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RESEARCH ARTICLE

Necessary and sufficient conditions for path-independence of Girsanov transformation for infinite-dimensional stochastic evolution equations

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Abstract Based on a recent result on linking stochastic differential equations on $\mathbb{R}^d$ to (finite-dimensional) Burger-KPZ type nonlinear parabolic partial differential equations, we utilize Galerkin type finite-dimensional approximations to characterize the path-independence of the density process of Girsanov transformation for the infinite-dimensional stochastic evolution equations. Our result provides a link of infinite-dimensional semi-linear stochastic differential equations to infinite-dimensional Burgers-KPZ type nonlinear parabolic partial differential equations. As an application, this characterization result is applied to stochastic heat equation in one space dimension over the unit interval.

Keywords Characterization theorem, Burgers-KPZ type nonlinear equations in infinite dimensions, infinite-dimensional semi-linear stochastic differential equations, Galerkin approximation, Girsanov transformation, stochastic heat equation, path-independence, Fréchet differentiation
MSC 60H15, 60H30, 35R60

1 Introduction and motivation

The aim of this paper is to characterize the path-independent property of the density process of Girsanov transformation for infinite-dimensional stochastic evolution equations (SEEs), a class of semi-linear stochastic differential equations (SDEs) in infinite dimensions. As a result, we establish a link between infinite-dimensional SEEs and the infinite-dimensional Burger-KPZ type nonlinear parabolic partial differential equations (PDEs), in the similar way as the

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link of finite-dimensional SDEs to the Burgers-KPZ type nonlinear parabolic PDEs carried out in [30]. Where in [30], a complete link of finite-dimensional Markovian type SDEs on $\mathbb{R}^d$ as well as on connected complete differential manifolds to Burgers-KPZ equations has been established, which gives a characterization of the path-dependence property of Girsanov transformation for finite-dimensional Markovian type SDEs. In [37], the simple case of one-dimensional SDEs (with more general conditions on the coefficients) was discussed in which a generalized Burgers equation has been linked to general Markovian type SDEs on $\mathbb{R}$.

We notice that in the derivations performed in [30,37], Itô formula plays a pivotal role. However, due to the complexity of infinite-dimensional stochastic differential equations, we cannot apply directly the infinite-dimensional Itô formula (see, e.g., [7, Theorem 4.17, Chapter 4] or [4, Theorem 4.1, Chapter 6]) to the infinite-dimensional SEEs concerned here. In this paper, our first task is to derive an Itô formula for certain regular functions of the solutions of our SEEs. We achieve this by utilizing Galerkin type finite-dimensional approximations of the infinite-dimensional SEEs. With the newly derive Itô formula in hand, we follow the same line of [30] to prove the necessary and sufficient conditions for the path-independent property of Girsanov transformation for the SEEs. We would like to point out that the Itô formula derived in this paper is also of interest in itself.

Before proceeding further, let us give some motivations for the present paper. There are two motivations towards our present study on infinite-dimensional stochastic differential equations. The first motivation comes from the mathematical study of economics and finance in conjunction with optimization problems. In recent years, due to the necessity of stochastic volatility as the measurement of uncertainty in modeling of financial markets, stochastic differential equations have received huge attention from both theoretical and practical aspects, cf. e.g., [17,22,23,28]. The primary point here is to model the price dynamics or the wealth growth by utilising SDEs, after having established a so-called real world probability space (e.g., the seminal paper [2] by Black and Scholes). To an equilibrium financial market, there must exist a so-called risk neutral probability measure which is absolutely continuous with the given real world probability measure and it is pivotal to determine the path-independent property for the associated density process defined by the Radon-Nikodym derivative [15,16]. It is often encountered in the economical and financial market models that one should consider agents in large scale that there are (at least) countably many stocks are treated together so that their pricing dynamics form an infinite-dimensional SDEs. From the view point of variational calculus, optimization problems—either in the pattern of maximizing the utility functions (and/or profits) or in the formulation of minimizing the cost functions (and/or risk factors)—are in fact linked with the path-independent property of the pricing trajectories, cf. e.g., [11,38]. Hence, characterizing the relevant path-independence of the SDEs in terms of (non-linear) PDEs would be interesting and useful.

Our second motivation is from the study of infinite interacting particle systems with stochastic dynamics. To illustrate this point, let us start with a bit more account of the appearance of classical nonlinear PDEs from mathematical physics. Since the pioneering work of Burgers in 1930s (cf. e.g., [3]), Burgers equation, the simplest nonlinear PDE,

$$\frac{\partial u(t,x)}{\partial t} + \lambda u(t,x)\frac{\partial u(t,x)}{\partial x} = \nu\frac{\partial^2 u(t,x)}{\partial x^2}, \quad (t,x) \in [0,+\infty) \times \mathbb{R},$$

has received a great attention both in mathematics and physics. Wherein the parameter $\lambda \in \mathbb{R}$ measures the strength of the nonlinearity, $\nu > 0$ stands for the viscosity, and the (linear) viscous dissipation term on the right-hand side of the equation is for the sake of softening shock wave phenomena.

Fix $d \in \mathbb{N}$, let $\mathbb{R}^d$ be the d-dimensional Euclidean space with the inner product being denoted by $\langle\cdot,\cdot\rangle$. The multidimensional analogue to the above Burgers equation is the so called higher-dimensional Burgers equation for a vorticity-free velocity field $\boldsymbol{u}\colon [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ (cf. e.g., [1]) which reads as follows:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \lambda(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u}$$

where ∇ stands for the space gradient, the dot product is

$$\boldsymbol{u}\cdot\nabla := \langle \boldsymbol{u}, \nabla\rangle,$$

and

$$\Delta := \nabla\cdot\nabla$$

is the Laplace operator on $\mathbb{R}^d$. Nowadays, Burgers equation is significant in the mathematical modeling of the large scale structure of the universe with complexity. The equation appears in many fields like aerodynamics, fluid dynamics (in particular, hydrodynamics), polymers and disordered systems, turbulence and propagation of chaos, as well as in shock wave and conservation laws—to name just a few. Among many interesting and important investigations, a breakthrough study has been made by three physicists Kardar, Parisi, and Zhang ([18]) for modeling the time evolution of the profile of a growing interface with the name of Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation describes the macroscopic properties of a wide variety of growth processes, such as growth by ballistic deposition and the Eden model (cf. [19]). For a more mathematical account of the KPZ equation, the reader is referred to [14]. The link of the KPZ equation to multidimensional Burgers equation can be explicated as follows. It is a natural assumption that the field $\boldsymbol{u}$ is often generated by a potential function (i.e., the profile) $u\colon [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$:

$$\boldsymbol{u}(t,\cdot) = -\nabla u(t,\cdot), \quad t \in [0,+\infty),$$

which, from the multidimensional Burgers equation, gives the following KPZ equation for the scalar function u:

$$\frac{\partial u(t,x)}{\partial t} = \nu\Delta u(t,x) + \frac{\lambda}{2}\,|\nabla u(t,x)|^2.$$

Clearly, the above KPZ equation describes the large-distance, long-time dynamics of the growth process specified by a single-valued height $u(t,x)$ on a substrate $x \in \mathbb{R}^d$. It reflects the competition between the surface tension smoothing forces $\nu\Delta u(t,x)$ and $\frac{\lambda}{2}|\nabla u(t,x)|^2$ (the nonlinear term of u represents the tendency for growth to occur preferentially in the local normal direction to the surface). On the other hand, when the diffusion coefficient $\sigma \equiv \sigma_0$, a constant, very interesting and new links of (stochastic) multidimensional Burgers' equations to (stochastic) Hamilton-Jacobi-Bellman (HJB) equations and the continuity equation have been thoroughly investigated by Truman and Zhao [31–33] (see also the early works [10,29] for a bridge between the diffusion equations and the Schrödinger equation, now called the Elworthy-Truman formula). In this content, Hamilton-Jacobi continuity equations provide the key to obtaining asymptotic expansions in ascending powers of σ_0 for solutions of the corresponding heat (and Schrödinger) wave functions in this setting.

Nowadays, because of their ubiquity, the Burgers equation, the KPZ equation, and the HJB equations (as well as any of their advances studies) maintain a very hot research topic on both theoretical and applied aspects in various fields involving disordered systems and non-equilibrium dynamics. The applied aspect links to many diverse areas ranging from physics, biochemistry, and climate and ocean studies (cf. e.g., [26,36]), to economical and financial studies (cf. [5,15,16,27,38]). There are many works in the literature devoted to analytic aspect of the equations themselves as well as to computational aspect (cf. e.g., [6,8,9,12,20,21,25] and references therein).

Go a step further, it is well known that a fairly rich class of the large scale systems is modeled by infinite-dimensional Markovian type semi-linear SDEs and the associated scaling limits of such systems are determined by KZP type nonlinear PDEs, cf. e.g., [18,26,36]. Thus, it is very natural to reveal an intrinsic link between the infinite-dimensional SDEs and nonlinear Burgers-KPZ type PDEs. In fact, our main result obtained in this paper does provide a direct link between infinite-dimensional stochastic equations and parabolic nonlinear PDEs in a persuasive manner, which shows that certain intrinsic properties of the (infinite) stochastic dynamical systems are indeed characterized by Burgers-KPZ type equations. This indicates in certain sense that the Burgers-KPZ type equations are ubiquitous for infinite systems of stochastically dynamical motions. Actually, this point inspired our investigation of the present work.

In the present paper, we will consider SDEs on a separable Hilbert space. To our aim, we notice that the methods employed in [37] and in [30] are Itô formula and Girsanov transformation. However, it is not straightforward to have Itô formula in infinite-dimensional so we have to use the finite-dimensional approximation approach here. Another important fact is that in [30,37], it is required that the diffusion coefficient of the (finite-dimensional) SDEs must be invertible so that the (unique) solution process of the initial value problem has a full support of the whole state space. While in infinite-dimensional situation, the SDEs are driven by cylindrical Brownian motion and the initial value problem is solvable in the integral formulation which requires that the diffusion coefficient as a family of operators (indexed by time parameter) must be Hilbert-Schmidt operators so that the relevant stochastic integrals against the cylindrical Brownian motion are well defined. As a matter of fact, Hilbert-Schmidt operators are degenerate which are no longer invertible operators. Apparently, this shows that it is not straight forward to lift the link for finite-dimensional SDEs to that for the infinite-dimensional SDEs. The current paper is devoted to study this problem for infinite-dimensional semi-linear SDEs involving a linear unbounded operator generating a contraction C_0-semigroup, where the stochastic equations are formulated in the mild manner and the Hilbert-Schmidt condition is posed for the convoluted diffusion coefficient with the associated C_0-seimgroup. By utilising Galerkin approximation, we can transfer our infinite-dimensional equation into a family of finite-dimensional equations and we can then derive an Itô formula for certain regular real-valued function of the solutions and further to establish a link of infinite-dimensional Burgers-KPZ nonlinear parabolic PDEs to infinite-dimensional Markovian type semi-linear SDEs. Extensions to more general infinite-dimensional spaces like Banach spaces, multi-Hilbertian spaces, as well as locally convex topological vector spaces are interesting and will be considered in the forthcoming works.

The rest of this paper is organized as follows. In the next section, we first give a brief account of the Galerkin type finite-dimensional approximations for SDEs on an (infinite-dimensional) separable Hilbert space H. Then we prove our main result on the characterization of path-independence of the Girsanov density of the SDEs. The final section is devoted to a consideration of parabolic stochastic partial differential equations as an example, where we demonstrate an application of our main result of Section 3.

2 Galerkin type approximation and Itô formula for infinite-dimensional SEEs

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a given filtered probability space satisfying the usual conditions that $(\Omega, \mathscr{F}, P)$ is a complete probability space and for each $t \geqslant 0$, $\mathscr{F}_t$ contains all P-null sets of $\mathscr{F}$ and

$$\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s = \mathscr{F}_t.$$

We use $\mathbb{E}$ to denote the expectation with respect to P.

Given a real separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$. Let $\{W_t\}_{t\geqslant 0}$ be a cylindrical Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ with the following expression:

$$W_t := W_t(\omega) := \sum_{i=1}^{+\infty} \beta_i(t,\omega) e_i, \quad \omega \in \Omega,\ t \in [0, +\infty),$$

where $\{\beta_i(t,\omega)\}_{i\geqslant 1}$ is a family of independent one-dimensional Brownian motions and $\{e_i\}_{i\geqslant 1}$ is a complete orthonormal basis for H which is fixed throughout the paper. We have

$$\mathbb{E}(\langle W_t, x\rangle_H, \langle W_s, y\rangle_H) = t \wedge s\langle x, y\rangle_H, \quad t, s \in [0, +\infty),\ x, y \in H.$$

Notice that the covariance operator of our cylindrical Brownian motion is just the identity operator I on H.

Let $L(H)$ be the collection of all bounded linear operators $L\colon H \to H$ equipped with the usual operator norm

$$\|L\| := \sup_{\|x\|=1} \|Lx\|_H.$$

Clearly, $(L(H), \|\cdot\|)$ is a Banach space.

Furthermore, we use $L_{\mathrm{HS}}(H)$ for the family of all Hilbert-Schmidt operators $L\colon H \to H$ endowed with the norm

$$\|L\|_{\mathrm{HS}} := \Big(\sum_{i=1}^{+\infty} \|Le_i\|_H^2\Big)^{1/2}.$$

Then $(L_{\mathrm{HS}}(H), \|\cdot\|_{\mathrm{HS}})$ is a Hilbert space.

Before proceed further, let us introduce the notion of Fréchet differentiation for infinite-dimensional spaces which is crucial in our paper. We state it in a little general form. Given two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we let $L(\mathbb{X}, \mathbb{Y})$ denote the totality of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. $L(\mathbb{X}, \mathbb{Y})$ is a Banach space endowed with the usual operator norm. A function $f\colon \mathbb{X} \to \mathbb{Y}$ is called *Fréchet differentiable* at $x \in \mathbb{X}$, if there exists a bounded linear operator $A_x\colon \mathbb{X} \to \mathbb{Y}$ such that

$$\lim_{\|h\|_{\mathbb{X}}\to 0} \frac{\|f(x+h) - f(x) - A_x h\|_{\mathbb{Y}}}{\|h\|_{\mathbb{X}}} = 0.$$

If the limit exists, we write $\nabla f(x) := A_x$ and call it the *Fréchet gradient* of f at x. A function $f\colon \mathbb{X} \to \mathbb{Y}$ that is Fréchet differentiable for any point $x \in \mathbb{X}$ is said to be C^1 if the function

$$\nabla f\colon x \in \mathbb{X} \mapsto \nabla f(x) \in L(\mathbb{X}, \mathbb{Y})$$

is continuous. Furthermore, $f\colon \mathbb{X} \to \mathbb{Y}$ is called a C^2 function if $\nabla f\colon \mathbb{X} \to L(\mathbb{X}, \mathbb{Y})$ is a C^1 function. Moreover, we let $\mathrm{Dom}(\nabla)$ denote the totality of all Fréchet differentiable functions $f\colon \mathbb{X} \to \mathbb{Y}$.

We would like to follow [35] to introduce the stochastic equation we are concerned. Let $(A, \mathscr{D}(A))$ be a linear, unbounded, negative definite, self-adjoint operator on H generating a contraction C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\geqslant 0}$. Let $L_A(H)$ be the totality of all densely defined closed linear operators $L\colon H \to H$ with domain $\mathrm{Dom}(L) \subset H$ such that for every $t > 0$, $\mathrm{e}^{tA}L$ extends to a unique Hilbert-Schmidt operator from H to H, while we use the same notation for the extension so $\mathrm{e}^{tA}L \in L_{\mathrm{HS}}(H)$. Namely,

$$L_A(H) := \{L\colon H \to H \mid \mathrm{e}^{tA}L \in L_{\mathrm{HS}}(H),\ \forall\, t > 0\}.$$

We endow $L_A(H)$ with the σ-algebra induced by the family

$$\{L \to \langle \mathrm{e}^{tA} Lx, y\rangle_H \mid t > 0,\ x, y \in H\}$$

from $\mathscr{B}(\mathbb{R})$ so that $L_A(H)$ is a measurable space.

We are concerned with the following initial value problem for a semi-linear stochastic differential equation on H:

$$\begin{cases} \mathrm{d}X_t = \{AX_t + b(t, X_t)\}\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t > 0, \\ X_0 = x \in H, \end{cases} \tag{2.1}$$

where

$$b\colon [0, +\infty) \times H \to H, \quad \sigma\colon [0, +\infty) \times H \to L_A(H),$$

are measurable mappings. In this paper, we require the two coefficients fulfill further that

$$b\colon [0, +\infty) \times H \to H, \quad (t, x) \in [0, +\infty) \times H \mapsto \mathrm{e}^{tA}\sigma(t, x) \in L_{\mathrm{HS}}(H),$$

are C^1 with respect to the first variable and C^2 with respect to the second variable, respectively. Here, we would like to point out that one should interpret $([0, +\infty), |\cdot|)$ and $(\mathbb{R}, |\cdot|)$ as Banach spaces and the differentiation with respect to $t \in [0, +\infty)$ or for $\mathbb{R}$-valued functions on any Banach space follows from above description. Throughout this paper, we shall assume the following two conditions.

(H1) Assume that $-A$ has discrete spectrum with eigenvalues

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$$

counting multiplicities such that

$$\sum_{j=1}^{+\infty} \frac{1}{\lambda_j} < +\infty.$$

We let $\{e_j\}_{j\in\mathbb{N}}$ be the corresponding eigen-basis of $-A$ throughout the paper.

(H2) There exist a constant $\varepsilon \in (0, 1)$ and an increasing function $L\colon [0, +\infty) \to (0, +\infty)$ such that

$$\sup_{t\in[0,T]} \left\{ \|b(t, 0)\|_H^2 + \int_0^t \|\mathrm{e}^{(t-s)A}\sigma(s, 0)\|_{\mathrm{HS}}^2 s^{-\varepsilon} \mathrm{d}s \right\} < +\infty, \quad \forall\, T > 0,$$

and

$$\|b(t, x) - b(t, y)\|_H + \|\mathrm{e}^{tA}(\sigma(t, x) - \sigma(t, y))\|_{\mathrm{HS}} \leqslant L(t)\|x - y\|_H, \qquad \forall\, t \geqslant 0,\ \forall\, x, y \in H.$$

Remark 2.1 Under assumption (H1), it clear that the space $L_A(H)$ allows to have invertible operators from H to H, such as the identity operator.

It is well known, e.g., from [4,7] and most recently [35], that (H1) and (H2) imply the existence and uniqueness of the mild solution to (2.1), that is, for any $x \in H$, there exists a unique H-valued $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted continuous process X_t, $t \geqslant 0$, such that $\mathbb{P}$-a.s.

$$X_t = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A} b(s, X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\sigma(s, X_s)\mathrm{d}W_s, \quad t \geqslant 0. \tag{2.2}$$

Moreover, we have

$$\mathbb{E}\big(\sup_{t\in[0,T]} \|X_t\|_H^2\big) < +\infty, \quad \forall\, T > 0.$$

For our purpose, we need a finite-dimensional approximation to (2.1) so that we can link the characterization theorem for finite-dimensional SDEs obtained in [30,37] to the present infinite-dimensional problem (2.1). To be more precise, we want to set a Galerkin approximation to (2.1), which is classical and efficient to get existence and uniqueness results for infinite-dimensional equations (see, e.g., [4, Chapter 6]). So let us follow [35] to set up the Galerkin approximation for (2.1). We notice that our assumption (H1) indicates that the operator A satisfies the coercivity condition and the monotonicity condition in [4, p. 178]. For simplicity, we assume that

$$\sigma\colon [0, +\infty) \times H \to L_A(H)$$

is diagonal with respect to the eigen-basis $\{e_i\}_{i\geqslant 1}$.

For any $n \geqslant 1$, let

$$\pi_n\colon H \to H_n := \mathrm{span}\{e_1, \dots, e_n\}$$

be the (orthogonal) projection operator, that is,

$$\pi_n x := \sum_{i=1}^n \langle x, e_i\rangle_H e_i, \quad x \in H.$$

We note that the project operator π_n commutes with the semigroup e^{tA}, $t \geqslant 0$. Furthermore, we let

$$A_n := A\,|_{H_n}, \quad b_n := \pi_n b, \quad \sigma_n := \pi_n\sigma.$$

We consider the following stochastic differential equation in H_n:

$$\begin{cases} \mathrm{d}X_t^n = \{A_n X_t^n + b_n(t, X_t^n)\}\mathrm{d}t + \sigma_n(t, X_t^n)\mathrm{d}W_t, \\ X^n(0) = \pi_n x. \end{cases} \tag{2.3}$$

As illustrated in [35], assumption (H2) implies that the coefficients b_n and σ_n fulfill the usuall growth and Lipschitz conditions so that there exists a unique strong solution $X_t^n \in H_n$, $t \in [0, +\infty)$, to (2.3). Furthermore, by [35, Theorem 3.1.2], one has

$$\lim_{n\to+\infty} \mathbb{E}\|X_t^n - X_t\|_H^2 = 0, \quad t \geqslant 0. \tag{2.4}$$

Finally in this section, we want to establish an Itô formula for real-valued functions of the solution X_t, $t \geqslant 0$, of (2.1). Here, we notice that the diffusion coefficient σ in (2.1) is not Hilbert-Schmidt, and thus, infinite-dimensional Itô formula given, e.g., in [7, Theorem 4.17, Chapter 4] or [4, Theorem 4.1, Chapter 6], cannot apply to the real-valued functions of the solutions of (2.1). As a matter of fact, it seems to us that so far, there is no Itô formula for functions of solutions of infinite-dimensional semi-linear SDEs containing our SEEs (2.1) which are only solved with mild solutions. Hence, it would be interesting to have Itô formula for this situation. In what follows, we will use the Galerkin finite-dimensional approximations (2.3) associated with our initial value problem (2.1) to complete our task.

Proposition 2.1 *Assume* (H1), (H2), *and let* $v\colon [0, +\infty) \times H \to \mathbb{R}$ *be in* $C_b^{1,2}([0, +\infty) \times H)$ *such that* $[\nabla v(t, x)] \in \mathrm{Dom}(A)$ *for any* $(t, x) \in [0, +\infty) \times H$ *and* $\|A\nabla v(t, \cdot)\|_H$ *is bounded locally and uniformly for* $t \in [0, +\infty)$. *Then we have*

$$\begin{aligned} v(t, X_t) = v(0, X_0) &+ \int_0^t \langle \sigma^*(s, X_s)\nabla v(s, X_s), \mathrm{d}W_s\rangle_H + \int_0^t \Big[\frac{\partial v(s, X_s)}{\partial s} \\ &+ \langle \nabla v(s, X_s), b(s, X_s)\rangle_H + \langle A\nabla v(s, X_s), X_s\rangle_H\Big]\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}[(\sigma\sigma^*)(s, X_s)\nabla^2 v(s, X_s)]\mathrm{d}s. \end{aligned} \tag{2.5}$$

Proof We start with (2.3) where we have derived an approximation sequence $\{X_t^n,\, t \geqslant 0\}_{n\in\mathbb{N}}$ for the solution $\{X_t,\, t \geqslant 0\}$ of (2.1), that is, $\{X_t^n,\, t \geqslant 0\}$ is indeed an n-dimensional (semimartingale) process (i.e., the process X_t^n, $t \geqslant 0$, lives on the finite-dimensional space H_n for each n, respectively) and the sequence $\{X_t^n,\, t \geqslant 0\}_{n\in\mathbb{N}}$ converges to $\{X_t,\, t \geqslant 0\}$ in $\|\cdot\|_H^2$. Furthermore, it is clear that for each $t \geqslant 0$,

$$\|\cdot\|_H - \lim_{n\to+\infty} v(t, \pi_n x) = v(t, x),$$

and therefore,

$$\lim_{n\to+\infty} v(t, X_t^n) = v(t, X_t).$$

Hence, we turn to the expression $v(t, X_t^n)$, which, for each fixed $n \in \mathbb{N}$, is a real-valued function of the finite-dimensional process X_t^n, $t \geqslant 0$, and we can apply the Itô formula to $v(t, X_t^n)$. To be more precise, viewing the expression $v(t, X_t^n)$ as the composition of the deterministic $C^{1,2}$-function

$$v\colon [0, +\infty) \times H_n \to \mathbb{R}$$

with the finite dimensional, continuous semi-martingale X_t^n with expression (i.e., from our previous (2.3))

$$\mathrm{d}X_t^n = [A_n X_t^n + b_n(t, X_t^n)]\mathrm{d}t + \sigma_n(t, X_t^n)\mathrm{d}W_t, \quad t \geqslant 0,$$

we can apply the Itô formula (see, e.g., [7, Theorem 4.17, Chapter 4] or [4, Theorem 4.1, Chapter 6]) to $v(t, X_t^n)$ with notice that here our W_t is (standard) cylindrical Brownian motion (with mean zero and covariance given by identity), which yields the following derivation:

$$\begin{aligned} v(t, X_t^n) &= v(0, \pi_n X_0) + \int_0^t \langle \nabla_n v(s, X_s^n), \sigma_n(s, X_s^n)\mathrm{d}W_s\rangle_H \\ &\quad + \int_0^t \Big[\frac{\partial v(s, X_s^n)}{\partial s} + \langle \nabla_n v(s, X_s^n), A_n X_s^n + b_n(s, X_s^n)\rangle_H\Big]\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \mathrm{Tr}[\nabla_n^2 v(s, X_s^n)(\sigma_n(s, X_s^n)(Id)^{1/2})(\sigma_n(s, X_s^n)(Id)^{1/2})^*]\mathrm{d}s \\ &= v(0, \pi_n X_0) + \int_0^t \langle \sigma_n^*(s, X_s^n)\nabla_n v(s, X_s^n), \mathrm{d}W_s\rangle_H \\ &\quad + \int_0^t \Big[\frac{\partial v(s, X_s^n)}{\partial s} + \langle \nabla_n v(s, X_s^n), A_n X_s^n + b_n(s, X_s^n)\rangle_H\Big]\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \mathrm{Tr}[(\sigma_n\sigma_n^*)(s, X_s^n)\nabla_n^2 v(s, X_s^n)]\mathrm{d}s, \end{aligned} \tag{2.6}$$

where

$$\nabla_n := \sum_{j=1}^n \nabla_{e_j} e_j, \quad \nabla_{e_j} := \langle \nabla, e_j\rangle_H, \ 1 \leqslant j \leqslant n,$$

and we have used the following identity in the above derivation:

$$\langle \nabla_n v(s, X_s^n), \sigma_n(s, X_s^n)\mathrm{d}W_s\rangle_H = \langle \sigma_n^*(s, X_s^n)\nabla_n v(s, X_s^n), \mathrm{d}W_t\rangle_H .$$

By our assumptions on v and that the operator A is self-adjoint, we have

$$\begin{aligned} \lim_{n\to+\infty}\int_0^t \langle [\sigma_n^*\nabla_n v](s, X_s^n), \mathrm{d}W_s\rangle_H &= \int_0^t \langle [\sigma^*\nabla v](s, X_s), \mathrm{d}W_s\rangle_H, \\ \lim_{n\to+\infty}\int_0^t \langle \nabla_n v(s, X_s^n), A_n X_s^n\rangle_H \mathrm{d}s &= \int_0^t \langle A\nabla v(s, X_s), X_s\rangle_H \mathrm{d}s, \\ \lim_{n\to+\infty}\int_0^t \langle \nabla_n v(s, X_s^n), b_n(s, X_s^n)\rangle_H \mathrm{d}s &= \int_0^t \langle \nabla v(s, X_s), b(s, X_s)\rangle_H \mathrm{d}s, \\ \lim_{n\to+\infty}\int_0^t \mathrm{Tr}[(\sigma_n\sigma_n^*)(s, X_s^n)\nabla_n^2 v(s, X_s^n)]\mathrm{d}s &= \int_0^t \mathrm{Tr}[(\sigma\sigma^*)(s, X_s)\nabla^2 v(s, X_s)]\mathrm{d}s, \end{aligned}$$

and

$$\lim_{n\to+\infty}\frac{\partial v(s, X_s^n)}{\partial s} = \frac{\partial v(s, X_s)}{\partial s}.$$

Therefore, letting $n \to +\infty$, we end up with (2.5) from (2.6). □

3 Characterization of path-independent property

We start with recalling the Girsanov theorem in infinite-dimensions (see [7, 10.2.1]). Notice that the covariance operator of our cylindrical Brownian motion $\{W_t\}_{t\geqslant 0}$ is the identity operator I on $(H, \|\cdot\|_H)$. One can then determine the infinite-dimensional Brownian motion on Itô's universal Wiener space with the reproducing kernel space H, cf. e.g., [13].

Next, assume that

$$\gamma\colon [0,+\infty)\times H\to H$$

is measurable such that for every $T>0$ (note that here T could take to be $+\infty$ as well)

$$\mathbb{E}\bigg(\exp\bigg[\frac{1}{2}\int_0^T\|\gamma(s,X_s)\|_H^2\mathrm{d}s\bigg]\bigg)<+\infty, \tag{3.1}$$

which is known as the Novikov condition. Then the process

$$\widetilde{W}_t := W_t-\int_0^t\gamma(s,X_s)\mathrm{d}s,\quad t\in[0,T],$$

is a cylindrical Brownian motion (i.e., having the identity operator I on H as its covariance operator) with respect to $\{\mathscr{F}_t\}_{t\in[0,T]}$ on the probability space $(\Omega,\mathscr{F},\widetilde{P}_T)$, where $\widetilde{P}_T$ is defined via the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\widetilde{P}_T}{\mathrm{d}P}(\omega):=\exp\bigg(\int_0^T\langle\gamma(s,X_s(\omega)),\mathrm{d}W_s(\omega)\rangle_H-\frac{1}{2}\int_0^T\|\gamma(s,X_s(\omega))\|_H^2\mathrm{d}s\bigg).$$

We refer the reader, e.g., to [7, Proposition 10.17] for an alternative sufficient condition instead of (3.1). The relation between W_t and $\widetilde{W}_t$ in the stochastic differentiation form is

$$\mathrm{d}\widetilde{W}_t=\mathrm{d}W_t-\gamma(t,X_t)\mathrm{d}t,$$

from which, in terms of the new cylindrical Brownian motion $\widetilde{W}_t$, the SDE in (2.1) reads

$$\mathrm{d}X_t=\{AX_t+b(t,X_t)+\sigma(t,X_t)\gamma(t,X_t)\}\mathrm{d}t+\sigma(t,X_t)\mathrm{d}\widetilde{W}_t,\quad t\in(0,T].$$

Furthermore, if $\sigma(t,x)$ is invertible for each $(t,x)\in[0,+\infty)\times H$, then we can specify

$$\gamma(t,x):=-\sigma^{-1}(t,x)b(t,x),\quad (t,x)\in[0,+\infty)\times H.$$

Thus, if the coefficients b and σ in our equation (2.1) fulfill the following condition:

$$\mathbb{E}\bigg(\exp\bigg[\frac{1}{2}\int_0^T\|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2\mathrm{d}s\bigg]\bigg)<+\infty,\quad\forall\, T>0,$$

or equivalently,

$$\mathbb{E}\bigg(\exp\bigg[-\int_0^T \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}W_s\rangle_H - \frac{1}{2}\int_0^T \|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2 \mathrm{d}s \bigg]\bigg) = 1, \quad T>0,$$

then our SDE in (2.1) becomes simply

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \sigma(t,X_t)\mathrm{d}\widetilde{W}_t, \quad t\in(0,T].$$

From now on, we assume further the following condition throughout the rest of the paper.

(H3) The operator $\sigma(t,x)$ is invertible for each $(t,x)\in[0,+\infty)\times H$ and the two coefficients b,σ in (2.1) fulfill

$$\mathbb{E}\bigg(\exp\bigg\{ \frac{1}{2}\int_0^T \|\sigma^{-1}(t,X_t)b(t,X_t)\|_H^2 \mathrm{d}t \bigg\}\bigg) < +\infty, \quad \forall\, T>0.$$

To summarize the above discussion, we conclude that under (H1)–(H3), the *Girsanov density*

$$\begin{aligned}\frac{\mathrm{d}\widetilde{P}_t}{\mathrm{d}P}(\omega) := \exp\bigg\{ &-\int_0^t \langle \sigma^{-1}(s,X_s(\omega))b(s,X_s(\omega)), \mathrm{d}W_s(\omega)\rangle_H \\ &-\frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s(\omega))b(s,X_s(\omega))\|_H^2 \mathrm{d}s\bigg\}, \quad t\geqslant 0, \end{aligned} \tag{3.2}$$

is a well-defined process for the SDE in (2.1).

Remark 3.1 Since the diffusion coefficient σ is invertible and the cylindrical Brownian motion W_t, $t\geqslant 0$, is determined by the inner product of H (the covariance operator is the identity operator on H), the solution X_t, $t>0$, of (2.1) lives on a large space containing the whole space H, namely, the support of the solution X_t, $t>0$, of (2.1) covers the whole space H. Moreover, it was proved in [35] that the Harnack inequality holds for the Markov semigroup of the solution X_t, $t>0$, which indicates that the law of the H-valued random variable X_t is fully supported by H for each $t>0$.

We are now in the position to state our main result. It gives necessary and sufficient conditions of the path-independence of the Girsanov density process for (infinite-dimensional) SDEs on separable Hilbert spaces.

Theorem 3.1 *Assume* (H1)–(H3), *and let*

$$v\colon [0,+\infty)\times H\to\mathbb{R}$$

be in $C_b^{1,2}([0,+\infty)\times H)$ *such that*

$$[\nabla v(t,x)]\in \mathrm{Dom}(A), \quad \forall\ (t,x)\in[0,+\infty)\times H, \tag{3.3}$$

and $\|A\nabla v(t,\cdot)\|_H$ *is bounded locally and uniformly for* $t \in [0,+\infty)$. *Then the Girsanov density* (3.2) *for* (2.1) *fulfills the following path-independent property*:

$$\frac{\mathrm{d}\widetilde{P}_t}{\mathrm{d}P} = \exp(v(0,X_0) - v(t,X_t), \quad t \geqslant 0, \tag{3.4}$$

if and only if v *satisfies*

$$\frac{\partial v(t,x)}{\partial t} = -\frac{1}{2}\left\{\mathrm{Tr}[(\sigma\sigma^*)\nabla^2 v](t,x) + \|\sigma^*\nabla v\|_H^2(t,x)\right\} - \langle x, A\nabla v(t,x)\rangle_H \tag{3.5}$$

and

$$b(t,x) = [(\sigma\sigma^*)\nabla v](t,x), \quad \forall\ (t,x) \in (0,+\infty)\times H. \tag{3.6}$$

Proof We start with the proof of the *sufficiency*. Namely, we assume that there is a $v \in C_b^{1,2}([0,+\infty)\times H)$ satisfying (3.3) and $\|A\nabla v(t,\cdot)\|_H$ is bounded locally and uniformly for $t \in [0,+\infty)$ such that (3.5) and (3.6) hold. We want to verify (3.4). We note that showing (3.4) is equivalent to verifying

$$\begin{aligned} v(t,X_t) = v(0,X_0) &+ \frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2 \mathrm{d}s \\ &+ \int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}W_s\rangle_H. \end{aligned} \tag{3.7}$$

By the Itô formula (2.5), we have

$$\begin{aligned} v(t,X_t) = v(0,X_0) &+ \int_0^t \langle \sigma^*(s,X_s)\nabla v(s,X_s), \mathrm{d}W_s\rangle_H \\ &+ \int_0^t \Big[\frac{\partial v(s,X_s)}{\partial s} + \langle \nabla v(s,X_s), b(s,X_s)\rangle_H + \langle A\nabla v(s,X_s), X_s\rangle_H\Big]\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}[(\sigma\sigma^*)(s,X_s)\nabla^2 v(s,X_s)]\mathrm{d}s. \end{aligned}$$

Now, from our assumption (3.6), we get

$$\begin{aligned} \|\sigma^*\nabla v\|_H^2(t,x) &= \langle [\sigma^*\nabla v](t,x), [\sigma^*\nabla v](t,x)\rangle_H \\ &= \langle [(\sigma\sigma^*)\nabla v](t,x), \nabla v(t,x)\rangle_H \\ &= \langle b(t,x), \nabla v(t,x)\rangle_H, \quad (t,x)\in[0,+\infty)\times H, \end{aligned} \tag{3.8}$$

and

$$\|\sigma^*\nabla v\|_H^2(t,x) = \|\sigma^{-1}b\|_H^2(t,x). \tag{3.9}$$

Putting the identity (3.8) into (3.5) yields

$$\frac{\partial v(t,x)}{\partial t} = -\frac{1}{2}\mathrm{Tr}[(\sigma\sigma^*)\nabla^2 v](t,x) - \frac{1}{2}\langle b(t,x), \nabla v(t,x)\rangle_H - \langle x, A\nabla v(t,x)\rangle_H,$$

and furthermore, along the path X_s, $s \geqslant 0$, we have

$$\frac{\partial v(s,X_s)}{\partial s} = -\frac{1}{2}\,\mathrm{Tr}[(\sigma\sigma^*)\nabla^2 v](s,X_s) - \frac{1}{2}\,\langle b(s,X_s), \nabla v(s,X_s)\rangle_H \\ - \langle X_s, A\nabla v(s,X_s)\rangle_H. \tag{3.10}$$

and by (3.9), we get

$$\|\sigma^*\nabla v\|_H^2(t,X_t) = \|\sigma^{-1}b\|_H^2(t,X_t). \tag{3.11}$$

Putting (3.10) and (3.11) into (2.5), we obtain

$$\begin{aligned} v(t,X_t) &= v(0,X_0) + \int_0^t \langle \sigma^*(s,X_s)\nabla v(s,X_s), \mathrm{d}W_s\rangle_H \\ &\quad + \frac{1}{2}\int_0^t \langle \nabla v(s,X_s), b(s,X_s)\rangle_H \mathrm{d}s \\ &= v(0,X_0) + \frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2 \mathrm{d}s \\ &\quad + \int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}W_s\rangle_H, \end{aligned}$$

which is exact (3.7), which we wanted for the *sufficiency*.

Now, let us show the *necessity*. That is, we assume that there is a $v \in C_b^{1,2}([0,+\infty)\times H)$ satisfying (3.3) and $\|A\nabla v(t,\cdot)\|_H$ is bounded locally and uniformly for $t\in[0,+\infty)$ such that (3.7) holds (which is equivalent to that (3.4) holds). We aim to verify that (3.5) and (3.6) hold. In fact, viewing $v(t,X_t)$ as a real-valued semimartingale, by the uniqueness of Doob-Meyer's decomposition theorem, we can compare the two expressions for $v(t,X_t)$ in (3.7) and in the Itô formula (2.5), respectively, to get for each $t>0$,

$$\int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s), \mathrm{d}W_s\rangle_H = \int_0^t \langle \sigma^*(s,X_s)\nabla v(s,X_s), \mathrm{d}W_s\rangle_H \tag{3.12}$$

and

$$\begin{aligned} &\frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s)b(s,X_s)\|_H^2 \mathrm{d}s \\ &\quad = \int_0^t \Big[\frac{\partial v(s,X_s)}{\partial s} + \langle \nabla v(s,X_s), b(s,X_s)\rangle_H + \langle A\nabla v(s,X_s), X_s\rangle_H\Big]\mathrm{d}s \\ &\quad\quad + \frac{1}{2}\int_0^t \mathrm{Tr}[(\sigma\sigma^*)(s,X_s)\nabla^2 v(s,X_s)]\mathrm{d}s. \end{aligned} \tag{3.13}$$

From (3.12), we get

$$\int_0^t \langle \sigma^{-1}(s,X_s)b(s,X_s) - \sigma^*(s,X_s)\nabla v(s,X_s), \mathrm{d}W_s\rangle_H = 0, \quad \forall\, t>0,$$

which implies

$$\sigma^{-1}(s, X_s)b(s, X_s) = \sigma^*(s, X_s)\nabla v(s, X_s), \quad \forall\ s > 0,$$

since W_s, $s \geqslant 0$, is non-degenerate. Moreover, due to the fact that the support of the solution X_s, $s \geqslant 0$, covers the whole space H, we can conclude that

$$b(s, x) = \sigma(s, x)\sigma^*(s, x)\nabla v(s, x), \quad \forall\ (s, x) \in (0, +\infty) \times H. \tag{3.14}$$

Hence, we obtain (3.6). Next, putting (3.14) into (3.13), we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|\sigma^*(s, X_s)\nabla v(s, X_s)\|_H^2 \mathrm{d}s \\
&\quad = \int_0^t \Big[\frac{\partial v(s, X_s)}{\partial s} + \|\sigma^*(s, X_s)\nabla v(s, X_s)\|_H^2 + \langle A\nabla v(s, X_s), X_s\rangle_H\Big]\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \mathrm{Tr}[(\sigma\sigma^*)(s, X_s)\nabla^2 v(s, X_s)]\mathrm{d}s
\end{aligned}$$

since

$$\begin{aligned}
\langle \nabla v(s, X_s), b(s, X_s)\rangle_H &= \langle \nabla v(s, X_s), \sigma(s, X_s)\sigma^*(s, X_s)\nabla v(s, X_s)\rangle_H \\
&= \langle \sigma^*(s, X_s)\nabla v(s, X_s), \sigma^*(s, X_s)\nabla v(s, X_s)\rangle_H \\
&= \|\sigma^*(s, X_s)\nabla v(s, X_s)\|_H^2.
\end{aligned}$$

Thus, the following holds for all $t > 0$:

$$\begin{aligned}
\int_0^t \Big[\frac{\partial v(s, X_s)}{\partial s} + \frac{1}{2}\,\mathrm{Tr}[(\sigma\sigma^*\nabla^2)(s, X_s)] - \frac{1}{2}\,\|\sigma^*\nabla v(s, X_s)\|_H^2 \\
+ \langle A\nabla v(s, X_s), X_s\rangle_H\Big]\mathrm{d}s = 0,
\end{aligned}$$

which further yields

$$\frac{\partial v(s, X_s)}{\partial s} + \frac{1}{2}\,\mathrm{Tr}[(\sigma\sigma^*\nabla^2)(s, X_s)] - \frac{1}{2}\,\|\sigma^*\nabla v(s, X_s)\|_H^2 + \langle A\nabla v(s, X_s), X_s\rangle_H = 0$$

for all $s > 0$. Using again the fact that the support of the solution X_s, $s \geqslant 0$, covers the whole space H, we derive

$$\frac{\partial v(s, x)}{\partial s} + \frac{1}{2}\,\mathrm{Tr}[(\sigma\sigma^*\nabla^2)(s, x)] - \frac{1}{2}\,\|\sigma^*\nabla v(s, x)\|_H^2 + \langle A\nabla v(s.x), x\rangle_H = 0,$$

from which we obtain (3.5). This completes the proof of the *necessity*. We are done. □

We end up this section with two examples on a link from finite-dimensional SDEs to infinite-dimensional SDEs. To illustrate our examples in its simplest manner, let us assume that the diffusion coefficient operator $\sigma(t, x)$ is diagonal for each $(t,x) \in [0,+\infty) \times H$ with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, i.e.,

$$\sigma(t,x) = \operatorname{diag}(\sigma_i(t,x))_{i\in\mathbb{N}}$$

with $(\sigma_i(t,x))_{i\in\mathbb{N}}$ being, for each $(t,x) \in [0,+\infty) \times H$, an (infinite-dimensional) $\mathbb{R}^{+\infty}$-vector with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$.

Example 3.1 Let $n \in \mathbb{N}$ be fixed. For equation (3.5), we let $v(t,x)$ depend on the first n components of $x = (x_1, x_2, ..., x_n, ...) \in H$, that is,

$$v(t,x) := v(t, x_1, x_2, \ldots, x_n).$$

Clearly, this is then similar to the case of finite-dimensions situation considered in [30]. In fact, for $x \in H_n$, recall that $Ae_i = -\lambda_i e_i$ (see our assumption (H1)). Then we have

$$-\langle Ax, \nabla v(t,x)\rangle_H = \sum_{i=1}^{n} \lambda_i \frac{\partial v(t,x)}{\partial x_i}, \quad \forall\ (t,x) \in [0,+\infty) \times H_n. \tag{3.15}$$

Furthermore, since for $i > n$,

$$\frac{\partial v(t,x)}{\partial x_i} = \frac{\partial^2 v(t,x)}{\partial x_i^2} = 0,$$

we have, for $b = (b_n^1, b_n^2, ..., b_n^n, b_n^{n+1}, ...)$,

$$b^i(t,x) = \sigma_i(t,x)^2 \frac{\partial v(t,x)}{\partial x_i} \tag{3.16}$$

and

$$\operatorname{Tr}[(\sigma\sigma^*)\nabla^2 v](t,x) = \sum_{i=1}^{n} \sigma_i(t,x)^2 \frac{\partial^2 v(t,x)}{\partial x_i^2}, \quad x \in H_n. \tag{3.17}$$

Similarly, set

$$\sigma_n(t,x) = \operatorname{diag}((\sigma_n)_i(t,x)).$$

Then we have

$$\|\sigma_n^* \nabla v\|_H^2(t,x) = \sum_{i=1}^{n} \sigma_i^2(t,x) \Big(\frac{\partial v(t,x)}{\partial x_i}\Big)^2, \quad x \in H_n. \tag{3.18}$$

Combining (3.15)–(3.18), equation (3.5) for such special $v\colon [t,x) \times H^n \to \mathbb{R}$ then becomes

$$\frac{\partial v(t,x)}{\partial t} = -\frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(t,x) \Big\{ \frac{\partial^2 v(t,x)}{\partial x_i^2} + \Big(\frac{\partial v(t,x)}{\partial x_i}\Big)^2 \Big\} + \sum_{i=1}^{n} \lambda_i \frac{\partial v(t,x)}{\partial x_i}. \tag{3.19}$$

Moreover, letting $n \to +\infty$, we arrive the straightforward infinite-dimensional analogy of the Burgers-KPZ equation:

$$\frac{\partial v(t,x)}{\partial t} = -\frac{1}{2}\sum_{i=1}^{+\infty}\sigma_i^2(t,x)\Big\{\frac{\partial^2 v(t,x)}{\partial x_i^2} + \Big(\frac{\partial v(t,x)}{\partial x_i}\Big)^2\Big\} + \sum_{i=1}^{+\infty}\lambda_i\frac{\partial v(t,x)}{\partial x_i},$$
$$(t,x) \in [0,+\infty) \times H. \quad (3.20)$$

The link to the Burgers-KPZ equation obtained in [30] (as well as from the one-dimensional equation derived in [37]) is that at there W_t, $t \geqslant 0$, is the standard Brownian motion with mean zero and covariance being the identity matrix, while as here our W_t, $t \geqslant 0$, is the (standard) cylindrical Brownian motion whose finite-dimensional projects are just the standard Brownian motion with mean zero and identity matrix covariance. It would be interest to study infinite-dimensional SDEs driven by cylindrical Wiener processes with more general covariance operators Q in the framework of abstract Wiener spaces (cf. e.g., [4,7,24,35]). We will consider this problem in our forthcoming work.

Example 3.2 Let $R\colon H \to L_{\rm HS}(H)$ be a fixed operator. For $m \in \mathbb{N}$, let $R_m\colon [0,+\infty) \times H \mapsto R_m(t,x) \in L_{\rm HS}(H)$ be bounded, i.e.,

$$\sup_{(t,x)\in[0,+\infty)\times H} \|R_m(t,x)\|_{\rm HS} < +\infty.$$

We set for the $\sigma(t,x) \in L_{\rm HS}(H)$, $(t,x) \in [0,+\infty) \times H$, in our Theorem 2.1 as the following perturbation:

$$\sigma^m(t,x) := R + 2^{-m}R_m(t,x), \quad (t,x) \in [0,+\infty) \times H.$$

That is, under the given orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$, the dependence of $\sigma^m(t,x)$ on the m-th coordinate $x_m = \langle x, e_m\rangle$ becomes weaker and weaker as m goes to sufficiently large and

$$\lim_{m\to+\infty} \|\sigma^m(t,x) - R\|_{\rm HS} = 0.$$

Next, we denote

$$(\sigma_i^m(t,x))_{i\in\mathbb{N}} := \mathrm{diag}((\sigma^m(t,x))_{\mathbb{N}\times\mathbb{N}}) = \mathrm{diag}((R + 2^{-m}R_m(t,x))_{\mathbb{N}\times\mathbb{N}}),$$

i.e., the real-valued coordinate

$$\sigma_i^m(t,x) := (R + 2^{-m}R_m(t,x))_{ii}$$

with

$$\lim_{m\to+\infty} \sigma_i^m(t,x) = \langle Re_i, Re_i\rangle =: r_i \in \mathbb{R}.$$

Then equation (3.5) in Theorem 2.1 for the $v^m(t,x)$ reads

$$\frac{\partial v^m(t,x)}{\partial t} = -\frac{1}{2}\left\{\sum_{i=1}^{+\infty}((R+2^{-m}R_m(t,x))_{ii})^2\frac{\partial^2 v^m(t,x)}{\partial x_i^2}\right.$$
$$\left.+\sum_{i=1}^{+\infty}((R+2^{-i}R_i(t,x))_{ii})^2(t,x)\Big(\frac{\partial v^m(t,x)}{\partial x_i}\Big)^2\right\}+\sum_{i=1}^{+\infty}\lambda_i\frac{\partial v^m(t,x)}{\partial x_i}.$$

As $m\to+\infty$, we have the real-valued (point wise) limit

$$v(t,x) := \lim_{m\to+\infty} v^m(t,x),$$

which satisfies the following infinite-dimensional Burger-KPZ equation (with constant coefficients)

$$\frac{\partial v(t,x)}{\partial t} = -\frac{1}{2}\left\{\sum_{i=1}^{+\infty} r_i^2\frac{\partial^2 v(t,x)}{\partial x_i^2}+\sum_{i=1}^{+\infty}\Big(r_i^2\frac{\partial v(t,x)}{\partial x_i}\Big)^2\right\}+\sum_{i=1}^{+\infty}\lambda_i\frac{\partial v(t,x)}{\partial x_i}.$$

4 Application to stochastic heat equation

In this final section, we will consider an example of space time inhomogeneous parabolic SPDEs. Here, we take for granted the familiarity with the introductory account on SPDEs presented, e.g., in [4,34] or [24]. Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},P)$ be the given probability set-up as in Section 2. We consider the following problem for a stochastic heat equation driven by space-time white noise on the bounded space domain $[0,1]\subset\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \phi(t,x,u(t,x)) + \psi(t,x,u(t,x)))\dfrac{\partial^2 B(t,x)}{\partial t\partial x}, \\ \qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in(0,1), \\ u(t,0)=u(t,1)=0, \qquad\qquad\qquad t\geqslant 0, \\ u(0,x)=u_0(x), \qquad\qquad\qquad x\in[0,1], \end{cases} \tag{4.1}$$

where

$$\phi,\psi\colon [0,+\infty)\times[0,1]\times\mathbb{R}\to\mathbb{R}$$

are space-time inhomogeneous coefficients, and $\{B(t,x)\}_{(t,x)\in[0,+\infty)\times[0,1]}$ is a Brownian sheet on $[0,+\infty)\times[0,1]$. The heuristic derivative $\frac{\partial^2 B}{\partial t\partial x}$ is interpreted as the space-time white noise, which can be made rigorously, e.g., by utilizing generalized functions ([34]).

It is sometimes also convenient, cf. e.g., [4,7], to link the space-time white noise to an $L^2([0,1])$-valued cylindrical Brownian motion on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},P)$. Let us elucidate this point a bit here. First, let $B(\mathrm{d}s,\mathrm{d}z)$ be such that

$$B(t,x)=\int_0^t\int_0^x B(\mathrm{d}s,\mathrm{d}z),\quad \forall\ (t,x)\in[0,+\infty)\times[0,1].$$

Next, it is clearly that the Hilbert space $H := L^2([0,1])$ is separable. Let $A := \frac{\partial^2}{\partial x^2}$ be the one-dimensional Laplace operator on $[0,1]$ with Dirichlet boundary condition so its domain is

$$\mathrm{Dom}(A) = H^2([0,1]) \cap H_0^1([0,1]),$$

where $H^k([0,1])$ stands for the L^2-Sobolev space of order k and $H_0^k([0,1])$ is the closure of $C_0^\infty([0,1])$ in H^k for $k=1,2$. Denote by $\{\theta_n\}_{n\in\mathbb{N}}$ the complete orthonormal basis in H consisting of the eigenfunctions of A, which is given by

$$\theta_n(x) := \sqrt{2}\,\sin(n\pi x), \quad n \in \mathbb{N},$$

so that

$$A\theta_n(x) = -n^2\pi^2\theta_n(x).$$

Then

$$W_t := \sum_{n=1}^{+\infty}\Big(\int_0^t\int_0^1 \theta_n(x)B(\mathrm{d}s,\mathrm{d}x)\Big)\theta_n$$

defines A-cylindrical Brownian motion on H (i.e., with covariance $Q = A$).

Problem (4.1) is solvable with a unique strong solution under the following assumption on the coefficients (cf. e.g., [4, Chapter 6] or [7, Chapter 7]).

I) The coefficients ϕ, ψ are Lipschitz continuous with linear growth in the sense that there exists $C > 0$ such that

$$|\phi(t,x,z)|^2 + |\psi(t,x,z)|^2 \leqslant C(1+|z|^2)$$

and

$$|\phi(t,x,z_1) - \phi(t,x,z_2)|^2 + |\psi(t,x,z_1) - \psi(t,x,z_2)|^2 \leqslant C|z_1 - z_2|^2$$

hold for all $(t,x) \in [0,+\infty) \times [0,1]$ and for arbitrarily given $z, z_1, z_2 \in \mathbb{R}$.

II) The diffusion coefficient ψ is uniformly bounded from below and above, i.e., there exist positive constants C_1 and C_2 such that for all $z \in \mathbb{R}$,

$$C_1 \leqslant |\psi(t,x,z)| \leqslant C_2$$

holds for all $(t,x,z) \in [0,+\infty) \times [0,1] \times \mathbb{R}$.

If I) is fulfilled, one can show that (4.1) has a unique (global) mild solution $u(t,x)$, $t \geqslant 0$, $x \in [0,1]$, i.e., u satisfies the following mild equation:

$$\begin{aligned} u(t,x) = &\int_0^1 p(t,x,y)u_0(y)\mathrm{d}y + \int_0^t\int_0^1 p(t-s,x,y)\phi(s,y,u(s,y))\mathrm{d}s\mathrm{d}y \\ &+ \int_0^t\int_0^1 p(t-s,x,y)\psi(s,y,u(s,y))B(\mathrm{d}s,\mathrm{d}y), \end{aligned}$$

with the property that

$$u(t) := u(t,\cdot)\colon [0,1] \to \mathbb{R} \in L^2([0,1]) = H,$$

$$\mathbb{E}\big(\sup_{t\in[0,+\infty)}\|u(t)\|_H^2\big)<+\infty,$$

where $p(t,x,y)$ stands for the fundamental solution of $\frac{\partial}{\partial t}-A$.

Now, we want to reformulate equation (4.1) in its abstract form. To this end, we set

$$X_t:=u(t,\cdot),\quad b(t,X_t):=\phi(t,\cdot,u(t,\cdot)),\quad \sigma(t,X_t)(v):=\psi(t,\cdot,u(t,\cdot))v(t,\cdot) \tag{4.2}$$

for $u(t,\cdot),v(t,\cdot)\in H$ for any $t\geqslant 0$. Then, equation (4.1) becomes

$$\begin{cases} \mathrm{d}X_t=\{AX_t+b(t,X_t)\}\mathrm{d}t+\sigma(t,X_t)\mathrm{d}W_t, & t\geqslant 0,\\ X_0=u_0\in H, \end{cases} \tag{4.3}$$

which is exactly in the form of (2.1). Since our one-dimensional Dirichlet Laplacian A fulfills assumption (H1), Theorem 2.1 goes to verbatim for sufficiency of the path-independent property of the Girsanov density process for (4.3), which can be further transferred to (4.1) via the links (4.2) in a straightforward manner.

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