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On the Smoothness of the Noncommutative Pillow and Quantum Teardrops*

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Abstract. Recent results by Krähmer [*Israel J. Math.* **189** (2012), 237–266] on smoothness of Hopf–Galois extensions and by Liu [arXiv:1304.7117] on smoothness of generalized Weyl algebras are used to prove that the coordinate algebras of the noncommutative pillow orbifold [*Internat. J. Math.* **2** (1991), 139–166], quantum teardrops $\mathcal{O}(\mathbb{WP}_q(1,l))$ [*Comm. Math. Phys.* **316** (2012), 151–170], quantum lens spaces $\mathcal{O}(L_q(l;1,l))$ [*Pacific J. Math.* **211** (2003), 249–263], the quantum Seifert manifold $\mathcal{O}(\Sigma_q^3)$ [*J. Geom. Phys.* **62** (2012), 1097–1107], quantum real weighted projective planes $\mathcal{O}(\mathbb{RP}_q^2(l;\pm))$ [*PoS Proc. Sci.* (2012), PoS(CORFU2011), 055, 10 pages] and quantum Seifert lens spaces $\mathcal{O}(\Sigma_q^3(l;-))$ [*Axioms* **1** (2012), 201–225] are homologically smooth in the sense that as their own bimodules they admit finitely generated projective resolutions of finite length.

Key words: smooth algebra; generalized Weyl algebra; strongly graded algebra; noncommutative pillow; quantum teardrop; quantum lens space; quantum real weighted projective plane

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Dedicated to Marc Rieffel on the occasion of his 75th birthday.

1 Introduction and tools

This note is intended to illustrate the claim that (often) a q-deformation of a non-smooth classical variety or an orbifold produces an algebra which has properties of the coordinate algebra of a non-commutative smooth variety or manifold. More precisely, we say that an algebra B (over an algebraically closed field $\mathbb{K}$) is *homologically smooth* or simply *smooth* provided that as a B-bimodule it has a finitely generated projective resolution of finite length; see [20, Erratum]. We prove that several classes of examples of coordinate algebras of q-deformed orbifolds are homologically smooth. To achieve this aim we use techniques developed in [13], which are applicable to principal comodule algebras [6], and those of [14], which are applicable to generalized Weyl algebras [1]. We summarize these presently.

In [13, Corollary 6] Krähmer gives a criterion for smoothness of quantum homogeneous spaces, which through an immediate extension to a more general class of Hopf–Galois extensions and then specification to strongly group-graded algebras, provides us with a tool for showing the smoothness of the noncommutative pillow algebra studied in [2], the quantum lens space algebras $\mathcal{O}(L_q(l;1,l))$ introduced in [12], the quantum teardrop algebras [5], the coordinate algebras of quantum real weighted projective planes $\mathcal{O}(\mathbb{RP}_q(l;-))$ defined in [3], the quantum Seifert manifold $\mathcal{O}(\Sigma_q^3)$ [7] and the quantum Seifert lens spaces $\mathcal{O}(\Sigma_q^3(l;-))$ [4]. More specifically, let G be a group (with the neutral element e). A G-graded algebra $A = \oplus_{g\in G} A_g$ is said to be *strongly*

*This paper is a contribution to the Special Issue on Noncommutative Geometry and Quantum Groups in honor of Marc A. Rieffel. The full collection is available at http://www.emis.de/journals/SIGMA/Rieffel.html *graded* if, for all $g, h \in G$, $A_g A_h = A_{gh}$. For such algebras, Krähmer's criterion for smoothness takes the following form.

Criterion 1. *Let A be a strongly G-graded algebra and set $B = A_e$. If the enveloping algebra $\mathcal{E}A := A \otimes A^{\mathrm{op}}$ of A is left Noetherian of finite global dimension, then the enveloping algebra of B is also left Noetherian of finite global dimension. Consequently, B is a homologically smooth algebra.*

Although [13, Corollary 6] is formulated for quantum homogeneous spaces obtained via surjective homomorphism of Hopf algebras with a cosemisimple codomain, the proof extends immediately to all faithfully flat Hopf–Galois extensions $B \subseteq A$ or principal comodule algebras such that B is a direct summand of A as a B-bimodule. In the case of G-graded algebras, $B = A_e$ is a direct summand of A as a B-bimodule, and [16, Proposition AI.3.6] ensures that a strongly G-graded algebra is a principal $\mathbb{K}G$-comodule algebra. In particular A is projective and faithfully flat as a left and right B-module. The way Criterion 1 is stated indicates its iterative nature which is implicit in the proof of [13, Corollary 6]. Note in passing that the assumption about the dimension of $\mathcal{E}A$ is not necessary to conclude that $\mathcal{E}B$ is left Noetherian.

An effective way of checking whether a graded G-algebra A is strongly graded is described in [16, Section AI.3.2]:

Lemma 1. *$A = \oplus_{g\in G} A_g$ is strongly graded if and only if there exists a function $\omega : G \to A \otimes A$ such that*

(a) for all $g \in G$, $\omega(g) \in A_{g^{-1}} \otimes A_g$,

(b) for all $g \in G$, $\mu \circ \omega(g) = 1$, where μ is the multiplication map of A.

Furthermore, if G is a cyclic group, then conditions (a) and (b) need only be checked for a generator g of G. If $\omega(g) = \sum_i \omega_i' \otimes \omega_i''$, satisfies (a) and (b), then ω is defined by setting $\omega(e) = 1 \otimes 1$ and

$$\omega\left(g^{n+1}\right) = \sum_i \omega_i' \omega(g^n) \omega_i'', \qquad \text{for all} \quad n > 0.$$

A function ω satisfying conditions (a) and (b) in Lemma 1 is a predecessor of a *strong connection form* on a principal comodule algebra; see [6, 8, 11].

Let R be an algebra, let p be an element of the centre of R and let π be an automorphism of R. The (*degree-one*) *generalized Weyl algebra* $R(\pi, p)$ is the extension of R by generators x_+, x_- subject to the relations, for all $r \in R$,

$$x_- x_+ = p, \qquad x_+ x_- = \pi(p), \qquad x_\pm r = \pi^{\pm 1}(r) x_\pm;$$

see [1]. In [14, Theorem 4.5] Liu gives the following criterion of smoothness of a generalized Weyl algebra over the polynomial algebra.

Criterion 2. *Let $R = \mathbb{K}[a]$ be a polynomial algebra and an automorphism $\pi : \mathbb{K}[a] \to \mathbb{K}[a]$ be determined by $\pi(a) = \kappa a + \chi$. Then the generalized Weyl algebra $R(\pi, p)$ is homologically smooth with homological dimension 2 if and only if the polynomial $p \in \mathbb{K}[a]$ has no multiple roots.*

Furthermore, Liu proves that if the smoothness Criterion 2 is satisfied, then $A = R(\pi, p)$ is a *twisted Calabi–Yau algebra* of dimension 2 with the Nakayama (twisting) automorphism $\nu : A \to A$ given by $\nu(x_\pm) = \kappa^{\pm 1} x_\pm$ and $\nu(a) = a$. This means that the Hochschild cohomology of A with values in its enveloping algebra is trivial in all degrees except degree 2, where it is equal to A with the A-bimodule structure $a \cdot b \cdot a' = ab\nu(a')$.

The reader should observe that, except for some special cases, the algebras described hereinafter are not smooth whenever the deformation parameters λ or q are equal to 1. By noting this they will fully grasp the main message of this note, namely that deformation may (and quite often does) result in smoothing classically singular objects.

2 Results

Throughout we work with associative complex $*$-algebras with identity. We write $\mathcal{E}A$ for the enveloping algebra $A \otimes A^{\mathrm{op}}$ of A. We often use the *q-Pochhammer symbol* which, for an indeterminate x and a complex number q, is defined as

$$(x;q)_n := \prod_{m=0}^{n-1} \left(1 - q^m x\right).$$

2.1 The noncommutative pillow

Let $\lambda = e^{2\pi i\theta}$, where θ is an irrational number. Recall that the coordinate $*$-algebra $\mathcal{O}(\mathbb{T}^2_\theta)$ of the *noncommutative torus* is generated by unitaries U, V, such that $UV = \lambda VU$; see [18]. The involutive algebra automorphism given by

$$\sigma : \ \mathcal{O}\big(\mathbb{T}^2_\theta\big) \to \mathcal{O}\big(\mathbb{T}^2_\theta\big), \qquad U \mapsto U^*, \qquad V \mapsto V^*,$$

makes $\mathcal{O}(\mathbb{T}^2_\theta)$ into a $\mathbb{Z}_2$-graded algebra. The fixed point (or degree-zero) subalgebra $\mathcal{O}(P_\theta)$ is generated by $U + U^*$ and $V + V^*$. It has been introduced and studied from a topological point of view in [2] (see also [9, Section 3.7]) as a deformation of the coordinate algebra of the *pillow orbifold* [19, Chapter 13] (an orbifold rather than manifold since, classically, the $\mathbb{Z}_2$-action determined by the automorphism σ is not free).

Theorem 1. *$\mathcal{O}(\mathbb{T}^2_\theta)$ is a strongly $\mathbb{Z}_2$-graded algebra and the noncommutative pillow algebra $\mathcal{O}(P_\theta)$ is homologically smooth.*

Proof. Set

$$\hat{x} = U - U^*, \qquad \hat{y} = V - V^*, \qquad \hat{z} = UV^* - U^*V.$$

Note that $\sigma(\hat{x}) = -\hat{x}$, $\sigma(\hat{y}) = -\hat{y}$ and $\sigma(\hat{z}) = -\hat{z}$, i.e. all these are homogeneous elements of $\mathcal{O}(\mathbb{T}^2_\theta)$ with the $\mathbb{Z}_2$-degree 1. A straightforward calculation affirms that these elements satisfy the following relation

$$\hat{x}^2 + \hat{y}^2 - \bar{\lambda}\hat{z}^2 - \hat{x}z\hat{y} = 2\big(\bar{\lambda}^2 - 1\big),$$

where $z = UV^* + U^*V \in \mathcal{O}(P_\theta)$. Therefore, the mapping $\omega : \mathbb{Z}_2 \to \mathcal{O}(\mathbb{T}^2_\theta) \otimes \mathcal{O}(\mathbb{T}^2_\theta)$, defined as $\omega(0) = 1 \otimes 1$ and

$$\omega(1) = \frac{1}{2\big(\bar{\lambda}^2 - 1\big)} \left(\hat{x} \otimes \hat{x} + \hat{y} \otimes \hat{y} - \bar{\lambda}\hat{z} \otimes \hat{z} - \hat{x}z \otimes \hat{y}\right),$$

satisfies conditions (a) and (b) in Lemma 1, and $\mathcal{O}(\mathbb{T}^2_\theta)$ is a strongly $\mathbb{Z}_2$-graded algebra.

Both $\mathcal{O}(\mathbb{T}^2_\theta)$ and $\mathcal{E}\mathcal{O}(\mathbb{T}^2_\theta)$ can be understood as iterated skew Laurent polynomial rings and hence they are left Noetherian by [15, Theorem 1.4.5]. Furthermore, the global dimension of the latter is less than or equal to 4 by [15, Theorem 7.5.3]. Therefore, the noncommutative pillow algebra $\mathcal{O}(P_\theta)$ is homologically smooth by Criterion 1. ■

In short, Theorem 1 means that for the irrational θ (or, more generally, for any real $\theta \in (0,1) \setminus \{\frac{1}{2}\}$) the action of $\mathbb{Z}_2$ on the noncommutative torus is free despite the fact that the corresponding action on the classical level is not free. The set of fixed points corresponds to a manifold rather than an orbifold.

2.2 Quantum teardrops and lens spaces

Here we deal with three (classes of) complex $*$-algebras given in terms of generators and relations.

The coordinate algebra of the quantum three-sphere, $\mathcal{O}(S^3_q)$, is generated by α and β such that

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \qquad \alpha\beta^* = q\beta^*\alpha, \qquad \beta\beta^* = \beta^*\beta,\\ &\alpha\alpha^* = \alpha^*\alpha + \left(q^{-2}-1\right)\beta\beta^*, \qquad \alpha\alpha^* + \beta\beta^* = 1, \end{aligned} \tag{1}$$

where $q \in (0,1)$; see [21]. For any positive integer l, the coordinate algebra of the quantum lens space $\mathcal{O}(L_q(l;1,l))$ is a $*$-algebra generated by c and d subject to the following relations:

$$cd = q^l dc, \qquad cd^* = q^l d^* c, \qquad dd^* = d^* d, \qquad cc^* = \left(dd^*; q^2\right)_l, \qquad c^*c = \left(q^{-2}dd^*; q^{-2}\right)_l,$$

see [12]. Finally, for a positive integer l, the coordinate algebra of the quantum teardrop $\mathcal{O}(\mathbb{WP}_q(1,l))$ is the $*$-algebra generated by a and b subject to the following relations

$$a^* = a, \qquad ab = q^{-2l} ba, \qquad bb^* = q^{2l} a\left(a; q^2\right)_l, \qquad b^*b = a\left(q^{-2}a; q^{-2}\right)_l;$$

see [5]. These algebras form a tower $\mathcal{O}(\mathbb{WP}_q(1,l)) \hookrightarrow \mathcal{O}(L_q(l;1,l)) \hookrightarrow \mathcal{O}(S^3_q)$ with embeddings $a \mapsto dd^*$, $b \mapsto cd$ and $c \mapsto \alpha^l$, $d \mapsto \beta$, respectively. We thus can and will think of $\mathcal{O}(\mathbb{WP}_q(1,l))$ and $\mathcal{O}(L_q(l;1,l))$ as subalgebras of $\mathcal{O}(S^3_q)$. $\mathcal{O}(S^3_q)$ is a $\mathbb{Z}_l$-graded algebra with grading given by $\deg(\alpha) = 1$, $\deg(\alpha^*) = l-1$, $\deg(\beta) = \deg(\beta^*) = 0$, and the above embedding identifies the degree-zero part of $\mathcal{O}(S^3_q)$ with $\mathcal{O}(L_q(l;1,l))$. By [5, Theorem 3.3], the latter is a strongly $\mathbb{Z}$-graded algebra with grading provided by $\deg(c) = \deg(d^*) = 1$, $\deg(c^*) = \deg(d) = -1$ and with the degree-zero part isomorphic to $\mathcal{O}(\mathbb{WP}_q(1,l))$.

That $\mathcal{O}(\mathbb{WP}_q(1,l))$ is homologically smooth can be argued as follows. $\mathcal{O}(S^3_q)$ is a coordinate algebra of the quantum group SU(2) and thus $\mathcal{E}\mathcal{O}(S^3_q)$ is left Noetherian and has a finite global dimension; see [10]. Hence, if it were a strongly $\mathbb{Z}_l$-graded algebra, then $\mathcal{E}\mathcal{O}(L_q(l;1,l))$ would be left Noetherian and would have a finite global dimension (so, in particular $\mathcal{O}(L_q(l;1,l))$ would be homologically smooth) by Criterion 1. Since, in turn $\mathcal{O}(L_q(l;1,l))$ is a strongly graded algebra, Criterion 1 would imply smoothness of the teardrop algebra $\mathcal{O}(\mathbb{WP}_q(1,l))$. This arguing leads to:

Theorem 2. *$\mathcal{O}(S^3_q)$ is a strongly $\mathbb{Z}_l$-graded algebra with the degree-zero subalgebra isomorphic to $\mathcal{O}(L_q(l;1,l))$. Consequently, both $\mathcal{O}(L_q(l;1,l))$ and $\mathcal{O}(\mathbb{WP}_q(1,l))$ are homologically smooth algebras.*

Proof. The case $l = 1$ is dealt with in [13], the remaining cases follow from

Lemma 2. *For all integers $l > 1$, there exist elements $\omega(1) \in \mathcal{O}(S^3_q)_{l-1} \otimes \mathcal{O}(S^3_q)_1$ such that $\mu(\omega(1)) = 1$.*

Proof. Set:

$$\omega(1) = x_1 \alpha^{l-1} \otimes \alpha^{*l-1} + \sum_{p=1}^{l-1} y_p a^{p-1} \alpha^* \otimes \alpha,$$

where $x_1, y_1, \dots, y_{l-1} \in \mathbb{C}$ are to be determined and $a = \beta\beta^* = dd^*$. Then $\omega(1) \in \mathcal{O}(S^3_q)_{l-1} \otimes \mathcal{O}(S^3_q)_1$. Using (1) one finds that

$$\alpha^m \alpha^{*m} = \left(a; q^2\right)_m =: \sum_{p=0}^{m} c^m_p a^p, \tag{2}$$

where c_p^m are the appropriate q-binomial coefficients (defined by the second equality in (2)). In view of (1), the condition $\mu(\omega(1)) = 1$ leads to

$$x_1 \sum_{p=0}^{l-1} c_p^{l-1} a^p + \sum_{p=0}^{l-2} y_{p+1} a^p - q^{-2} \sum_{p=1}^{l-1} y_p a^p = 1.$$

By comparing the powers of a, this is converted into an inhomogeneous system of l equations with unknown $x_1, y_1, \ldots, y_{l-1}$, whose determinant is

$$\Delta_{l:1} = \begin{vmatrix} c_0^{l-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1^{l-1} & -q^{-2} & 1 & 0 & \cdots & 0 & 0 \\ c_2^{l-1} & 0 & -q^{-2} & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \\ c_{l-2}^{l-1} & 0 & 0 & 0 & \cdots & -q^{-2} & 1 \\ c_2^{l-1} & 0 & 0 & 0 & \cdots & 0 & -q^{-2} \end{vmatrix}.$$

$\Delta_{l:1}$ can be evaluated by expanding by the first column to give

$$\Delta_{l:1} = (-1)^{l-1} \big(q^{-2(l-1)} c_0^{l-1} + q^{-2(l-2)} c_1^{l-1} + \cdots + c_{l-1}^{l-1} \big) = \big(-q^2 \big)^{l-1} \prod_{p=1}^{l-1} \big(1 - q^{2p} \big) \neq 0.$$

The final equality follows from the definition of the q-binomial coefficients (2). This proves the existence of $\omega(1)$ as stated. ■

Since 1 is a generator of $\mathbb{Z}_l$, Lemma 2 ensures the existence of mappings $\omega : \mathbb{Z}_l \to \mathcal{O}(S_q^3) \otimes \mathcal{O}(S_q^3)$ that satisfy conditions (a) and (b) in Lemma 1. Hence $\mathcal{O}(S_q^3)$ is a strongly $\mathbb{Z}_l$-graded algebra, and the second assertion of the theorem follows by Criterion 1. ■

Therefore, for any $q \in (0,1)$ the action of $\mathbb{Z}_l$ on the quantum three-sphere described above is free despite the fact that the corresponding action on the classical level is not free (unless, obviously, $l = 1$). The fixed points correspond to a manifold rather than an orbifold.

2.3 Odd weighted real projective planes $\mathbb{RP}_q^2(l; -)$ and quantum Seifert lens spaces

For a positive integer l, the coordinate $*$-algebra $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ of the odd quantum weighted real projective plane is generated by a, b, c_- which satisfy the relations:

$$\begin{gathered} a = a^*, \qquad ab = q^{-2l} ba, \qquad ac_- = q^{-4l} c_- a, \qquad b^2 = q^{3l} ac_-, \qquad bc_- = q^{-2l} c_- b, \\ bb^* = q^{2l} a \big(a; q^2\big)_l, \qquad b^* b = a \big(q^{-2} a; q^{-2}\big)_l, \qquad b^* c_- = q^{-l} \big(q^{-2} a; q^{-2}\big)_l b, \\ c_- b^* = q^l b \big(a; q^2\big)_l, \qquad c_- c_-^* = \big(a; q^2\big)_{2l}, \qquad c_-^* c_- = \big(q^{-2} a; q^{-2}\big)_{2l}, \end{gathered}$$

see [3]. To prove homological smoothness of $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ we make use of Criterion 1 and build a tower of strongly graded algebras with $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ as the foundation.

The coordinate $*$-algebra of the quantum Seifert manifold $\mathcal{O}(\Sigma_q^3)$ is generated by a central unitary ξ and elements ζ_0, ζ_1 such that

$$\zeta_0 \zeta_1 = q \zeta_1 \zeta_0, \qquad \zeta_0 \zeta_0^* = \zeta_0^* \zeta_0 + \big(q^{-2} - 1\big) \zeta_1^2 \xi, \qquad \zeta_0 \zeta_0^* + \zeta_1^2 \xi = 1, \qquad \zeta_1^* = \zeta_1 \xi. \tag{3}$$

It has been shown in [7, proof of Proposition 5.2] that $\mathcal{O}(\Sigma_q^3)$ can be understood as the degree-zero part of a $\mathbb{Z}_2$-grading of $\mathcal{O}(S_q^2)[u,u^{-1}]$, where $u^{-1}=u^*$ and $\mathcal{O}(S_q^2)$ is the coordinate $*$-algebra of the equatorial Podleś sphere [17], generated by z_0 and self-adjoint z_1 such that

$$z_0z_1 = qz_1z_0, \qquad z_0z_0^* = z_0^*z_0 + \left(q^{-2}-1\right)z_1^2, \qquad z_0z_0^* + z_1^2 = 1. \tag{4}$$

The $\mathbb{Z}_2$-grading of $\mathcal{O}(S_q^2)[u,u^{-1}]$ is determined by setting, for all monomials w of degree k in the basis $\{z_0^r z_1^s, z_0^{*r}z_1^s \,|\, r,s\in\mathbb{N}\}$ of $\mathcal{O}(S_q^2)$, $\deg(wu^m) = (k+m) \mod 2$. $\mathcal{O}(\Sigma_q^3)$ can be identified with the degree-zero part of $\mathcal{O}(S_q^2)[u,u^{-1}]$ by $*$-embedding $\zeta_i\mapsto z_iu$, $\xi\mapsto u^{-2}$. Thanks to the last of equations (4), the function

$$\omega:\mathbb{Z}_2\to\mathcal{O}\left(S_q^2\right)\left[u,u^{-1}\right]\otimes\mathcal{O}\left(S_q^2\right)\left[u,u^{-1}\right], \qquad 0\mapsto 1\otimes 1, \qquad 1\mapsto z_0\otimes z_0^* + z_1\otimes z_1,$$

satisfies conditions (a) and (b) in Lemma 1, hence $\mathcal{O}(S_q^2)[u,u^{-1}]$ is a strongly $\mathbb{Z}_2$-graded algebra. Since $\mathcal{E}\mathcal{O}(S_q^3)$ is Noetherian, and there is a surjective $*$-algebra homomorphism $\mathcal{O}(S_q^3)\to\mathcal{O}(S_q^2)$, $\alpha\mapsto z_0$, $\beta\mapsto z_1^*$, both $\mathcal{E}\mathcal{O}(S_q^2)$ and $\mathcal{E}\mathcal{O}(S_q^2)[u,u^{-1}]$ and hence also $\mathcal{E}\mathcal{O}(\Sigma_q^3)$ are Noetherian.

As explained in [4], $\mathcal{O}(\Sigma_q^3)$ is a $\mathbb{Z}_l$-graded algebra with grading given by

$$\deg(\zeta_0)=1, \qquad \deg(\zeta_0^*)=l-1, \qquad \deg(\zeta_1)=\deg(\xi)=0.$$

The degree-zero part of $\mathcal{O}(\Sigma_q^3)$ is isomorphic to the $*$-algebra $\mathcal{O}(\Sigma_q^3(l;-))$ generated by x, y and central unitary z subject to the following relations

$$y^*=yz, \qquad xy=q^lyx, \qquad xx^*=\left(y^2z;q^2\right)_l, \qquad x^*x=\left(q^{-2}y^2z;q^{-2}\right)_l.$$

The embedding of $\mathcal{O}(\Sigma_q^3(l;-))$ into $\mathcal{O}(\Sigma_q^3)$ is given by $x\mapsto\zeta_0^l$, $y\mapsto\zeta_1$ and $z\mapsto\xi$. The similarity of relations (3) and (1) leads immediately to equations (2) with α replaced by ζ_0 and $a=\zeta_1^2\xi$. This allows one to use the same arguments as in Lemma 2 to prove that there exist $x_1,y_1,\ldots,y_{l-1}\in\mathbb{C}$ such that

$$\omega(1) = x_1\zeta_0^{l-1}\otimes\zeta_0^{*l-1} + \sum_{i=1}^{l-1} y_ia^{i-1}\zeta_0^*\otimes\zeta_0 \in \mathcal{O}\left(\Sigma_q^3\right)_{l-1}\otimes\mathcal{O}\left(\Sigma_q^3\right)_l,$$

has the required property $\mu(\omega(1))=1$. Therefore, $\mathcal{O}(\Sigma_q^3)$ is a strongly graded $\mathbb{Z}_l$-algebra.

Finally, it is proven in [4] that $\mathcal{O}(\Sigma_q^3(l;-))$ is a strongly $\mathbb{Z}$-graded algebra with grading given by $\deg(x)=\deg(y)=1$, $\deg(x^*)=-1$ and $\deg(z)=-2$. The degree-zero subalgebra of $\mathcal{O}(\Sigma_q^3(l;-))$ can be identified with the coordinate algebra of weighted real projective plane $\mathcal{O}(\mathbb{RP}_q^2(l;-))$ via the map $a\mapsto y^2z$, $b\mapsto xyz$ and $c_-\mapsto x^2z$.

Summarizing, we have presented in this section a tower of $*$-algebras

$$\mathcal{O}\left(\mathbb{RP}_q^2(l;-)\right)\hookrightarrow\mathcal{O}\left(\Sigma_q^3(l;-)\right)\hookrightarrow\mathcal{O}\left(\Sigma_q^3\right)\hookrightarrow\mathcal{O}\left(S_q^2\right)\left[u,u^{-1}\right]. \tag{5}$$

The second, third and fourth terms are strongly group graded algebras. Each antecedent term is the degree-zero part of the subsequent one. Since the enveloping algebra of $\mathcal{O}(S_q^2)[u,u^{-1}]$ is Noetherian, so are the enveloping algebras of all its predecessors. By [14, Corollary 4.6] the global dimension of $\mathcal{E}\mathcal{O}(S_q^2)$ is finite, hence so is the global dimension of $\mathcal{E}\mathcal{O}(S_q^2)[u,u^{-1}]$, and, by Criterion 1, the global dimensions of enveloping algebras of all its predecessors in (5). This proves the following

Theorem 3. *The algebras $\mathcal{O}(\Sigma_q^3)$, $\mathcal{O}(\mathbb{RP}_q^2(l;-))$ and $\mathcal{O}(\Sigma_q^3(l;-))$ are homologically smooth.*

2.4 Quantum real weighted projective planes $\mathbb{RP}^2_q(l;+)$ and teardrops (revisited)

Let k be a natural number and l be a positive integer. Write $\mathcal{A}(k,l)$ for the $*$-algebra generated by a and b subject to the following relations

$$a^* = a, \qquad ab = q^{-2kl}ba, \qquad bb^* = q^{2kl}a^k\left(a;q^2\right)_l, \qquad b^*b = a^k\left(q^{-2}a;q^{-2}\right)_l.$$

If k and l are coprime then $\mathcal{A}(k,l)$ is the coordinate algebra of the quantum weighted projective line or the *quantum spindle* $\mathcal{O}(\mathbb{WP}_q(k,l))$ introduced in [5]. The special case $k=1$ is simply the *quantum teardrop*; see Section 2.2. For l odd, $\mathcal{A}(0,l)$ is the coordinate algebra of the quantum weighted even real projective plane $\mathcal{O}(\mathbb{RP}^2_q(l;+))$ introduced in [3]. The following theorem is a consequence of Criterion 2.

Theorem 4. *The algebras $\mathcal{A}(k,l)$ are homologically smooth* (*of dimension* 2) *if and only if $k=0,1$.*

Proof. We only need to observe that each $\mathcal{A}(k,l)$ is a generalized Weyl algebra over the polynomial algebra $\mathbb{C}[a]$ given by the automorphism $\pi(a) = q^{2l}a$, element $p = a^k \prod_{m=1}^{l}(1-q^{-2m}a)$ and generators $x_- = b$, $x_+ = b^*$. Since p has no multiple roots if and only if $k=0,1$, the assertion follows by Criterion 2. ■

Furthermore, for $k=0,1$, $\mathcal{A}(k,l)$ are twisted Calabi–Yau algebras with the twisting automorphism $\nu(b) = q^{-2l}b$, $\nu(b^*) = q^{2l}b^*$ and $\nu(a) = a$. Hence they enjoy the Poincaré duality in the sense of Van den Bergh [20].

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