{Labels}(R_2)$ are disjoint, the only possible connection between σ_1 and σ_2 is via the expansion of ρ_1^1 in the left-upper part of $\rho_\wedge$. But this would have as a consequence that also

$$\operatorname{lnk}_{S_1^*}(\sigma_1) \cap \operatorname{lnk}_{S_2^*}(\sigma_2) \neq \varnothing,$$

as they are then also connected in the original deduction, a contradiction to the assumption that R_1 was a GLHN.

(ii) and (iii) are done similarly.

contraction

In the case of the contraction rule, we are starting from slightly different assumptions, namely that the first two pre-derivations have been contracted into one. Again, we have to show that the new set is a GLHN. As in the previous case, it is easy to see that all conditions but condition 6 (indep. premises) are trivially satisfied. What remains to show is that for all other critical rules in R', condition 6 is still satisfied. We consider again only the case of $\wedge$-i, the other cases are similar. If an application of $\wedge$-i occurs in ρ_1 or ρ_2, then the definition of the condition ignores the part below the binary rule, and thus the contraction anyway, so there are no changes for these rules.

On the other hand, if it occurs in some ρ_k for $k \geq 3$, then we have the following figure:

$$\begin{array}{ccc} \dfrac{\overset{\Gamma_1}{\underset{\rho_1\,\vdots}{A}} \quad \overset{\Gamma_2}{\underset{\rho_2\,\vdots}{A}}}{A}\;\mathit{contr} & & \begin{array}{c} \dfrac{\overset{\Delta}{\underset{\sigma_1\,\vdots}{F}} \quad \overset{\Xi}{\underset{\sigma_2\,\vdots}{G}}}{F \wedge G}\;\wedge\text{-}i \\ \vdots \end{array} \\ \rho_c, & \dots & \rho_k \end{array}$$

Let us assume without loss of generality that $k = n_1$ and let $n = n_1$. Let

$$\begin{aligned} R_1 &= \{\rho_1, \rho_2, \dots, \rho_{n-1}, \rho_n\} \\ S_1 &= \{\rho_1, \rho_2, \dots, \rho_{n-1}, \sigma_1\} \\ S_2 &= \{\rho_1, \rho_2, \dots, \rho_{n-1}, \sigma_2\}\ . \end{aligned}$$

Since R_1 is a GLHN, it satisfies condition 6 (indep. premises) by assumption. Hence we have that

$$\operatorname{lnk}_{S_1^*}(\sigma_1) \cap \operatorname{lnk}_{S_2^*}(\sigma_2) = \varnothing$$

Consider the corresponding sets after applying the contraction rule:

$$\begin{aligned} R &= \{\rho_c, \rho_3, \dots, \rho_{n-1}, \rho_n\} \\ S_1' &= \{\rho_c, \rho_3, \dots, \rho_{n-1}, \sigma_1\} \\ S_2' &= \{\rho_c, \rho_3, \dots, \rho_{n-1}, \sigma_2\} \end{aligned}$$

Reviewing the definition of expansion, we see that the expansion of S_i', and the expansion of the original S_i, coincide: $S_i'^* = S_i^*$. As a consequence we obtain that the condition 6 (indep. premises) is also satisfied in R.

Remark. Here we see the importance of defining the equivalence via the expansion. Without this, it is easy to give a counter-example: Consider the proof in GLC given in Figure 2. The intended translation into GLHN would be:

$$\dfrac{\mathrm{com}^x_{A,B}\,\dfrac{A}{B} \qquad \mathrm{com}^y_{A,C}\,\dfrac{A}{C}}{B \wedge C}\;\wedge{:}i$$

$$\dfrac{\mathrm{com}^{\bar{x}}_{B,A}\,\dfrac{\wedge{:}e\,\dfrac{B \wedge C}{B}}{A} \qquad \mathrm{com}^{\bar{y}}_{C,A}\,\dfrac{\wedge{:}e\,\dfrac{B \wedge C}{C}}{A}}{A}\;\mathit{contr}$$

Here, condition 6 (indep. premises) would *not* be satisfied for the binary rule $\wedge{:}i$ if we drop expansions, as both upper sub-pre-derivations are connected to the same pre-derivation due to the merge of contraction. By unwinding the contractions before checking the equivalence relations we can ensure that the addition of a binary rule does not disturb other binary rules.

Similar treatment of the other cases completes the proof. ■

The Hyper Natural Deduction system given by Baaz et.al. [18] can be viewed as a description on the hyper rule level, with resulting derived hypersequent.

Lemma 2. *Conditions 2 (separate derivations) to 6 (indep. premises) from Definition 7 are stable when taking a subset of sub-pre-derivations. That is, if a set R of pre-derivations*

$$
(\wedge{:}r)\,\dfrac{(com)\,\dfrac{A \Rightarrow A \qquad (\wedge{:}l)\,\dfrac{B \Rightarrow B}{B \wedge C \Rightarrow B}}{B \wedge C \Rightarrow A \mid A \Rightarrow B} \qquad (com)\,\dfrac{A \Rightarrow A \qquad (\wedge{:}l)\,\dfrac{C \Rightarrow C}{B \wedge C \Rightarrow C}}{B \wedge C \Rightarrow A \mid A \Rightarrow C}}{EC\,\dfrac{B \wedge C \Rightarrow A \mid B \wedge C \Rightarrow A \mid A \Rightarrow B \wedge C}{B \wedge C \Rightarrow A \mid A \Rightarrow B \wedge C}}
$$

Fig. 2. GLC derivation exhibiting the necessity for expansion in corresponding GLHN deduction

satisfies Conditions 2 to 6 from Definition 7, then so does any collection of sub-pre-derivations of sub-derivations in R.

Proof sketch: All conditions can be easily checked. ■

The following retraction lemma provides a means for inductive proofs on the length of GLHNs.

Lemma 3 (Retraction)**.** *Let $R = \{\rho_1, \dots, \rho_n\}$ be a* GLHN*.*

(i) If ρ_1 ends in a unary logical rule, let σ be the immediate sub-pre-derivation of ρ. Then $\{\sigma, \rho_2, \dots, \rho_n\}$ forms a GLHN*.*

(ii) If ρ_1 ends in a non-unary logical rule ($\wedge$-i, $\vee$-e, $\to$-e) or contraction, let σ_i for $i = 1, 2(, 3)$ be the immediate sub-pre-derivations of ρ. Using the notations from Condition 6 (indep. premises) of Definition 7, the equivalence classes $[\sigma_i]_{\approx_{R_i}}$ are GLHN*s, which as sets are pairwise disjoint.*

(iii) If ρ_1 and ρ_2 end in a pair of dually labelled communication rules, let σ_i be the immediate sub-pre-derivations of ρ_i, $i = 1, 2$. Again employing the notations from Condition 6 of Definition 7, the equivalence classes $[\sigma_1]_{\approx_{R_1}}$ and $[\sigma_2]_{\approx_{R_2}}$ are GLHN*s, which as sets are disjoint.*

(iv) If ρ_1 and ρ_2 end in a pair of dually labelled splitting rules, let σ be the immediate sub-pre-derivation of ρ_1 (observe that σ is also the immediate sub-pre-derivations of ρ_2 according to Condition 4 (consistent split) of Definition 7). Then $\{\sigma, \rho_3, \dots, \rho_n\}$ forms a GLHN*.*

Proof sketch: By inspection of the conditions and using Lemma 2. ■

The following lemma provides us with a way to combine two GLHNs by concatenating them.

Lemma 4 (Concatenation)**.** *Assume we have a* GLHN $R_1 = \{\rho_i^1 : i = 1, \dots, n_1\}$ *with* $\mathrm{Conc}(\rho_1^1) = A$*, and another* GLHN $R_2 = \{\rho_i^2 : i = 1, \dots, n_2\}$ *with* $A \in \mathrm{Assum}(\rho_1^2)$*. Furthermore, assume that the sets of labels of R_i are disjoint. Generate ρ_0 from ρ_1^2 by attaching at each open assumption A the pre-derivation ρ_1^1. Then,*

$$R = \{\rho_0, \rho_j^i : i = 1, 2, \quad j = 2, \dots, n_i\}$$

is again a GLHN*. Furthermore,* $\mathrm{Conc}(\rho_0) = \mathrm{Conc}(\rho_1^2)$ *and*

$$\mathrm{Assum}(\rho_0) = \mathrm{Assum}(\rho_1^1) \cup (\mathrm{Assum}(\rho_1^2) \setminus \{A\}) \ .$$

Proof sketch: By inspection of the conditions. ■

Our last lemma concerns the relation between equivalence classes in the non-unary case of logical rules. Let us assume the same notions as in Condition 6 (indep. premises) of Definition 7.

Lemma 5. *Let $R = \{\rho_1, \dots, \rho_n\}$ be a* GLHN*, and assume we are in case ii or iii of the Retraction Lemma 3, that is, we have a non-unary logical rule or dually labelled communication rules as final rules of some pre-derivation in R. With the notations from Condition 6 (indep. premises) of Definition 7, we can obtain all of R from the equivalence classes of $[\sigma_i]_{\approx_{R_i}}$, disregarding ρ_1 and σ_i:*

$$[\rho_1]_{\approx_R} \setminus \{\rho_1\} = \bigcup_{i=1}^{2(3)} \left([\sigma_i]_{\approx_{R_i}} \setminus \{\sigma_i\}\right)$$

Proof sketch: By analysing the equivalence chains that connect different labels. ■

V. Translations between Hypersequent Calculus and Hyper Natural Deduction

A. *Translation from* GLC *to* GLHN

We will translate GLC derivations into GLHN. As an illustration consider the linearity formula $C = (A \to B) \vee (B \to A)$, which has the following GLC derivation:

$$
EC\,\dfrac{\vee,r\,\dfrac{\vee,r\,\dfrac{\to,r\,\dfrac{\to,r\,\dfrac{com\,\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}}{\Rightarrow A \to B \mid B \Rightarrow A}}{\Rightarrow A \to B \mid \ \Rightarrow B \to A}}{\Rightarrow C \mid \ \Rightarrow B \to A}}{\Rightarrow C \mid \ \Rightarrow C}}{\Rightarrow C}
$$

The corresponding GLHN deduction would look like this:

$$
contr\,\dfrac{\vee\text{-}i\,\dfrac{{}^1 \to\text{-}i\,\dfrac{\mathrm{com}^x_{A,B}\,\dfrac{{}^1[A]}{B}}{A \to B}}{C} \qquad \vee\text{-}i\,\dfrac{{}^2 \to\text{-}i\,\dfrac{\dfrac{}{\mathrm{com}^x_{A,B}}\,\dfrac{{}^2[B]}{A}}{B \to A}}{C}}{C}
$$

Definition 10. Let

$$\mathcal{H} = \Gamma_1 \Rightarrow A_1 \mid \ \dots \ \mid \Gamma_k \Rightarrow A_k$$

and

$$\mathcal{H}' = \Delta_1 \Rightarrow B_1 \mid \ \dots \ \mid \Delta_\ell \Rightarrow B_\ell \ .$$

$\mathcal{H}$ is a *syntactic sub hypersequent* of $\mathcal{H}'$, denoted $\mathcal{H} \sqsubseteq \mathcal{H}'$, if and only if there exists an injection $f \colon \{1, \dots, k\} \to \{1, \dots, \ell\}$ such that for all $i = 1, \dots, k$,

$$A_i = B_{f(i)} \qquad \text{and} \qquad \Gamma_i \subseteq \Delta_{f(i)} \ .$$

Remark: If $\mathcal{H} \sqsubseteq \mathcal{H}'$ and $\mathcal{H}$ is valid, then also $\mathcal{H}'$ is valid.

Definition 11. The *derived sequent* of a pre-derivation ρ, denoted $\mathrm{Seq}(\rho)$, is given by $\mathrm{Assum}(\rho) \Rightarrow \mathrm{Conc}(\rho)$.

The *derived hypersequent* of a finite set of pre-derivations $R = \{\rho_1, \ldots, \rho_k\}$, denoted $\mathrm{HypSeq}(R)$, is given by $\mathrm{Seq}(\rho_1) \mid \ldots \mid \mathrm{Seq}(\rho_k)$.

Theorem 2. *Assume a hypersequent $\mathcal{H}$ has a* GLC *derivation. Then there exists a deduction R in* GLHN *such that the derived hypersequent of R is a syntactic sub hypersequent of $\mathcal{H}$.*

Proof sketch: Fix a GLC derivation D of $\mathcal{H}$. We prove the claim by induction on the length of D. If D is an axiom $A \Rightarrow A$, we use the GLHN A. If D is an axiom of the form $\bot \Rightarrow A$, then we use the rule $\bot_I$ of GLHN to form the deduction $\bot/A$.

For the other rules of GLC we use the corresponding GLHN hyper rules in combination with the Stability Lemma 1 and the Concatenation Lemma 4. ■

A formula A is GLC derivable if the hypersequent $\Rightarrow A$ is GLC derivable. A is GLHN derivable if there is a GLHN deduction consisting of one pre-derivation ρ, which has no free assumptions and ends in A, that is, $\mathrm{Assum}(\rho) = \emptyset$ and $\mathrm{Conc}(\rho) = A$.

Corollary 1. *If A is* GLC *derivable, then A is also* GLHN *derivable.*

B. Translation from GLHN *to* GLC

We will show by induction on the total number of inference steps over all pre-derivations, that any GLHN deduction $R = \{\rho_1, \ldots, \rho_n\}$ can be translated into a GLC derivation. We see that as long as two branches are not connected with communication or split, the translation needs to generated independent hypersequent proofs, while as soon as they become connected, the two independent proofs need to be merged into one hypersequent proof with multiple sequents.

Which branches go into different hypersequent proofs, and which branches go into the same as different sequents, is determined by the equivalence relation $\approx_R$ and its equivalence classes.

Theorem 3. *Let R be a* GLHN *deduction. Then there exists a* GLC *derivation of the derived hypersequent* $\mathrm{HypSeq}(R)$ *of R.*

Proof sketch: Fix a total order $\precsim$ on $\mathrm{Labels}(R)/\_$ which comes with R being a GLHN deduction. The proof is by induction on the total number of nodes over all pre-derivations in R. As GLC has external weakening, it will be sufficient to provide a GLC-derivation for a sub-multi-set of $\mathrm{HypSeq}(R)$. The proof layout is as follows: We distinguish cases according to the final rules occurring in all pre-derivations in R, with the intention to apply the Retraction Lemma 3 to obtain shorter GLHNs to which we can apply the induction hypothesis. For logical rules and contractions this is always possible. The problematic cases are communication and split as we need, in order to apply the Retraction Lemma, a pair of dually labelled rules at the end of two pre-derivations, which is not guaranteed to exist in general. Therefore, we will deal with logical rules and contraction as long as this is possible, and will see that if only communication and split labels occur at the end of pre-derivations, a pair of dually labelled rules is guaranteed to exist due to Condition 5 (label order) of Definition 7.

If one of the pre-derivations in R has the form of a single node deduction consisting of just a formula A, then the GLC axiom $A \Rightarrow A$ provides the required derivation.

Now assume that at least one pre-derivation in R ends in an application of a logical rule or contraction. In this case, by using the Retraction Lemma 3 and induction hypothesis, we obtain a hypersequent derivation, which we follow with the respective rules of GLC.

Finally, assume that all pre-derivations in $R = \{\rho_1, \ldots, \rho_k\}$ end in an application of communication or splitting. Let x_i be the label of the last inference in ρ_i, and let $L = \{x_1, \ldots, x_k\}$. In order to be able to apply the Retraction Lemma 3 we need in L a pair of dual labels: Let $\preceq$ be the pre-order on $\mathrm{Labels}(R)$ induced by $\precsim$ according to Definition 6. Choose a largest element in L according to $\preceq$, w.l.o.g. assume this is x_1. We will argue that the dual of x_1 is also in L. For the sake of contradiction assume that this is not the case. Then, due to Condition 1 (dual labels), $\overline{x_1}$ has to appear somewhere in R, say in ρ_i. By assumption it cannot be the label of the last rule in ρ_i, x_i. As there is a path through ρ_i which contains both labels, we must have $\overline{x_1} \prec x_i$ using Condition 5 (label order). This contradicts the maximality of x_1 (and thus also of $\overline{x_1}$) in L.

Thus, we know that there is a pair of dual labels in L, w.l.o.g. assume these are x_1 and x_2. Again, by using the Retraction Lemma 3 and induction hypothesis, we obtain GLC derivation(s) that can be extended with the corresponding communication rule or splitting rule of GLC. ■

As a consequence of the previous theorem we obtain the following corollary:

Corollary 2. *If A is* GLHN *derivable, then A is also* GLC *derivable.*

Corollaries 1 and 2 together show, using the completeness of GLC for infinitary propositional Gödel logic [11]:

Corollary 3. GLHN *is sound and complete for infinitary propositional Gödel logic.*

VI. Normalisation

The natural deduction calculus NJ allows derivations to be normalised. We will describe a similar results for GLHN.

In general, we will follow [30, Chapter 6] for proving normalisation. The notion of normal derivation depends on notions like segment and cuts, which in turn need the notions minor and major premise of rules. We thus start by defining the latter for communication, split and contraction rules. As communication and split are linking different pre-derivations, we cannot define the minor premise of such a rule locally, i.e. dependent on just this rule, anymore, but have to define them in context of a GLHN deduction.

Definition 12 (Minor premises for communication, split and contraction rules)**.** Let R be a (finite) set of pre-derivations.

A *minor premise w.r.t.* R of a communication rule of the form

$$\mathrm{com}^{x}_{A,B}\ \frac{A}{B}$$

is any occurrence of B in R as the premise of a rule labelled by $\overline{\mathrm{com}^{x}_{A,B}}$.

A *minor premise w.r.t.* R of a splitting rule of the form

$$S^{x}_{\Gamma,\Delta}\ \frac{A}{A}$$

is any occurrence of A in R as the premise of a rule labelled by $S^{x}_{\Gamma,\Delta}$ or $\overline{S^{x}_{\Gamma,\Delta}}$.

Both premises of a contraction rule are *minor premises* of that contraction occurrence.

During normalisation, elimination rules are permuted over minor premises of other disjunction elimination like rules (del-rules, see below) until they reach an introduction rule. For communication, split and contraction the reason for calling premises "minor" is the same as above, during nomalisation elimination rules are just "permuted". But due to the non-local nature of minor premises for communication and split the situation now is much more involved.

We adapt the notion of segment, cut, cutrank, and critical cut from [30, Def. 6.1.2] to take the additional minor premises into account, which can be conveniently done by just defining the "del-rules" of GLHN [30, Def. 6.1.1]. For the benefit of the reader we will restate the definition of the former notions from [30, Def. 6.1.2] as well. Let *I-rules* denote the introduction rules for logical connectives, and *E-rules* denote the elimination rules for logical connectives. With $|A|$ we denote the length of formula A, given by the number of occurrences of logical connectives in A.

Definition 13. The *del-rules* of GLHN are $\vee$-e, contr, com and split.

Definition 14 ([30, Def. 6.1.2])**.** A *segment* (of *length* n) in a GLHN deduction R is a sequence $A_1, \dots, A_n$ of consecutive occurrences of a formula A in R such that

- for $1 < n$, $i < n$, A_i is a minor premise of a del-rule application in R, with conclusion A_{i+1},
- A_n is not a minor premise of a del-rule application,
- A_1 is not the conclusion of a del-rule application.

A segment is *maximal*, or a *cut (segment)* if A_n is the major premise of an E-rule, and either $n > 1$, or $n = 1$ and $A_1 = A_n$ is the conclusion of an I-rule. The *cutrank* $\mathrm{CR}(s)$ of a maximal segment s with formula A is $|A|$. The *cutrank* $\mathrm{CR}(R)$ of a GLHN deduction R is the maximum of the cutranks of cuts of R. If there is no cut, the cutrank of R is zero. A *critical* cut of R is a cut of maximal cutrank among all cuts in R. We shall use s, s' for segments.

A deduction without critical cuts is said to be *normal*.

A. Conversions

We now extend the conversions defined in [30, Chap. 6.1] to deal with our additional cases involving communication, splitting and contraction rules.

The *detour conversions* $\wedge$-conversion, $\vee$-conversion and $\to$-conversion, and the *simplification conversions* from [30, Chap. 6.1] stay the same. The *permutation conversion* $\vee$-perm conversion for E-rules from [30, Chap. 6.1] also stays the same.

We now define three more permutation conversions which deal with permuting an E-rule over a contraction, communication and split rule.

contr-perm conversion:

$$\textit{E-rule}\ \dfrac{\textit{contr}\ \dfrac{\overset{\sigma_1}{A} \quad \overset{\sigma_2}{A}}{A} \qquad \sigma}{B}$$

contracts to

$$\textit{contr}\ \dfrac{\textit{E-rule}\ \dfrac{\overset{\sigma_1}{A} \quad \sigma}{B} \qquad \textit{E-rule}\ \dfrac{\overset{\sigma_2}{A} \quad \sigma}{B}}{B}$$

com-perm conversion and *split-perm conversion* differ from the previous cases in that they have to be defined w.r.t. a GLHN deduction $R = \{\rho_1, \dots, \rho_n\}$. We will define them only for $\to$-e, the cases for other E-rules are similar.

com-perm conversion: Consider a pair of dually labelled communication rules occurring in R. Assume the label is of the form $\ell = \mathrm{com}^{x}_{A\to B,C}$, and that the two occurrences are of the form

$$\rho_1:\ \begin{array}{c}\Gamma\\ \sigma_0 \vdots\\ \ell\ \dfrac{A \to B}{C}\\ \vdots\end{array} \qquad\qquad \rho_2:\ \begin{array}{c} \to\text{-}e\ \dfrac{\bar{\ell}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\vdots\\ C\end{array}}{A \to B} \qquad \begin{array}{c}\Pi\\ \sigma_2\vdots\\ A\end{array}}{B}\\ \vdots\end{array}$$

Let μ be

$$\to\text{-}e\ \dfrac{\begin{array}{c}\Gamma\\ \sigma_0\vdots\\ A \to B\end{array} \qquad \begin{array}{c}\Pi\\ \sigma_2\vdots\\ A\end{array}}{B}$$

and define the following three pre-derivations:

$$\delta_1:\ \begin{array}{c}\Gamma, {}^{1}[\Pi]\\ \mu\vdots\\ \mathrm{com}^{x}_{B,C}\ \dfrac{{}^{1}S_{\Gamma,\Pi}\ \dfrac{B}{B}}{C}\end{array} \qquad\qquad \delta_3:\ \begin{array}{c}\Delta\\ \sigma_1\vdots\\ \dfrac{\mathrm{com}^{\bar{x}}_{C,B}\ \dfrac{C}{B}}{A \to B}\end{array}$$

$$\delta_2:\ \textit{contr}\ \dfrac{\begin{array}{c}{}^{1}[\Gamma], \Pi\\ \mu\vdots\\ {}^{1}S_{\Pi,\Gamma}\ \dfrac{B}{B}\end{array} \qquad \begin{array}{c}\Delta\\ \sigma_1\vdots\\ \mathrm{com}^{\bar{x}}_{C,B}\ \dfrac{C}{B}\end{array}}{B}$$

Then w.r.t. this pair of dual label occurrences, R contracts to R' which is obtained by applying the following three steps to R, where by "sub-tree rooted in a label-occurrence" we mean "sub-pre-derivation rooted in the conclusion of the rule carrying this label-occurrence":

- Replace *all* sub-trees rooted in occurrences of $\mathrm{com}^x_{A\to B,C}$ with δ_1
- replace the sub-tree rooted in the displayed occurrence of $\to$-e in ρ_2 with δ_2
- for *all remaining* occurrences of $\overline{\mathrm{com}^x_{A\to B,C}}$, replace the sub-trees rooted in them with δ_3.

These substitutions are possible since the proof trees rooted in dual communications labels are separated (Condition 2 (separate derivations) of Definition 7).

split-perm conversion: Consider a pair of dually labelled splitting rules occurring in R. Assume the label is of the form $\ell = S^x_{\Gamma,\Delta}$, and that the two occurrences are of the form

$$\begin{array}{c} {}^1[\Gamma],\Delta \\ \sigma_0\,\vdots \\ {}^1\ell\ \dfrac{A\to B}{A\to B} \\ \ddots\vdots\cdot \\ \rho_1 \end{array} \qquad\qquad \begin{array}{c} \begin{array}{cc} \Gamma,{}^2[\Delta] & \\ \sigma_0\,\vdots & \Pi \\ {}^2\bar\ell\ \dfrac{A\to B}{A\to B} & \sigma_1\,\vdots \end{array} \\ \to\text{-}e\ \dfrac{A\to B \qquad A}{B} \\ \ddots\vdots\cdot \\ \rho_2 \end{array}$$

Abbreviating sub-pre-derivations as

$$\sigma_2:\quad \begin{array}{c} {}^1[\Gamma],\Delta \\ \sigma_0\,\vdots \\ {}^1\ell\ \dfrac{A\to B}{A\to B} \end{array} \qquad\qquad \sigma_3:\quad \begin{array}{c} \Gamma,{}^2[\Delta] \\ \sigma_0\,\vdots \\ {}^2\bar\ell\ \dfrac{A\to B}{A\to B} \end{array}$$

$$\sigma_4:\quad \begin{array}{c} \begin{array}{cc} \Gamma & \Pi \\ \sigma_3\,\vdots & \sigma_1\,\vdots \end{array} \\ \to\text{-}e\ \dfrac{A\to B \qquad A}{B} \end{array}$$

ρ_1 and ρ_2 can be written as

$$\begin{array}{c} \sigma_2 \\ \ddots\vdots\cdot \\ \rho_1 \end{array} \qquad\qquad \begin{array}{c} \sigma_4 \\ \ddots\vdots\cdot \\ \rho_2 \end{array}$$

Let μ be the following pre-derivation:

$$\begin{array}{c} \begin{array}{cc} \Gamma,\Delta & \Pi \\ \sigma_0\,\vdots & \sigma_1\,\vdots \end{array} \\ \to\text{-}e\ \dfrac{A\to B \qquad A}{B} \end{array}$$

We consider cases depending on whether there are occurrences of $S^{\bar x}_{\Delta,\Gamma}$ other than those in occurrences of σ_4.

Case I) $S^{\bar x}_{\Delta,\Gamma}$ only occurs within occurrences of σ_4 in R. Using μ, define the following two pre-derivations

$$\delta_1:\quad \begin{array}{c} {}^1[\Gamma],\Delta,{}^1[\Pi] \\ \mu\,\vdots \\ {}^1S^x_{(\Gamma,\Pi),\Delta}\ \dfrac{B}{B} \\ \to\text{-}i\ \overline{\quad A\to B \quad} \end{array} \qquad\qquad \delta_2:\quad \begin{array}{c} \Gamma,{}^1[\Delta],\Pi \\ \mu\,\vdots \\ {}^1S^{\bar x}_{\Delta,(\Gamma,\Pi)}\ \dfrac{B}{B} \end{array}$$

W.r.t. the fixed pair of dual label occurrences, R contracts to R' which is obtained by applying the following two steps to R:

- Replace *all* occurrences of σ_2 with δ_1,
- replace *all* occurrences of σ_4 with δ_2.

Case II) $S^{\bar x}_{\Delta,\Gamma}$ occurs outside occurrences of σ_4. Using μ, define the following three pre-derivations

$$\delta_1:\quad \begin{array}{c} {}^3[\Gamma],\Delta,{}^1[\Pi] \\ \mu\,\vdots \\ {}^1S^x_{\Pi,(\Gamma,\Delta)}\ \dfrac{B}{B} \\ {}^3S^{y}_{\Gamma,\Delta}\ \dfrac{}{B} \\ \to\text{-}i\ \overline{\quad A\to B \quad} \end{array} \qquad\qquad \delta_2:\quad \begin{array}{c} {}^2[\Gamma,\Delta],\Pi \\ \mu\,\vdots \\ {}^1S^{\bar x}_{(\Gamma,\Delta),\Pi}\ \dfrac{B}{B} \end{array}$$

$$\delta_3:\quad \begin{array}{c} \Gamma,{}^4[\Delta],{}^1[\Pi] \\ \mu\,\vdots \\ {}^1S^x_{\Pi,(\Gamma,\Delta)}\ \dfrac{B}{B} \\ {}^4S^{\bar y}_{\Delta,\Gamma}\ \dfrac{}{B} \\ \to\text{-}i\ \overline{\quad A\to B \quad} \end{array}$$

W.r.t. the fixed pair of dual label occurrences, R contracts to R' which is obtained by applying the following three steps to R:

- Replace *all* occurrences of σ_2 with δ_1;
- replace *all* occurrences of σ_4 with δ_2.
- replace *all remaining* occurrences of σ_3 with δ_3.

These substitutions are possible since the proof trees rooted in splitting rules carrying the same label (modulo duality) are separated (Condition 2 (separate derivations) of Definition 7).

We now show that in the above cases, R' again constitutes a GLHN.

Lemma 6. *Contr-perm, com-perm and split-perm conversions convert* GLHN *deductions into* GLHN *deductions.*

Proof sketch: For all three conversions we have to check all conditions for GLHN. All conditions besides Condition 6 (indep. premises) follow immediately from the assumption that the original figure formed a GLHN. What remains is to show that due to the reshuffling and extension of sub-pre-derivations, Condition 6 has not been violated, which can be proven by inspecting the new binary rules one by one. ■

We now turn the central theorem on normalisation:

Theorem 4 (Normalisation)**.** *Each* GLHN *deduction R reduces to a normal* GLHN *deduction.*

Proof: We adapt the proof of normalisation [30, Theorem 6.1.8] to our setting.

We use main induction on the cutrank n of R, with a side-induction on

$$\alpha = \sum_{\text{critical cut } s \text{ in } R} \omega^{\text{length of } s}$$

By a suitable choice of the critical cut to which we apply a conversion we can achieve that either n decreases, or that n remains constant but α decreases. Let us call s a *m.c.c.* (max critical cut) in R if it is a critical cut *of maximal length*. Choose a m.c.c. s for which no m.c.c. occurs in a branch of R above *the lowest formula occurrence in* s. Applying a conversion to s, the resulting R' has a lower cutrank (if s has length 1, and it is the only maximal segment in R), or has the same cutrank but a lower value for α.

To see this in the case of com-perm conversion, using the notation form above, μ does not contain a m.c.c. (the original segment s has been reduced in length). The performed substitution may create many more critical cuts, but not of original maximal length, thus the number of m.c.c.'s has been reduced (which is reflected by α decreasing). ∎

VII. Conclusion

We have provided a new system of Hyper Natural Deductions which is sound and complete for infinitary propositional Gödel logic. It improves the previously known system by Baaz et.al. [18] by giving a "low level" description of Hyper Natural Deduction which is closer to Natural Deduction NJ — the system of Baaz et.al. [18] can be viewed as a description on, what we call, the hyper rule level with resulting derived hypersequent.

The advantage of our system is that it allows to extend the usual conversions used to prove normalisation of NJ, to our setting, and using them to prove the existence of normal forms (weak normalisation) for Hyper Natural Deduction. In the standard Curry-Howard correspondence, a detour-conversion step ($\to$-conversion) for natural deduction corresponds to β-reduction for λ-calculus. Thus, if we are aiming for a computational interpretation of Hyper Natural Deduction like a Curry-Howard correspondence, we need a procedural normalisation procedure like the one based on conversions. Another advantage in line with the conjecture about communication inherent in Hyper Natural Deduction mentioned in the introduction, is that the conversion rules for communication and split can be viewed as "communicating" sub-pre-derivations over a "channel" build by dual labels.

A lot of our set-up and new conversions resemble elements of process algebras. In ongoing work we are investigating this point of view by trying to provide a direct connection to some process algebra like the π-calculus. A first step will be to create a term system for the presented Hyper Natural Deduction system, with the aim to have a simpler set-up to investigate normalisation.

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