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New sufficient conditions of existence, moment estimations and non confluence for SDEs with non-Lipschitzian coefficients

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Abstract

The objective of the present paper is to find new sufficient conditions for the existence of unique strong solutions to a class of (time-inhomogeneous) stochastic differential equations with random, non-Lipschitzian coefficients. We give an example to show that our conditions are indeed weaker than those relevant conditions existing in the literature. We also derive moment estimations for the maximum process of the solution. Finally, we present a sufficient condition to ensure the non confluence property of the solution of time-homogeneous SDE which, in one dimension, is nothing but stochastic monotone property of the solution.

MSC: 60H10

Keywords: Stochastic differential equations; Non-Lipschitzian; Existence; Non explosion; Non confluence; Moment estimations for the maximum process; Test function

1. Introduction and main results

The theory of stochastic differential equations (SDEs) has been very well developed since the seminal work of the great mathematician Kiyosi Itô in the mid 1940s. Fundamental conditions

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like linear growth and Lipschitzian type conditions on the both drift and diffusion coefficients to ensure the existence and uniqueness of solutions of SDEs with any given initial data. The proofs are either based on Picard iteration (see. e.g., [6]) or via martingale problem formulation (cf. [15]). Since the remarkable paper [3], SDEs (as well as stochastic functional differential equations) with non-Lipschitzian coefficients have received much attention widely, see, e.g., [9,12,5,14], just mention a few. In the present paper, we aim to issue new sufficient conditions for the existence and uniqueness of strong solutions to SDEs with random non-Lipschitzian coefficients. We also derive the moment estimation of the solution. Furthermore, we will give sufficient conditions for the non confluence property (also known as non contact property, cf. [17]) of the solution of time-homogeneous SDE.

Given a probability space $(\Omega, \mathscr{F}, P)$ endowed with a complete filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $d, m \in \mathbb{N}$ be arbitrarily fixed. We are concerned with the following stochastic differential equation

$$dX_t(\omega) = \sigma(t, \omega, X_t)dB_t(\omega) + b(t, \omega, X_t)dtX_0(\omega) = x_0, \quad \text{a.s.} \tag{1.1}$$

where the initial $x_0 \in \mathbb{R}^d$, $(B_t)_{t\geq 0}$ is an m-dimensional standard $\mathscr{F}_t$-Brownian motion, and $\sigma : (t, \omega, x) \in [0, \infty)\times\Omega\times\mathbb{R}^d \mapsto \sigma(t, \omega, x) \in \mathbb{R}^d\otimes\mathbb{R}^m$ and $b : (t, \omega, x) \in [0, \infty)\times\Omega\times\mathbb{R}^d \mapsto b(t, \omega, x) \in \mathbb{R}^d$ are progressively measurable, respectively, continuous with respect to the third variable x.

In order that the integrals in the definition of the solutions of Eq. (1.1) are well-defined, we make the following assumption which is enforced throughout the paper

$$\mathbb{E}\int_0^T \sup_{|x|\leq R} (|b(s, \cdot, x)| + \|\sigma(s, \cdot, x)\|^2)ds < \infty, \quad \forall T, R > 0 \tag{1.2}$$

where the norm $\|\cdot\|$ stands for the Hilbert–Schmidt norm $\|\sigma\|^2 := \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2$ for any $d \times m$-matrix $\sigma = (\sigma_{ij}) \in \mathbb{R}^d \otimes \mathbb{R}^m$ and $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^d$.

Let us first discuss the sufficient conditions under which there is a unique strong solution of Eq. (1.1). Fix $R > 0$ arbitrarily, let $\eta_R : [0, 1) \to \mathbb{R}_+$ be an increasing continuous function on a very small interval $[0, \varepsilon_0)(0 < \varepsilon_0 \leq c_0)$ which satisfies

$$\eta_R(0) = 0, \quad \int_{0+} \frac{dx}{\eta_R(x)} = \infty.$$

Our first main result is the following.

Theorem 1.1. *Let $R > 0$ be fixed arbitrarily. Assume that for all $t \geq 0, \omega \in \Omega$, $|x| \vee |y| \leq R$, the following locally weak monotonicity condition*

$$\begin{aligned}&\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 + 2\langle x - y, b(t, \omega, x) - b(t, \omega, y)\rangle\\ &\quad\leq g(t, \omega)\eta_R(|x - y|^2)\end{aligned} \tag{1.3}$$

holds for $|x - y| \leq c_0 < 1$, with g being a progressively measurable and non negative function such that

$$\mathbb{E}\int_0^t g(s, \cdot)ds < \infty, \quad t \geq 0.$$

Then, there is a unique strong solution of SDE (1.1).

We will use Euler's approximation method to prove Theorem 1.1. To this end, we give briefly Euler's approximation for our SDE (1.1). For any fixed n, set $X^{(n)}(0) := x_0$, for nonnegative integer k, and $t \in [\frac{k}{n}, \frac{k+1}{n})$, define

$$X^{(n)}(t) := X^{(n)}\left(\frac{k}{n}\right) + \int_{\frac{k}{n}}^{t} b\left(s, \cdot, X^{(n)}\left(\frac{k}{n}\right)\right) ds + \int_{\frac{k}{n}}^{t} \sigma\left(s, \cdot, X^{(n)}\left(\frac{k}{n}\right)\right) dB_s. \quad (1.4)$$

To prove Theorem 1.1, we need the following lemmas.

Lemma 1.2. *Assume that (1.3) holds. Then there is at least one uniformly convergent subsequence of $X^{(n)}(\cdot)$.*

Lemma 1.3. *Assume that (1.3) holds. If $X(\cdot)$ is the limit process of a subsequence of $X^{(n)}(\cdot)$, then $X^{(n)}(\cdot)$ is a solution of Eq. (1.1).*

Our next main result concerns the non explosion property of the solution of SDE (1.1). Let $\gamma : [0, \infty) \to \mathbb{R}_+$ be a continuous, increasing function satisfying

$$\text{(i)} \lim_{x\to\infty} \gamma(x) = \infty; \qquad \text{(ii)} \int_K^\infty \frac{dx}{\gamma(x) + 1} = \infty.$$

Our second main result is the following.

Theorem 1.4. *Assume that there is a constant $K > 0$ such that*

$$\|\sigma(t, \omega, x)\|^2 + 2\langle x, b(t, \omega, x)\rangle \le f(t, \omega)(\gamma(|x|^2) + 1), \quad |x| \ge K \quad (1.5)$$

with f being a progressively measurable and non negative function satisfying

$$\mathbb{E} \int_0^t f(s, \cdot) ds < \infty, \quad t \ge 0.$$

Then, the solution of Eq. (1.1) is global, namely, the lifetime

$$\zeta := \inf\{t > 0, |X_t| = \infty\} = +\infty.$$

We now give an example to support our conditions (1.3) and (1.5). However, our example does not fulfill the conditions $(H1)$ and $(H2)$ in [3], respectively, neither does for the conditions of Theorem 4 in [16]. This then indicates that our conditions are indeed weaker than those known conditions.

Example. When the equation reduces to time independent case, let $m = d$, and $\sigma(x) = \text{diag}(\sigma_1(x), \ldots, \sigma_d(x))$, $b(x) = (b_1(x), \ldots, b_d(x))^T$, where $\sigma_i(x) = x_i^{\frac{2}{3}}$, $b_i(x) = -x_i^{\frac{1}{3}}$, here "$T$" denotes the transpose of the vector or a matrix. Then it is easy to see that neither conditions $(H1)$, $(H2)$ in [3] nor conditions in Theorem 4 in [16] hold since in this case the coefficients σ and b are Hölder continuous with orders $\frac{2}{3}$ and $\frac{1}{3}$, respectively. But our conditions (1.3) and (1.5) are satisfied for such defined σ and b. Indeed,

$$\begin{aligned}&\|\sigma(x) - \sigma(y)\|^2 + 2\langle x - y, b(x) - b(y)\rangle \\ &\quad = \sum_{i=1}^d \left(x_i^{\frac{2}{3}} - y_i^{\frac{2}{3}}\right)^2 - 2\sum_{i=1}^d (x_i - y_i)\left(x_i^{\frac{1}{3}} - y_i^{\frac{1}{3}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{d} \left(x_i^{\frac{1}{3}} - y_i^{\frac{1}{3}} \right)^2 \left[\left(x_i^{\frac{1}{3}} + y_i^{\frac{1}{3}} \right)^2 - 2 \left(x_i^{\frac{2}{3}} + x_i^{\frac{1}{3}} y_i^{\frac{1}{3}} + y_i^{\frac{2}{3}} \right) \right] \\
&= - \sum_{i=1}^{d} \left(x_i^{\frac{1}{3}} - y_i^{\frac{1}{3}} \right)^2 \left(x_i^{\frac{2}{3}} + y_i^{\frac{2}{3}} \right) \\
&\leq \eta_R(|x-y|^2)
\end{aligned}
$$

where we have used the fact that the second last line is clearly non-positive in the last derivation. Similarly, we have

$$
\|\sigma(x)\|^2 + 2\langle x, b(x)\rangle = \sum_{i=1}^{d} \left(x_i^{\frac{2}{3}} \right)^2 - 2 \sum_{i=1}^{d} x_i x_i^{\frac{1}{3}} = - \sum_{i=1}^{d} x_i^{\frac{4}{3}} \leq \gamma(|x|^2).
$$

This shows that our conditions (1.3) and (1.5) are fulfilled. Hence, by Theorems 1.1 and 1.4, there is a unique strong global solution of the SDE (1.1).

We would like to point out here that in the above example the conditions are given for the two coefficients b and σ jointly, which guarantee the Hölder continuity condition. For instance, putting the drift coefficient b to be zero, the diffusion coefficient σ in our example is clearly not Hölder continuous.

Remark 1.5. In [9], the first author studied pathwise uniqueness and non explosion properties of the solution of Eq. (1.1). We have generalized Fang and Zhang's results to a more general case. But it is not clear at that time whether there is a solution of Eq. (1.1) under the given condition. However, we can get the same conclusion when (1.5) and (1.3) hold. Moreover, we can prove that there is really a unique strong global solution of Eq. (1.1) if (1.5) and (1.3) hold.

In [5], Hofmanová and Seidler proved the following result. Assume b and σ are Borel functions such that $b(t,\cdot)$ and $\sigma(t,\cdot)$ are continuous for any $t \in [0,T]$ and the linear growth hypothesis is satisfied, that is

$$
\exists K_* < \infty,\ \forall t \in [0,T],\ \forall x \in \mathbb{R}^m, \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \leq K_*(1+\|x\|).
$$

Let ν be a Borel probability measure on $\mathbb{R}^m$. Then there exists a weak solution to the problem

$$
dX = b(t,X)dt + \sigma(t,X)dW, \qquad X(0) \sim \nu
$$

where W is a standard Brownian motion.

Remark 1.6. Let us comment our conditions with those of Hofmanová and Seidler in [5]. On one hand, our coercive condition (1.5) on the coefficients b and σ is weaker than the corresponding linear growth hypothesis given by Hofmanová and Seidler in [5], as we do not need the linear growth hypothesis here. In [5], however, the authors have proved the existence of weak solutions under the spatial continuity of the coefficients plus linear growth hypothesis. In our paper, we show the existence of a unique strong solution under the spatial continuity of the coefficients together with a locally weak monotonicity condition.

Let us give some comparison of our conditions with those existing in the literature. There are many works dealing with the existence and uniqueness of SDEs. Stroock and Varadhan (see [15]) proved the weak existence and uniqueness of Eq. (1.1) by using the martingale problem method when σ is bounded continuous and uniformly elliptic and b is bounded and measurable, and both the coefficients are independent with t and ω.

In Watanabe and Yamada [16,18], the authors gave sufficient conditions on σ and b for the strong uniqueness and existence of stochastic differential equation

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t).$$

If we take $\eta_R(x) = R(\rho^2(\sqrt{x}) + \sqrt{x}\bar{\rho}(\sqrt{x}))$, where ρ and $\bar{\rho}$ are the same as that of [16], it is obvious that condition (1.3) holds for such defined η_R. Note that we do not need the concave condition on ρ and $\bar{\rho}$.

More recently, Fang and Zhang [3] gave the sufficient conditions on σ and b under which the degenerated time-homogeneous equation of (1.1) has no explosion, pathwise uniqueness and non confluence. They proved a special case in [2] for non explosion and pathwise uniqueness of the equation. Since $\sigma,\ b$ are both continuous, according to Ikeda and Watanabe [6, Chapter IV, Theorem 2.3], the solution does exist under Fang and Zhang's conditions. By taking $\eta_R(x) = Rxr(x)$, $\gamma(x) = x\rho(x) + 1$, where $r(x)$ and ρ are the same as that of conditions $(H1)$ and $(H2)$ in Fang and Zhang [3], our conditions (1.3) and (1.5) are satisfied. By Theorems 1.1 and 1.4, there is a unique strong solution of Eq. (1.1), which is non explosive. And the above example shows that $(H1)$ and $(H2)$ in [3] do not hold, but (1.3) and (1.5) hold. Moreover, they must assume that the control function be differentiable to make sure the Gronwall lemma can be used, but we can drop the differentiability condition by using a new test function. $\eta_R(x) = Rx\log(1/x)$, $(x < 1)$ is a typical example for our η_R.

In [11], Prévöt and Röckner (see also [7]) proved that when σ, b satisfy (1.2) and the so called weak coercivity and local weak monotonicity, there exists a unique (up to P-indistinguishability) solution to the stochastic differential equation (1.1). Both [7,11] had to use the linearity of the control function to prove that the approximation sequence $X^{(n)}(t)$ is uniformly convergent. Since η_R in our condition (1.3) may not be linear function, and there is no weak coercivity, their method cannot be used in our case either. Krylov and Röckner [8] proved existence and uniqueness of strong solutions to stochastic equations in domains $G \subset \mathbb{R}^d$ with singular time dependent drift b up to an explosion time, but they must assume the unit diffusion.

Recently, Shao, Wang and Yuan [14] proved that there exists a unique non explosive solution for (1.1) when the coefficients satisfy certain global conditions. Actually, their assumptions guarantee the local coercivity condition in [11,7]. Thus, the condition is too strong in a certain sense for the existence of (1.1).

We now turn to the moment estimation for the following Markovian type stochastic differential equation

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt, \qquad X_0 = x_0. \tag{1.6}$$

In [4], the authors get the upper bound of pth moment of maximum process $\sup_{0\le s\le t} |X_t|$ for X_t being the unique strong solution of the time-homogeneous SDE. When the diffusion term is Lipschitz continuous, and the drift term satisfies one-sided Lipschitz condition, they prove that for $p \ge 2$, there exists $C(p, t) > 0$ such that

$$\mathbb{E}(\sup_{0\le s\le t} |X_t|^p) \le C(p, t)(1 + |x_0|^p).$$

Starting with that X_t is the unique solution of our Eq. (1.6), we will investigate the pth moment of the solution under more general and weaker conditions.

Theorem 1.7. *Assume that the coefficients σ and b satisfy*

$$\Big(\|\sigma(t,x)\|^2 + 2\langle x, b(t,x)\rangle\Big) \vee |\sigma^T(t,x)x| \le f(t)(|x|^2+1). \tag{1.7}$$

Let $p > 2$ be fixed arbitrarily. We have the following

(i) *If*

$$0 \le f \in L^p_{\text{loc}}(\mathbb{R}_+) := \left\{ f : \mathbb{R}_+ \to \mathbb{R}_+, \int_0^t f(s)^p ds < \infty,\ \forall t > 0 \right\},$$

then we have

$$\mathbb{E}(\sup_{0\le s\le t} |X_s|^p) \le A \exp\left\{ B \int_0^t f(s)^{\frac{p}{2}} ds + C \int_0^t f^p(s) ds \right\}$$

where A, B, C are only dependent on p, t and function f.

(ii) *If $0 \le f \in L^{\frac{2p}{p-2}}_{\text{loc}}(\mathbb{R}_+)$, then for any fixed $t > 0$,*

$$\mathbb{E}(\sup_{0\le s\le t} |X_s|^p) \le A_1 e^{B_1 t}$$

where A, B are still only dependent of p, t and function f.

Remark 1.8. We would like to point out that when p is close to 2, $L^{\frac{2p}{p-2}}_{\text{loc}}$ is close to L^∞_{loc}, so $f \in L^{\frac{2p}{p-2}}_{\text{loc}}$ is essentially that f is bounded. But $f \in L^p_{\text{loc}}$ is not necessarily bounded. On the other hand, for p sufficiently large, $f \in L^{\frac{2p}{p-2}}_{\text{loc}}$ is near that $f \in L^2_{\text{loc}}$, but $f \in L^p_{\text{loc}}$ is near essentially bounded.

Finally, let us consider the property of non confluence of the following time-homogeneous SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x_0. \tag{1.8}$$

We say that the solution X_t of Eq. (1.8) has non confluence, if for all $x_0 \neq y_0$,

$$P(X_t(x_0) \neq X_t(y_0), \forall t > 0) = 1.$$

Such kind of non confluence property was studied by Emery in an early work [1] for general stochastic differential equations under Lipschitzian conditions, and by Yamada and Ogura for non-Lipschitz case in [17]. However the mixing condition imposed in [17] for coefficients σ and b is difficult to be checked and not natural.

Fix $R > 0$ arbitrarily, let $\gamma_R : [0,1) \to \mathbb{R}_+$ be a differentiable function on a very small interval $[0, \varepsilon_0)(0 < \varepsilon_0 \le c_0)$ such that

$$\gamma_R(0) = 0, \quad \int_{0+} \frac{dx}{\gamma_R(x)} = \infty$$

and

$$\frac{x(\gamma_R'(x)+1)}{\gamma_R(x)} \le K, \quad \forall |x| \le c_0$$

for some constant $K > \frac{1}{2}$ which is independent of x and R.

Then by using a new test function, we can show the following result.

Theorem 1.9. *If the coefficients σ and b satisfy that for $|x| \vee |y| \le R$,*

$$\|\sigma(x)-\sigma(y)\|^2 - \frac{2}{2K-1}\langle x-y, b(x)-b(y)\rangle \le \gamma_R(|x-y|^2), \quad |x-y| \le c_0, \tag{1.9}$$

then the solution X_t of Eq. (1.8) has non confluence property before lifetime ζ. That is, for any $x_0 \neq y_0$, the conditional probability

$$P(\{\omega \in \Omega : X_t(x_0,\omega) \neq X_t(y_0,\omega), t > 0\} | \{\omega \in \Omega : t < \zeta(\omega)\}) = 1.$$

For the case of one dimension, the non confluence corresponds to stochastic monotonicity, that is, if $x_0 \le y_0$, then $X_t(x_0) \le X_t(y_0)$ for all $t \ge 0$ almost surely.

Corollary 1.10. *Suppose $d = m = 1$. If σ and b are locally Lipschitzian continuous, then the solution X_t of Eq. (1.8) is stochastic monotone before lifetime.*

The rest of the paper is organized as follows. We show there exists uniformly convergent subsequence of Euler approximation which is our Lemma 1.2 in Section 2. Then we prove that the limit process is a solution of Eq. (1.1) in Section 3. In Section 4, we will prove non explosion result which is our Theorem 1.4. Then in Section 5 we will get the upper bound of the pth moment of the maximum process. Finally, we prove non confluence of the solution of time-homogeneous SDE in Section 6.

2. Uniform convergence of Euler approximation

Proof of Lemma 1.2. If we denote $\kappa(n,t) := [tn]/n$, Eq. (1.4) is equivalent to

$$X^{(n)}(t) = x_0 + \int_0^t b(s,\cdot,X^{(n)}(s)+p^{(n)}(s))ds + \int_0^t \sigma(s,\cdot,X^{(n)}(s)+p^{(n)}(s))dB_s \tag{2.1}$$

where

$$\begin{aligned} p^{(n)}(t) &:= X^{(n)}(\kappa(n,t)) - X^{(n)}(t) \\ &= -\int_{\kappa(n,t)}^t b(s,\cdot,X^{(n)}(\kappa(n,s)))ds \\ &\quad - \int_{\kappa(n,t)}^t \sigma(s,\cdot,X^{(n)}(\kappa(n,s)))dB_s, t \in [0,\infty). \end{aligned} \tag{2.2}$$

In what follows, we want to prove that there is a subsequence of $X^{(n)}(t)$ that converges to some process $X(t)$. Define

$$\tau^{(n)}(R) := \inf\{t > 0, |X^{(n)}(t)| \ge R\},$$

then by the definition of $p^{(n)}(t)$ in (2.2),

$$\begin{aligned} -\langle e_i, p^{(n)}(t)\rangle = &\int_{\kappa(n,t)}^t \langle e_i, b(s,\cdot,X^{(n)}(\kappa(n,s)))\rangle ds \\ &+ \int_{\kappa(n,t)}^t \langle e_i, \sigma(s,\cdot,X^{(n)}(\kappa(n,s)))dB_s\rangle \end{aligned}$$

where e_i, $1 \le i \le d$, is the canonical basis of $\mathbb{R}^d$. It follows that

$$\begin{aligned} P(|\langle e_i, p^{(n)}(t)\rangle| \ge 2\varepsilon, t \le \tau^{(n)}(R)) &\le P\left(\int_{\kappa(n,t)}^{t} \sup_{|x|\le R} |b(s,x)| ds \ge \varepsilon\right) \\ &\quad + P\left(\sup_{\tilde{t}\in[0,t]} \left| \int_0^{\tilde{t}\wedge\tau^{(n)}(R)} 1_{[\kappa(n,t),\infty)}(s)\langle e_i, \sigma(s,\cdot, X^{(n)}(\kappa(n,s)))dB_s\rangle \right| \ge \varepsilon\right) \\ &\le P\left(\int_{\kappa(n,t)}^{t} \sup_{|x|\le R} |b(s,x)| ds \ge \varepsilon\right) + \frac{1}{\varepsilon^2}\mathbb{E}\int_{\kappa(n,t)}^{t\wedge\tau^{(n)}(R)} \sup_{|x|\le R} \|\sigma(s,x)\|^2 ds. \end{aligned} \tag{2.3}$$

The last inequality is derived by utilizing the martingale inequality. Then for any fixed $t \in [0,\infty)$ and R, we have

$$1_{\{t\le\tau^{(n)}(R)\}} p^{(n)}(t) \xrightarrow{P} 0, \quad n \to \infty. \tag{2.4}$$

Hence, there exists a subsequence of $\{n\}$ depending on R and t (which is also denoted as $\{n\}$ for the sake of simplicity) such that

$$1_{\{t\le\tau^{(n)}(R)\}} p^{(n)}(t) \xrightarrow{\text{a.s.}} 0, \quad n \to \infty. \tag{2.5}$$

Now let $Y_t = X^{(n)}(t) - X^{(m)}(t)$, $\xi_t = |Y_t|^2$. Define the following test function:

$$\varphi_\delta(x) = \int_0^x \frac{ds}{\eta_R(s)+\delta}. \tag{2.6}$$

It is obvious that for any $0 < x < \varepsilon_0$, when $\delta \downarrow 0$,

$$\varphi_\delta(x) \uparrow \varphi_0(x) = \int_0^x \frac{ds}{\eta_R(s)} = \infty. \tag{2.7}$$

Since

$$\frac{\varphi_\delta(x_2)-\varphi_\delta(x_1)}{x_2-x_1} \ge \frac{\varphi_\delta(x_3)-\varphi_\delta(x_2)}{x_3-x_2}, \quad 0 < x_1 < x_2 < x_3 < \varepsilon_0, \tag{2.8}$$

$\varphi_\delta(x)$ is a concave function on the interval $[0,\varepsilon_0)$. Note here that since $\lim_{x\to 0}\varphi'_\delta(x) = \frac{1}{\delta}$, there is a concave extension of $\varphi_\delta(x)$ on the real line. Let $\bar{\varphi}_\delta$ be a concave function on $\mathbb{R}$, and satisfy $\bar{\varphi}_\delta(x) = \varphi_\delta(x)$, $x \in [0,\varepsilon_0)$. Then the second order derivative of $\bar{\varphi}_\delta$ in the sense of distributions $\bar{\varphi}''_\delta$ is a non positive Radon measure (see [13] Appendix 3). Let

$$\tau_{n,m} := \inf\{t > 0, \xi_t \ge \varepsilon_0\}.$$

By using Itô-Tanaka's formula, we have

$$\begin{aligned} \bar{\varphi}_\delta(\xi_{t\wedge\tau_{n,m}}) = 2\int_0^{t\wedge\tau_{n,m}} & \bar{\varphi}'_\delta(\xi_s)\langle Y_s, (\sigma(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\ & - \sigma(s,\omega,X^{(m)}(s)+p^{(m)}(s)))dB_s\rangle \\ & + 2\int_0^{t\wedge\tau_{n,m}} \bar{\varphi}'_\delta(\xi_s)\langle Y_s, b(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\ & - b(s,\omega,X^{(m)}(s)+p^{(m)}(s))\rangle ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t\wedge\tau_{n,m}} \bar{\varphi}'_\delta(\xi_s)\|\sigma(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- \sigma(s,\omega,X^{(m)}(s)+p^{(m)}(s))\|^2 ds + \frac{1}{2}\int_{\mathbb{R}} L^a_{t\wedge\tau_{n,m}}(\xi)\bar{\varphi}''{}_\delta(da) \\
\le\ & 2\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\langle Y_s, (\sigma(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- \sigma(s,\omega,X^{(m)}(s)+p^{(m)}(s)))dB_s\rangle \\
&+ 2\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\langle Y_s, b(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- b(s,\omega,X^{(m)}(s)+p^{(m)}(s))\rangle ds \\
&+ \int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\|\sigma(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- \sigma(s,\omega,X^{(m)}(s)+p^{(m)}(s))\|^2 ds.
\end{aligned}
$$

The above inequality holds because the second derivative $\bar{\varphi}''{}_\delta$ in the sense of distributions is a non positive Radon measure and the local time is always non positive. Since $\varphi_\delta(\xi_{t\wedge\tau}) = \bar{\varphi}_\delta(\xi_{t\wedge\tau})$, taking expectation on both sides, we have

$$
\begin{aligned}
\mathbb{E}\varphi_\delta(\xi_{t\wedge\tau_{n,m}}) \le\ & 2\mathbb{E}\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\langle Y_s, b(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- b(s,\omega,X^{(m)}(s)+p^{(m)}(s))\rangle ds + \mathbb{E}\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\|\sigma(s,\omega,X^{(n)}(s) \\
&+ p^{(n)}(s)) - \sigma(s,\omega,X^{(m)}(s)+p^{(m)}(s))\|^2 ds.
\end{aligned} \tag{2.9}
$$

By the definition of $p^{(n)}(t)$ given in (2.2), we know that the above quantity could be dominated by

$$
\begin{aligned}
&2\mathbb{E}\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)\langle p^{(m)}(s)-p^{(n)}(s), b(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- b(s,\omega,X^{(m)}(s)+p^{(m)}(s))\rangle ds + \mathbb{E}\int_0^{t\wedge\tau_{n,m}} \varphi'_\delta(\xi_s)g(s,\omega)\eta_R(|X^{(n)}(s) \\
&+ p^{(n)}(s) - X^{(m)}(s) - p^{(m)}(s)|^2)ds.
\end{aligned} \tag{2.10}
$$

Denote

$$
\begin{aligned}
H(s) :=\ & |\varphi'_\delta(\xi_s)\langle p^{(m)}(s)-p^{(n)}(s), b(s,\omega,X^{(n)}(s)+p^{(n)}(s)) \\
&- b(s,\omega,X^{(m)}(s)+p^{(m)}(s))\rangle|,
\end{aligned} \tag{2.11}
$$

and

$$
G(s) := \varphi'_\delta(\xi_s)\eta_R(|X^{(n)}(s)+p^{(n)}(s)-X^{(m)}(s)-p^{(m)}(s)|^2). \tag{2.12}
$$

Then

$$H(s) \leq 2 \sup_{|x| \leq R} |b(s, \omega, x)| \times \frac{1}{\eta_R(\xi_s) + \delta} |p^{(n)}(s) - p^{(m)}(s)|, \quad s \leq \tau^{(n,m)}(R)$$

where

$$\tau^{(n,m)}(R) := \tau^{(n)}(R) \wedge \tau^{(m)}(R).$$

Denote

$$T_{n,m}(R) := \tau_{n,m} \wedge \tau^{(n,m)}(R). \tag{2.13}$$

Then, by (2.5), for fixed δ, let m, n be large enough (dependent on δ), we have

$$\begin{aligned} \frac{1}{\eta_R(\xi_s) + \delta} |p^{(n)}(s) - p^{(m)}(s)| &\leq \frac{1}{\delta} |p^{(n)}(s) - p^{(m)}(s)| \\ &\leq 1, \quad s \in [0, t \wedge T_{n,m}(R)), \quad \text{a.s.} \end{aligned}$$

So

$$H(s) \leq 2 \sup_{|x| \leq R} |b(s, \omega, x)|, \quad s \in [0, t \wedge T_{n,m}(R)), \quad \text{a.s.} \tag{2.14}$$

Similarly, by the continuity of function η_R, for fixed δ, and m, n be large enough (dependent on δ), we have

$$\begin{aligned} G(s) &= \frac{\eta_R(|X^{(n)}(s) - X^{(m)}(s) + p^{(n)}(s) - p^{(m)}(s)|^2)}{\eta_R(|X^{(n)}(s) - X^{(m)}(s)|^2) + \delta} \\ &\leq 1 + \frac{|\eta_R(|X^{(n)}(s) - X^{(m)}(s) + p^{(n)}(s) - p^{(m)}(s)\,|^2) - \eta_R(|X^{(n)}(s) - X^{(m)}(s)\,|^2)|}{\eta_R(|X^{(n)}(s) - X^{(m)}(s)|^2) + \delta} \\ &\leq 2, \quad s \in [0, t \wedge T_{n,m}(R)), \quad \text{a.s.} \end{aligned} \tag{2.15}$$

Then by Fatou's lemma, we have

$$\begin{aligned} &\limsup_{m,n \to \infty} \mathbb{E} \int_0^{t \wedge \tau_{n,m}} \varphi_\delta'(\xi_s) g(s, \omega) \eta_R(|X^{(n)}(s) + p^{(n)}(s) - X^{(m)}(s) - p^{(m)}(s)|^2) ds \\ &\quad \leq \mathbb{E} \int_0^t \limsup_{m,n \to \infty} \mathbf{1}_{[0, \tau_{n,m}]}(s) \varphi_\delta'(\xi_s) g(s, \omega) \eta_R(|X^{(n)}(s) \\ &\qquad + p^{(n)}(s) - X^{(m)}(s) - p^{(m)}(s)|^2) ds \\ &\quad \leq 2\mathbb{E} \int_0^t g(s, \omega) ds. \end{aligned} \tag{2.16}$$

In the last inequality above we used the fact that

$$\limsup_{m,n \to \infty} \varphi_\delta'(\xi_s) \eta_R(|X^{(n)}(s) + p^{(n)}(s) - X^{(m)}(s) - p^{(m)}(s)|^2) ds$$

is bounded almost surely. Hence

$$\mathbb{E} \int_0^{t \wedge \tau_{n,m}} \varphi_\delta'(\xi_s) g(s, \omega) \eta_R(|X^{(n)}(s) + p^{(n)}(s) - X^{(m)}(s) - p^{(m)}(s)|^2) ds$$

is uniformly bounded when m, n are large. Now obviously the choice of m and n is independent of sample points. An argument with similar spirit may also work for the first term of right hand side in (2.10). Hence

$$\limsup_{n,m\to\infty} \mathbb{E}\varphi_\delta(\xi_{t\wedge T_{n,m}(R)}) \le C(t) < \infty \tag{2.17}$$

where $C(t)$ is a positive constant independent of δ and R, $m(\delta)$ and $n(\delta)$.

Next, suppose that there are subsequences $\{m_k\}$, $\{n_k\}$ such that

$$\lim_{k\to\infty} \xi_{t\wedge T_{(n_k,m_k)}(R)} = \limsup_{n,m\to\infty} \xi_{t\wedge T_{n,m}(R)} = \xi_0, \quad \text{as } k\to\infty.$$

Since

$$|\varphi_\delta(\xi_{t\wedge T_{n,m}(R)})| \le \frac{\varepsilon_0}{\delta}$$

for any n, m when δ is fixed, by using the dominated convergence theorem, we have

$$\begin{aligned}\mathbb{E}\varphi_\delta(\limsup_{n,m\to\infty} \xi_{t\wedge T_{n,m}(R)}) &= \lim_{k\to\infty} \mathbb{E}\varphi_\delta(\xi_{t\wedge T_{(n_k,m_k)}(R)}) \\ &\le \limsup_{n,m\to\infty} \mathbb{E}\varphi_\delta(\xi_{t\wedge T_{n,m}(R)}) \le C(t) < \infty.\end{aligned}$$

Then let $\delta \downarrow 0$. Since $C(t)$ is independent of δ, we have

$$\mathbb{E}\int_0^{\limsup_{n,m\to\infty} \xi_{t\wedge T_{n,m}(R)}} \frac{1}{\eta_R(s)} ds \le C(t) < \infty.$$

By using $\int_{0+} \frac{ds}{\eta_R(s)} = \infty$, it follows that for any fixed $t > 0$,

$$P(\limsup_{n,m\to\infty} \xi_{t\wedge T_{n,m}(R)} = 0) = 1. \tag{2.18}$$

Now by Fatou's lemma we have

$$0 \le \limsup_{n,m\to\infty} \mathbb{E}\xi_{t\wedge T_{n,m}(R)} \le \mathbb{E}\limsup_{n,m\to\infty} \xi_{t\wedge T_{n,m}(R)} = 0, \tag{2.19}$$

that is, $\lim_{n,m\to\infty} \mathbb{E}\,\xi_{t\wedge T_{n,m}(R)}$ does exist and the limit is 0. Therefore,

$$\begin{aligned}P(\sup_{s\le t\wedge T_{n,m}(R)} |X^{(n)}(s) - X^{(m)}(s)|^2 \ge \varepsilon) &\le P(\sup_{s\le t\wedge \tau^{(n,m)}(R)} |X^{(n)}(s) - X^{(m)}(s)|^2 \ge \varepsilon) \\ &\le \frac{\mathbb{E}\xi_{t\wedge T_{n,m}(R)}}{\varepsilon} \to 0.\end{aligned} \tag{2.20}$$

The last inequality holds by using Lemma 3.1.3 in [11]. We now can select a subsequence, which will again be denoted by $X^{(n)}$ such that

$$P(\sup_{s\le t\wedge T_{n,m}(R)} |X^{(n)}(s) - X^{(m)}(s)|^2 \ge 2^{-m\wedge n}) \le 2^{-m\wedge n}. \tag{2.21}$$

Since t is arbitrary, we have

$$P(\sup_{s\le T_{n,m}(R)} |X^{(n)}(s) - X^{(m)}(s)|^2 \ge 2^{-m\wedge n}) \le 2^{-m\wedge n}. \tag{2.22}$$

Denote

$$\tau_R := \liminf_{n,m\to\infty} T_{n,m}(R).$$

Due to (2.22) there is a subsequence $X^{(n_k)}$ which is convergent to an $\mathscr{F}_t$ adapted process X defined in $[0, \tau_R]\mathbb{P}$-almost surely in $C([0, \tau_R]; \mathbb{R}^d)$.

Let $R \to \infty$, we obtain that there is a subsequence of $X^{(n_k)}$ (still denotes $X^{(n_k)}$ for simplicity) such that for any fixed $T > 0$,

$$\sup_{t \le \tau(T)} |X^{(n_k)}(t) - X(t)| \xrightarrow{\text{a.s.}} 0, \quad k \to \infty \tag{2.23}$$

where

$$\tau(T) := \liminf_{R\to\infty} \tau_R \wedge T.$$

The proof is complete. □

3. The limit process is a solution of Eq. (1.1)

Proof of Lemma 1.3. We are now going to prove that the limit process $X(t)$ is a solution of SDE (1.1). By (2.23) we only need to prove that there is a subsequence of the right hand side of (2.1) that converges to

$$X_0 + \int_0^t b(s, \cdot, X(s))ds + \int_0^t \sigma(s, \cdot, X(s))dB_s, \quad t \le \tau(T).$$

Since the convergence in (2.23) is uniform, by equicontinuity we have

$$\sup_{t \le \tau(T)} |X^{(n_k)}(\kappa(n_k, t)) - X(t)| \xrightarrow{\text{a.s.}} 0, \quad k \to \infty. \tag{3.1}$$

In particular, for $S(t) := \sup_{k\in\mathbb{N}} |X^{(n_k)}(\kappa(n_k, t))|$,

$$\sup_{t \le \tau(T)} S(t) < \infty, \quad P\text{-a.s.} \tag{3.2}$$

For $R \in [0, \infty)$, define the $(\mathscr{F}_t)$-stopping time

$$\tilde{\tau}(R, T) := \inf\{t \ge 0, S(t) > R\} \wedge \tau(T).$$

By the continuity of b in $x \in \mathbb{R}^d$ and by local integrability condition (1.2)

$$\lim_{k\to\infty} \int_0^t b(s, \cdot, X^{(n_k)}(\kappa(n_k, s)))ds = \int_0^t b(s, \cdot, X(s))ds,$$
$$P\text{-a.s. on } \{t \le \tilde{\tau}(R, T)\}. \tag{3.3}$$

For the stochastic integrals part we construct another sequence of stopping times. For $R, N \ge 0$, define

$$\tau_N(R, T) := \inf\left\{t \ge 0, \int_0^T \sup_{|x|\le R} \|\sigma(s, \cdot, x)\|^2 ds \ge N\right\} \wedge \tilde{\tau}(R, T).$$

Now by the continuity of σ in $x \in \mathbb{R}^d$, (1.2) and Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \mathbb{E} \int_0^{\tau_N(R,T)} \|\sigma(s, \cdot, X^{(n_k)}(\kappa(n_k, s))) - \sigma(s, \cdot, X(s))\|^2 ds = 0,$$

hence

$$P\text{-}\lim_{k\to\infty}\int_0^t \sigma(s,\cdot,X^{(n_k)}(\kappa(n_k,s)))dB_s = \int_0^t \sigma(s,\cdot,X(s))dB_s \tag{3.4}$$

on $t \le \tau_N(R,T)$.

Thus, there exists a subsequence of $\{n_k\}$ (which is also denoted as $\{n_k\}$ for the sake of simplicity) such that

$$\lim_{k\to\infty}\int_0^t \sigma(s,\cdot,X^{(n_k)}(\kappa(n_k,s)))dB_s = \int_0^t \sigma(s,\cdot,X(s))dB_s, \quad \text{a.s.} \tag{3.5}$$

on $t \le \tau_N(R,T)$. By the integrability condition (1.2) for every $\omega \in \Omega$ there exists $N(\omega) \ge 0$ such that $\tau_N(R,T) = \tilde{\tau}(R,T)$ for all $N \ge N(\omega)$, so

$$\bigcup_{N\in\mathbb{N}} \{t \le \tau_N(R,T)\} = \{t \le \tilde{\tau}(R,T)\}.$$

Therefore, (3.5) holds on $t \le \tilde{\tau}(R,T)$ almost surely. But by the definition of $\tilde{\tau}(R,T)$ we have

$$\lim_{R\to\infty} \tilde{\tau}(R,T) = \tau(T).$$

It follows that

$$X(t) = X_0 + \int_0^t b(s,\cdot,X(s))ds + \int_0^t \sigma(s,\cdot,X(s))dB_s, \quad t \le \tau(T). \tag{3.6}$$

So the SDE (1.1) has a solution $X(t)$ at least before $t \le \tau(T)$ for any $T > 0$. $\square$

Proof of Theorem 1.1. By the same method that we have already used in [9] and cut by a stopping time $\tau_R = \inf\{t > 0, |X_t| \vee |Y_t| \ge R\}$, we can prove that SDE (1.1) has pathwise uniqueness when the coefficients σ, b satisfy condition (1.3). So by Lemmas 1.2 and 1.3 there exists a unique strong solution of Eq. (1.1) with lifetime. $\square$

4. Non explosion of the solution

Proof of Theorem 1.4. First we show that

$$\liminf_{R\to\infty} \tau_R = \zeta, \quad \text{a.s.}$$

By the definition of $\tau^{(n,m)}(R)$, it follows that $\tau^{(n,m)}(R) \to \zeta$ as $m, n, R \to \infty$ subsequently, according to (2.18), we have

$$\limsup_{k\to\infty} \xi_{t\wedge\tau_{n_k,m_k}\zeta} = 0, \quad \text{P-a.s.} \tag{4.1}$$

If $P(\liminf_{k\to\infty} \tau_{n_k,m_k} < \zeta) > 0$, then there exists a subsequence $\{k_l\}$ such that

$$P(\lim_{l\to\infty} \tau_{n_{k_l},m_{k_l}} < \zeta) > 0.$$

Therefore, for $0 < T < \zeta$ close to ζ enough, there exists sufficiently large $l > 0$ such that

$$P(\tau_{n_{k_l},m_{k_l}} \le T < \zeta) > 0.$$

It follows that on $\{\tau_{n_{k_l},m_{k_l}} \le T < \zeta\}$,

$$\xi_{T\wedge\tau_{n_{k_l},m_{k_l}}\wedge\zeta} = \xi_{\tau_{n_{k_l},m_{k_l}}}.$$

According to (4.1), $\xi_{T\wedge\tau_{n_{k_l},m_{k_l}}\wedge\zeta}$ is smaller than any fixed positive ε for sufficiently large l, we have

$$\xi_{\tau_{n_{k_l},m_{k_l}}} < \varepsilon_0 \quad \text{as } l \to \infty$$

holds with positive probability which is absurd since $\xi_{\tau_{n,m}} \equiv \varepsilon_0 > 0$ by definition. So $\liminf_{k\to\infty} \tau_{n_k,m_k} = \zeta$, a.s.. By definition of τ_R, we have

$$\liminf_{R\to\infty} \tau_R = \liminf_{R\to\infty} \liminf_{n,m\to\infty} \tau_{n,m} \wedge \tau^{(n,m)}(R) = \zeta.$$

Now we only need to show $\zeta = \infty$.

Define

$$\varphi(x) := \int_0^x \frac{ds}{\gamma(s)+1}.$$

Then φ is a concave function on $[0,\infty)$ since γ is an increasing function and

$$\varphi'(x) = \frac{1}{\gamma(x)+1}.$$

As in the proof of Lemma 1.2, we can extend φ to a function $\bar\varphi$ on $\mathbb{R}$ which is still concave.

Let $\tilde\xi_t := |X_t|^2$. Define

$$\hat\tau_R := \inf\{t > 0,\ |X_t| \ge R\}, \quad R > 0,$$

then $\hat\tau_R \uparrow \zeta$ as $R \uparrow \infty$. By Itô-Tanaka's formula, we have

$$\begin{aligned}
\varphi(\tilde\xi_{t\wedge\hat\tau_R}) = \bar\varphi(\tilde\xi_{t\wedge\hat\tau_R}) &= \bar\varphi(\tilde\xi_0) + 2\int_0^{t\wedge\hat\tau_R} \bar\varphi'(\tilde\xi_s)\langle X_s, \sigma(s,\omega,X_s)dB_s\rangle \\
&\quad + 2\int_0^{t\wedge\hat\tau_R} \bar\varphi'(\tilde\xi_s)\langle X_s, b(s,\omega,X_s)\rangle ds \\
&\quad + \int_0^{t\wedge\hat\tau_R} \bar\varphi'(\tilde\xi_s)\|\sigma(s,\omega,X_s)\|^2 ds + \frac{1}{2}\int_R L^a_{t\wedge\hat\tau_R}(\tilde\xi)\bar\varphi''(da) \\
&\le \varphi(\tilde\xi_0) + 2\int_0^{t\wedge\hat\tau_R} \varphi'(\tilde\xi_s)\langle X_s, \sigma(s,\omega,X_s)dB_s\rangle \\
&\quad + \int_0^{t\wedge\hat\tau_R} \varphi'(\tilde\xi_s)\big(2\langle X_s, b(s,\omega,X_s)\rangle + \|\sigma(s,\omega,X_s)\|^2\big)ds.
\end{aligned}$$

The inequality holds because $\bar\varphi''$ in the sense of distribution is a non positive Radon measure and the local time is non positive (see [13, Appendix]). Furthermore, taking expectation on both sides, we have

$$\begin{aligned}
\mathbb{E}\varphi(\tilde\xi_{t\wedge\hat\tau_R}) &\le \varphi(\tilde\xi_0) + \mathbb{E}\left(\int_0^{t\wedge\hat\tau_R} \varphi'(\tilde\xi_s)(2\langle X_s, b(s,\omega,X_s)\rangle + \|\sigma(s,\omega,X_s)\|^2)ds\right) \\
&\le \varphi(\tilde\xi_0) + \mathbb{E}\int_0^t f(s,\omega)ds =: C_t < \infty.
\end{aligned}$$

Note that C_t is independent of R. Letting $R \uparrow \infty$, and using Fatou's lemma, we get

$$\mathbb{E}\int_0^{\tilde\xi_{t\wedge\zeta}} \frac{dx}{\gamma(x)+1} \le C_t < \infty.$$

Now if $P(\zeta < \infty) > 0$, then for some $T > 0$, $P(\zeta < T) > 0$. Taking $t = T$ in (2.4), we get

$$P(\zeta < T)\varphi(\tilde{\xi}_\zeta) = \mathbb{E}\big(\mathbf{1}_{\{\zeta<T\}}\varphi(\tilde{\xi}_\zeta)\big) \le C_T,$$

which is impossible since $\varphi(\tilde{\xi}_\zeta) = \infty$. Thus, for any $T > 0$,

$$X(t) = X_0 + \int_0^t b(s,\cdot,X(s))ds + \int_0^t \sigma(s,\cdot,X(s))dB_s, \quad t \le T. \tag{4.2}$$

It follows that the solution is non-explosive (that is, the lifetime $\zeta = \infty$). We complete the proof. □

5. Moment inequality of the solution

Proof of Theorem 1.7. Denote $Y_t := \sup_{0\le s\le t} |X_s|$. By Itô's formula, we have

$$|X_t|^2 = x_0^2 + \int_0^t (2\langle X_s, b(s,X_s)\rangle + \|\sigma(s,X_s)\|^2)ds + M_t \tag{5.1}$$

where $M_t := 2\int_0^t \langle X_s, \sigma(s,X_s)dB_s\rangle$. Then

$$Y_t^2 \le |x_0|^2 + \int_0^t f(s)(|Y_s|^2+1)ds + \sup_{0\le s\le t} |M_s|. \tag{5.2}$$

So there exists $C_p > 0$ such that

$$Y_t^p \le C_p\left(|x_0|^p + \left(\int_0^t f(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}} + \sup_{0\le s\le t} |M_s|^{\frac{p}{2}}\right). \tag{5.3}$$

Thus,

$$\mathbb{E}(Y_t^p) \le C_p\left(|x_0|^p + \mathbb{E}\left(\int_0^t f(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\sup_{0\le s\le t} |M_s|^{\frac{p}{2}}\right)\right). \tag{5.4}$$

By the Burkholder–Davis–Gundy inequality, there exists $C_p' > 0$ such that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le t} |M_s|^{\frac{p}{2}}\right) &\le C_p'\mathbb{E}\left[\left(\int_0^t |\sigma^T(s,X_s)X_s|^2 ds\right)^{\frac{p}{4}}\right] \\
&\le C_p'\mathbb{E}\left((Y_t^2+1)^{\frac{p}{4}}\left(\int_0^t f(s)|\sigma^T(s,X_s)X_s|ds\right)^{\frac{p}{4}}\right) \\
&\le C_p'\left(\frac{1}{2K}\mathbb{E}\left((Y_t^2+1)^{\frac{p}{2}}\right) + \frac{K}{2}\mathbb{E}\left[\left(\int_0^t f^2(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right]\right).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}(Y_t^p) \le\ & C_p|x_0|^p + C_p\mathbb{E}\left(\int_0^t f(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}} \\
&+ C_pC_p'\left(\frac{1}{2K}\mathbb{E}\left((Y_t^2+1)^{\frac{p}{2}}\right) + \frac{K}{2}\mathbb{E}\left[\left(\int_0^t f^2(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right]\right)
\end{aligned}$$

$$\le C_p|x_0|^p + \frac{C_pC'_pC''_p}{2K}(\mathbb{E}(Y_t^p)+1) + C_p\mathbb{E}\left(\int_0^t f(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}} + \frac{C_pC'_pK}{2}\mathbb{E}\left[\left(\int_0^t f^2(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right].$$

Taking $K := C_pC'_pC''_p$, we have

$$\mathbb{E}(Y_t^p) \le 1 + 2C_p|x_0|^p + 2C_p\mathbb{E}\left(\int_0^t f(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}} + C_p^2C_p'^2C''_p\mathbb{E}\left[\left(\int_0^t f^2(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right].$$

If $f \in L^p_{\text{loc}}(\mathbb{R}_+)$, then

$$\mathbb{E}\left[\left(\int_0^t f^r(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right] \le C''_p\left(\left(\int_0^t f^r(s)ds\right)^{\frac{p}{2}} + \mathbb{E}\left[\left(\int_0^t f^r(s)Y_s^2ds\right)^{\frac{p}{2}}\right]\right) \le C''_p\left(\left(\int_0^t f^r(s)ds\right)^{\frac{p}{2}} + \int_0^t f^{\frac{rp}{2}}(s)\mathbb{E}(Y_s^p)ds\right)$$

where $r = 1, 2$. The last inequality holds because of Hölder's inequality. Thus,

$$\mathbb{E}(Y_t^p) \le A + B\int_0^t f^{\frac{p}{2}}(s)\mathbb{E}(Y_s^p)ds + C\int_0^t f^p(s)\mathbb{E}(Y_s^p)ds$$

where

$$A := 1 + 2C_p|x_0|^p + 2C_pC''_p\left(\int_0^t f(s)ds\right)^{\frac{p}{2}} + C_p^2{C'}_p^2{C''}_p^2\left(\int_0^t f^2(s)ds\right)^{\frac{p}{2}}$$

$$B := 2C_pC''_p, \qquad C := C_p^2{C'}_p^2{C''}_p^2.$$

By Gronwall's lemma, we have

$$\mathbb{E}(Y_t^p) \le A\exp\left\{B\int_0^t f(s)^{\frac{p}{2}}ds + C\int_0^t f^p(s)ds\right\}.$$

On the other hand, if $f \in L^{\frac{2p}{p-2}}_{\text{loc}}((R)_+)$, then

$$\mathbb{E}\left[\left(\int_0^t f^r(s)(|Y_s|^2+1)ds\right)^{\frac{p}{2}}\right] \le C''_p\left(\left(\int_0^t f^r(s)ds\right)^{\frac{p}{2}} + \mathbb{E}\left[\left(\int_0^t f^r(s)Y_s^2ds\right)^{\frac{p}{2}}\right]\right)$$

$$\le C''_p\left(\left(\int_0^t f^r(s)ds\right)^{\frac{p}{2}} + \left(\int_0^t f(s)^{\frac{rp}{p-2}}ds\right)^{\frac{p-2}{p}}\int_0^t \mathbb{E}(Y_s^p)ds\right)$$

where $r = 1, 2$. So we arrive at

$$\mathbb{E}(Y_t^p) \le A_1 + B_1\int_0^t \mathbb{E}(Y_s^p)ds \tag{5.5}$$

where

$$A_1 := A = 1 + 2C_p|x_0|^p + 2C_pC''_p\left(\int_0^t f(s)ds\right)^{\frac{p}{2}} + C_p^2C'^2_pC''^2_p\left(\int_0^t f^2(s)ds\right)^{\frac{p}{2}},$$

and

$$B_1 := 2C_pC''_p\left(\int_0^t f^{\frac{p}{p-2}}(s)ds\right)^{\frac{p-2}{p}} + C_p^2C'^2_pC''^2_p\left(\int_0^t f^{\frac{2p}{p-2}}(s)ds\right)^{\frac{p-2}{p}}.$$

By Gronwall's lemma, we have

$$\mathbb{E}(Y_t^p) \leq A_1e^{B_1t}. \quad \square$$

Example. We just consider the time-homogeneous case for simplicity. Suppose $d = 2, m = 1$. For any $r > 0$, define

$$\sigma(x) = |x|^r(-x_2, x_1)^T, \qquad b(x) = -|x|^{2r}x^T.$$

It is obvious that there exists a unique strong solution for the giving stochastic differential equation since the local Lipschitzian condition holds for both σ and b. On the other hand,

$$\Big(|\sigma(x)|^2 + 2\langle x, b(x)\rangle\Big) \vee |\sigma^T(x)x| = (|x|^{2r+2} - 2|x|^{2r+2}) \vee 0 = 0 \leq K(|x|^2 + 1).$$

So by Theorem 1.7, we can get the upper bound of the pth moment of the maximum process. But there is no $K > 0$ such that

$$|\sigma(x)|^2 = |x|^{2r+2} \leq K(|x|^2 + 1).$$

So we have given a sufficient condition for the boundedness of the pth moment of the maximum process, which is weaker than that of [4,10].

6. Non confluence of the solution

Proof of Theorem 1.9. Assume that $X_t(x_0)$ is a solution of Eq. (1.8) starting from x_0. Without loss of generality, we assume that $0 < \varepsilon < |x_0 - y_0| < c_0/2$, then define

$$\hat{\tau}_\varepsilon := \inf\{t > 0, |X_t(x_0) - X_t(y_0)| \leq \varepsilon\}, \qquad \hat{\tau} := \inf\{t > 0, X_t(x_0) = X_t(y_0)\}. \tag{6.1}$$

Clearly, $\hat{\tau}_\varepsilon \to \hat{\tau}$, as $\varepsilon \to 0$.

Let

$$\begin{aligned}\tau &:= \inf\left\{t > 0, |X_t(x_0) - X_t(y_0)| \geq \frac{3}{4}c_0\right\},\\ \tau_R &:= \inf\{t > 0, |X_t(x_0)| \vee |X_t(y_0)| \geq R\}\end{aligned} \tag{6.2}$$

and

$$Y_t := X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau_R}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau_R}(y_0), \qquad \xi_t := |Y_t|^2.$$

We take the test function

$$\Phi_\delta(x) := \exp\left(\int_x^{c_0} \frac{ds}{\gamma_R(s) + \delta}\right). \tag{6.3}$$

By Itô's formula, we have

$$\begin{aligned}\Phi_\delta(\xi_{t\wedge\tau}) &= \Phi_\delta(\xi_0) + \int_0^{t\wedge\tau} \Phi'_\delta(\xi_s)d\xi_s + \frac{1}{2}\int_0^{t\wedge\tau} \Phi''{}_\delta(\xi_s)d\langle\xi,\xi\rangle_s \\ &= \Phi_\delta(\xi_0) + M_t + \int_0^{t\wedge\tau} \Phi_\delta(\xi_s)\left(\frac{-1}{\gamma_R(\xi_s)+\delta}\right)(2\langle Y_s, f_s\rangle + \|e_s\|^2)ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau} \Phi_\delta(\xi_s)\frac{\gamma'_R(\xi_s)+1}{(\gamma_R(\xi_s)+\delta)^2}\cdot 4\xi_s\|e_s\|^2 ds\end{aligned}$$

where

$$\begin{aligned}M_t &:= 2\int_0^{t\wedge\tau} \Phi'_\delta(\xi_s)\langle Y_s, e_s dB_s\rangle, \\ e_s &:= \sigma(X_s(x_0)) - \sigma(X_s(y_0)), \\ h_s &:= b(X_s(x_0)) - b(X_s(y_0)),\end{aligned}$$

then taking expectation on both sides, we get

$$\begin{aligned}\mathbb{E}(\varphi_\delta(\xi_{t\wedge\tau})) &= \Phi_\delta(\xi_0) + \mathbb{E}\int_0^{t\wedge\tau} \Phi_\delta(\xi_s)\Big[\frac{2\xi_s(\gamma'_R(\xi_s)+1)\|e_s\|^2}{(\gamma_R(\xi_s)+\delta)^2} - \frac{2\langle Y_s, h_s\rangle + \|e_s\|^2}{\gamma_R(\xi_s)+\delta}\Big]ds \\ &\le \Phi_\delta(\xi_0) + \mathbb{E}\int_0^{t\wedge\tau} \Phi_\delta(\xi_s)\frac{2K\|e_s\|^2 - (2\langle Y_s, h_s\rangle + \|e_s\|^2)}{\gamma_R(\xi_s)+\delta}ds \\ &\le \Phi_\delta(\xi_0) + (2K-1)\mathbb{E}\int_0^{t\wedge\tau} \Phi_\delta(\xi_s)\frac{\|e_s\|^2 - \frac{2}{2K-1}\langle Y_s, h_s\rangle}{\gamma_R(\xi_s)+\delta}ds \\ &\le \Phi_\delta(\xi_0) + (2K-1)\mathbb{E}\int_0^t \Phi_\delta(\xi_s)ds.\end{aligned}$$

The last inequality holds because of condition (1.9). Then by Gronwall's lemma, we have

$$\mathbb{E}(\Phi_\delta(\xi_{t\wedge\tau})) \le \Phi_\delta(\xi_0)e^{(2K-1)t}.$$

Thus

$$\mathbb{E}(\Phi_\delta(|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(y_0)|^2)) \le \Phi_\delta(\xi_0)e^{(2K-1)t}. \tag{6.4}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}(\Phi_\delta(|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(y_0)|^2)) \\ &\quad\ge \mathbb{E}(\Phi_\delta(|X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(x_0) - X_{t\wedge\hat{\tau}_\varepsilon\wedge\tau\wedge\tau_R}(y_0)|^2)1_{\hat{\tau}_\varepsilon\le t\wedge\tau\wedge\tau_R}) \\ &\quad= \Phi_\delta(\varepsilon^2)P(\hat{\tau}_\varepsilon \le t\wedge\tau\wedge\tau_R).\end{aligned}$$

Thus,

$$P(\hat{\tau}_\varepsilon \le t\wedge\tau\wedge\tau_R) \le C_t\exp\left(-\int_{\varepsilon^2}^{\xi_0}\frac{ds}{\gamma(s)+\delta}\right) \tag{6.5}$$

where C_r is independent of R. Let $R\to\infty, \delta\to 0, \varepsilon\to 0$ subsequently. We have for any nonnegative t, $P(\hat{\tau}\le t\wedge\tau\wedge\zeta) = 0$. Let $t\to\infty$, it follows that $P(\hat{\tau}\le\tau\wedge\zeta) = 0$. Therefore, $\xi_\cdot$ is positive almost surely on the interval $[0, \tau\wedge\zeta]$. Now we define

$$T_0 := 0, \qquad T_1 := \tau\wedge\zeta, \qquad T_2 := \inf\left\{t > \tau\wedge\zeta, |X_t(x_0) - X_t(y_0)| \le \frac{c_0}{2}\right\}\wedge\zeta \tag{6.6}$$

and generally

$$\begin{aligned} T_{2n} &:= \inf\left\{t > T_{2n-1}, |X_t(x_0) - X_t(y_0)| \le \frac{c_0}{2}\right\} \wedge \zeta, \\ T_{2n+1} &:= \inf\left\{t > T_{2n}, |X_t(x_0) - X_t(y_0)| \ge \frac{3c_0}{4}\right\} \wedge \zeta. \end{aligned} \tag{6.7}$$

Similar to Fang and Zhang [3], it is clear that $T_n \to \zeta$, a.s. as $n \to \infty$. By definition $\xi_.$ is positive almost surely on the interval $[T_{2n-1}, T_{2n}]$. Since $X_t(x)$ is stochastic continuous with respect to the initial value x (see Theorem 3, [9]), according to Corollary 5.3, [3], the diffusion process $X_t(x)$ is Feller. By pathwise uniqueness of solutions, $\{X_t\}_{t\ge 0}$ has the strong Markovian property (see Mao [10]). Starting from T_{2n} and applying the same arguments as in the first part of the proof, one can show that $\xi_.$ is positive almost surely on the interval $[T_{2n}, T_{2n+1}]$, this ends the proof. □

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